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ON THE JOINT ASYMPTOTIC DISTRIBUTION OF EXTREME MIDRANGES[1]

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We derive the joint asymptotic distribution of the k midranges formed by averaging the ith smallest normalized order statistic with the ith largest normalized order statistic, $i = 1, \cdots, k$. We then derive the distribution of the maximum midrange among these k extreme midranges and the limiting distribution of this maximum as $k \to \infty$. These results imply that, even in infinite samples, different distributions in the class of symmetric, unimodal distributions with tails that die at least as fast as a double exponential distribution may have different maximum likelihood estimates for the location parameter. We also discuss the application of these results to a test of symmetry suggested by Wilk and Gnanadesikan (1968).

1. Introduction. We present here two results on the distribution of midranges. The midranges are defined to be the averages of symmetrically placed order statistics. We will refer to the midranges formed by averaging symmetrically placed extreme order statistics as extreme midranges. The first result gives the joint asymptotic distribution of the k extreme normalized midranges. The second result gives the asymptotic distribution of the maximum of these k extreme normalized midranges and the limiting distribution of this maximum as $k \to \infty$.

These theorems were derived while investigating the following problem. Let $X_1, \cdots, X_n$ be a sample of size n drawn independently from a distribution with density of the form

$$f_g(x; \mu) = c_g e^{-g(x-\mu)}, \quad -\infty < x < \infty,$$

where μ is a location parameter, c_g is a normalizing constant and

$$g \in G = \{g \mid R^1 \to R^1, g \textit{ has two continuous derivatives}, g(x) = g(-x),$$
$$g(0) = 0, \quad g''(x) > 0 \; \forall x \neq 0, \quad \lim_{x\to\infty} g''(x)/[g'(x)]^2 = 0\}.$$

Let $\hat{\mu}$ be a maximum likelihood estimate of the location parameter, and let

$$B(G) = \{\hat{\mu} \mid g \in G\}$$

be the set of all possible maximum likelihood estimates $\hat{\mu}$ for a given data set and for the family of distributions corresponding to $g \in G$. Note that this is a family of unimodal, symmetric distributions with tails that die at least as fast as the double exponential distribution. Leamer (1981) has shown that

$$B(G) = \{\hat{\mu} \mid \min_{l<i<\bar{n}} M_i \leq \hat{\mu} \leq \max_{1\leq i\leq \bar{n}} M_i\},$$

where (i) M_i is the ith midrange, (ii) $\bar{n}$ is the greatest integer less than or equal to $(n + 1)/2$, and (iii) the equalities hold if and only if all the midranges are equal. This result says that given the data set, any point in the interval between the smallest and largest midrange is a maximum likelihood estimate for some distribution in the family corresponding to $g \in G$. The problem considered is: does the range of the set $B(G)$ tend to zero in

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probability as the number of data points increases to infinity regardless of from which distribution in the above family we happen to be sampling? In fact, we will show below that this is not the case. This implies that within the family of distributions given above, different distributional assumptions may lead to different estimates even in infinite samples.

Since the range of the midranges need not tend to zero in probability as $n \to \infty$, we investigate the asymptotic distribution of $R_M = \max_{1 \le i \le \tilde{n}} M_i - \min_{1 \le i \le \tilde{n}} M_i$. The investigation of the asymptotic distribution of R_M led to the two results referred to above.

In Section 2, we will present the two main theorems. In Section 3, we will discuss further the problem presented above as well as discuss an application of these results to a test of symmetry suggested by Wilk and Gnanadesikan (1968).

2. The distribution of extreme midranges. In this section, we derive the joint asymptotic distribution of the normalized k extreme midranges and the asymptotic distribution of the maximum of these k midranges. These results will be proved for a very general class of underlying distributions. This section is divided into three subsections. The asymptotic distribution of extreme order statistics will be reviewed, and the joint asymptotic density of the normalized k largest and k smallest order statistics will be derived in Section 2.1. The joint asymptotic density of the k extreme midranges and the asymptotic density of the maximum of these k midranges will be derived in Sections 2.2 and 2.3 respectively.

2.1 *The distribution of extreme order statistics.* Let $X_1, \cdots, X_n$ be an independent identically distributed sequence of random variables drawn from a symmetric distribution which satisfies the following assumption.

ASSUMPTION 2.1. Let $F(x)$ be a distribution function. Assume that there is a real number x_1, such that, for all $x_1 \le x < \infty$, $f(x) = F'(X)$ and $F''(x)$ exist and $f(x) \ne 0$. Furthermore, assume

$$\lim_{x\to\infty} \frac{d}{dx}\left[\frac{1 - F(x)}{f(x)}\right] = 0.$$

Distributions for which Assumption 2.1 holds are generally referred to as being of exponential-type because the tails die at an exponential rate. We denote by $Z_1, \cdots, Z_n$ the order statistics of the sample. The ith midrange, M_i, is defined as $M_i = (Z_i + Z_{n+1-i})/2$.

It is well known (v. Galambos (1978) Theorem 2.7.2) that Assumption 3.1 is a sufficient condition for there to exist normalizing sequences a_n and $b_n > 0$ such that

$$\lim_{n\to\infty} P(Z_n < a_n + b_n x) = \exp(-e^{-x}), \quad -\infty < x < \infty.$$

Furthermore, it is also known that the kth extremes can be normalized to have nondegenerate limiting distributions if and only if the maximum can be normalized and that the same normalizing sequences can be used.

We now give the joint asymptotic density of the normalized upper k and lower k order statistics. Since we are dealing with symmetric distributions, the normalizing sequences for the lower k order statistics can be taken as $-a_n$ and b_n. Also noting that the upper and lower extremes are asymptotically independent (v. David, 1970), we may find their joint density as the product of the asymptotic density for the normalized upper k and normalized lower k order statistics. These asymptotic densities have been given by Dwass (1966) and Weissman (1978), who investigated extremal processes. The results can also be obtained by a direct use of Scheffe's Theorem on the convergence of densities.

Let $V_i = (Z_{n+1-i} - a_n)/b_n$ denote the ith normalized upper order statistic, where a_n and $b_n > 0$ are the normalizing sequences for the upper extreme order statistics. Similarly let $U_i = (Z_i + a_n)/b_n$ denote the ith normalized lower order statistic. The joint asymptotic density of the normalized lower k and upper k order statistics is given by

$$(1)\quad f_{U_1,\dots,U_k,V_1,\dots V_k}(u_1,\dots,u_k,v_1,\dots,v_k) = \exp[-e^{u_k} - e^{-v_k} + \textstyle\sum_{j=1}^{k}(u_j - v_j)]1_{(u_l<\cdots<u_k)}1_{(v_k<\cdots<v_1)}.$$

2.2 *The joint asymptotic distribution of extreme midranges.* We now must make a change of variables to get the joint asymptotic distribution of the normalized midranges, $\tilde{M}_j = (U_j + V_j)/2 = (Z_i + Z_{n+1-i})/2b_n$. Note that these midranges are normalized by the sequence b_n.

THEOREM 2.2. *Define* $(a)^+ = \max(0, a)$. *The joint asymptotic density of the first* k *normalized midranges as* $n \to \infty$ *is given by*

$$(2)\quad f_{\tilde{M}_1,\dots,\tilde{M}_k}(m_1,\cdots,m_k) = \frac{(2k)!}{k!}\frac{\exp[-2\sum_{j=1}^{k} m_j - 4\sum_{j=1}^{k-1} j(m_{j+1}-m_j)^+]}{(1+e^{-2m_k})^{2k}},$$
$$-\infty < m_1,\cdots,m_k < \infty.$$

PROOF. We first make a change of variables in (1) by setting $v_j = 2m_j - u_j$, $1 \le j \le k$. The Jacobian of the transformation is 2^k. Then,

$$f_{U_1,\dots U_k,\tilde{M}_1,\dots,\tilde{M}_k}(u_1,\cdots,u_k,m_1,\cdots,m_k) = \exp[-e^{u_k} - e^{u_k-2m_k} + 2\textstyle\sum_{j=1}^{k}(u_j - m_j)]2_k$$
$$\times 1_{(u_1<\cdots\le u_k)}1_{(2m_k-u_k<2m_{k-1}-u_{k-1}<\cdots<2m_1-u_1)}.$$

We now must integrate out the variables $U_1, \cdots, U_k$. We will integrate the variables out in the order u_1 then $u_2, u_3, \cdots, u_k$. Note that the range of u_i is $-\infty < u_i < \min(u_{i+1}, u_{i+1} + 2(m_i - m_{i+1}))$ for $1 \le i \le k-1$ and $-\infty < u_k < \infty$. We start by integrating out u_1; there are two cases,

$$\int_{-\infty}^{u_2} f_{U_1,\dots,U_k,\tilde{M}_1,\dots,\tilde{M}_k}\,du_1,\quad m_1 \ge m_2,$$

and

$$\int_{-\infty}^{u_2+2(m_1-m_2)} f_{U_1,\dots,U_k,\tilde{M}_1,\dots,\tilde{M}_k}\,du_1,\ m_1 < m_2.$$

The two cases can be conveniently written together as

$$\int_{-\infty}^{u_2-2(m_2-m_1)^+} f_{U_1,\dots,U_k,\tilde{M}_1,\dots,\tilde{M}_1,\dots,\tilde{M}_k}\,du_1.$$

Carrying out the integration, we obtain

$$f_{U_2,\dots,U_k,\tilde{M}_1,\dots,\tilde{M}_k} = \frac{2^k}{2}\exp[-e^{u_k} - e^{u_k-2m_k} - 2\textstyle\sum_{j=1}^{k} m_j + 2\sum_{j=3}^{k} u_j + 4u_2 - 4(m_2-m_1)^+]$$
$$\times 1_{(u_1<\cdots<u_k)}1_{(u_i<u_{i+1}+2(m_i-m_{i+1}),i=2,\cdots,k-2)}.$$

Similarly, integrating over u_2 yields

$$f_{U_3,\dots U_k,\tilde{M}_1,\dots,\tilde{M}_k} = \frac{2^k}{2(4)}\exp[-e^{u_k} - e^{u_k-2m_k} - 2\textstyle\sum_{j=1}^{k} m_j + 2\sum_{j=4}^{k} u_j$$
$$+ 6u_3 - 4(m_2-m_1)^+ - 8(m_3-m_2)^+]$$
$$\times 1_{(u_3<\cdots<u_k)}1_{(u_i<u_{i+1}+2(m_i-m_{i+1}),i=3,\cdots,k-1)}.$$

Continuing in this way through $k - 1$, we arrive at

$$f_{U_k,\tilde{M}_1,\cdots,\tilde{M}_k} = \frac{2^k}{2(4)\cdots(2k-2)} \exp[-e^{u_k} - e^{u_k - 2m_k} + 2ku_k - 2\textstyle\sum_{j=1}^{k} m_j \tag{3}$$
$$- 4\textstyle\sum_{j=1}^{k-1} j(m_{j+1}-m_j)^+], \quad -\infty < u_k < \infty.$$

We now integrate over that part of (3) which involves u_k:

$$\int_{-\infty}^{\infty} \exp[-e^{u_k} - e^{u_k-2m_k} + 2ku_k]\, du_k = \int_0^{\infty} t^{2k-1} e^{-t(1+e^{-2m_k})}\, dt = \frac{\Gamma(2k)}{(1+e^{-2m_k})^{2k}}, \tag{4}$$

where the substitution, $t = e^{u_k}$, has been used. Substituting this into (3) and simplifying, we arrive at the joint asymptotic density given in the theorem.

Theorem 2.2 generalizes the result of Gumbel (1944), who gave the asymptotic marginal distribution of $\tilde{M}_k$.

2.3 *The asymptotic distribution of the maximum of the midranges.* With the joint asymptotic distribution of the extreme midranges in hand, we are ready to attack the problem of finding the asymptotic distribution of the maximum of the extreme midranges. Let $T_k = \max_{l \le i \le k} \tilde{M}_k$. We want to find $P(T_k < x)$. The difficulty arises from the terms involving $(m_{i+1} - m_i)^+$, since these terms require the k-fold integral to be broken up into integrals over many different regions. Theorem 2.3.1 gives a one-dimensional integral expression for $P(T_k < t)$. Lemma 2.3.3 then shows that this integral can be written as a summation of a finite number of terms. Finally, Theorem 2.3.2 shows that as $k \to \infty$, T_k tends toward an exponential distribution.

THEOREM 2.3.1. *The distribution of T_k is given by the following integral:*

$$P(T_k < t) = F_{T_k}(t) = \int_{e^{-2t}}^{\infty} \binom{2k-1}{k} \frac{ku^{k-1} - e^{-2t}(k-1)u^{k-2}}{(1+u)^{2k}}\, du, \quad k \ge 1. \tag{5}$$

The proof of the above theorem is a consequence of the following two lemmas. For convenience, we write X for e^{-2t}.

LEMMA 2.3.1. *Define $I_j(s, t)$ recursively by $I_1 = 1$ and*

$$I_{j+1}(s,t) = \int_X^s I_j(u,t)\, du + \int_s^{\infty} s^{2j} u^{-2j} I_j(u,t)\, du. \tag{6}$$

Then,

$$F_{T_k}(t) = \int_X^{\infty} \frac{(2k)!}{2^k k!} \frac{I_k(u,t)}{(1+u)^{2k}}\, du. \tag{7}$$

PROOF. We first make a change of variables in (2) by setting $s_i = \exp(-2m_i)$, $i = 1, \cdots, k$. The Jacobian of the transformation is $(-\frac{1}{2})^k / \prod_{i=1}^{k} s_i$. Then,

$$f_{s_1,\cdots,s_k}(s_1,\cdots,s_k) = \frac{(2k)!(-1)^k}{2^k k!} (1+s_k)^{-2k} \textstyle\prod_{j=1}^{k-1} [s_{j+1} s_j^{-1}]^{2jW_j},$$

where

$$W_j = \begin{cases} 1, & \text{if } s_{j+1} \le s_j \\ 0, & \text{if } s_{j+1} > s_j. \end{cases}$$

Now, in terms of the S_j's, we can write

$$F_{T_k}(t) = \int_X^\infty \cdots \int_X^\infty \frac{(2k)!}{2^k k!} (1 + s_k)^{-2k} \prod_{j=1}^{k-1} (s_{j+1}s_j^{-1})^{2jW_j}\, ds_1 \cdots ds_k,$$

where the factor $(-1)^k$ is used to invert the limits of integration.

We perform the integration in the order s_1 then $s_2, \cdots, s_k$. At each stage, because of the factor W_j in the exponent, the integral breaks into two parts. We suppress the factors not involved in the first integration and write

$$I_2(s_2, t) = \int_X^{s_2} 1\, ds_1 + \int_{s_2}^\infty s_2^2 s_1^{-2}\, ds_1,$$

where the definition of I_2 given in (6) has been used. Note that in the first integral $s_1 < s_2$, so that $W_1 = 0$, while in the second integral $s_1 \geq s_2$, so that $W_1 = 1$.

At the next integration the integral again breaks into two parts. Suppressing the factors not involved in the second integration, and using (6), we can write

$$I_3(s_3, t) = \int_X^{s_3} I_2\, ds_2 + \int_{s_3}^\infty s_3^4 s_2^{-4} I_2\, ds_2.$$

At each stage the integration breaks into two regions and depends on the value of the integral from the previous stage. In general, we have

$$I_{j+1}(s_{j+1}, t) = \int_X^{s_{j+1}} I_j(s_j, t)\, ds_j + \int_{s_{j+1}}^\infty s_{j+1}^{2j} s_j^{-2j} I_j(s_j, t)\, ds_j.$$

Then for the final stage, we find

$$F_{T_k}(t) = \int_X^\infty \frac{(2k)!}{2^k k!} \frac{I_k(s_k, t)}{(1 + s_k)^{2k}}\, ds_k.$$

We next find a closed form expression for $I_j(s, t)$.

LEMMA 2.3.2

(8) $$I_j(s, t) = \frac{2^{j-1}}{j!} [js^{j-1} - X(j-1)s^{j-2}].$$

The proof follows easily by induction.

PROOF OF THEOREM 3.3.1. By substituting the value of $I_k(u, t)$ given in (8) into (7), we can write

$$F_{T_k}(t) = \int_X^\infty \frac{(2k)!}{2^k k!} \frac{2^{k-1}}{k!} \frac{[ku^{k-1} - X(k-1)u^{k-2}]}{(1 + u)^{2k}}\, du,$$

which simplifies to the form given in the theorem.

We next evaluate the integral expression for $F_{T_k}(t)$ by using the fact that

(9) $$F_{T_k}(t) = F_{T_1}(t) - \sum_{j=1}^{k-1} [F_{T_j}(t) - F_{T_{j+1}}(t)].$$

To simplify the notation we will denote $F_{T_j}(t)$ by F_j. The next lemma gives a closed form expression for the value of $F_j - F_{j+1}$.

LEMMA 2.3.3.

(10) $$F_j - F_{j+1} = \frac{(2j)!}{j!(j+1)!} \frac{X^{j+1}}{(1 + X)^{2j+1}}.$$

PROOF. In the integral for F_j in (5) let $v = 1/(1+u)$. The integral becomes (cf. Abramowitz and Stegun, 1970, page 263):

$$F_j = \int_0^{1/(1+X)} \frac{(2j-1)!}{j!(j-2)!}\left[\frac{j}{j-1}(1-v)^{j-1}v^{j-1} - X(1-v)^{j-2}v^j\right] dv$$

$$(11)\quad = \frac{1}{(1+X)^{2j-1}}\left[\sum_{i=j}^{2j-1}\binom{2j-1}{i}X^{2j-1-i} - X\sum_{i=j+1}^{2j-1}\binom{2j-1}{i}X^{2j-1-i}\right]$$

$$= \frac{1}{(1+X)^{2j-1}}\left[\sum_{i=0}^{j-1}\binom{2j-1}{i}X^i - X\sum_{i=0}^{j-2}\binom{2j-1}{i}X^i\right]$$

$$= \frac{1}{(1+X)^{2j-1}}\sum_{i=0}^{j-1}\binom{2j}{i}\left(\frac{j-1}{j}\right)X^i.$$

We now use (11) to evaluate $F_j - F_{j+1}$. By using the properties of the binomial coefficients, we can show that for $0 \le i \le j$, the coefficient of $X^i(1+X)^{-2j-1}$ in $F_j - F_{j+1}$ is zero, while for $i = j$ the coefficient is equal to $(2j)!/j!(j+1)!$, and the lemma follows.

Using (9) and (10), we can write

$$(12)\quad F_{T_k}(t) = \frac{1}{1+X} - \sum_{j=1}^{k-1}\frac{(2j)!}{j!(j+1)!}\frac{X^{j+1}}{(1+X)^{2j+1}}.$$

We now look for $F_T = \lim_{k\to\infty} F_{T_k}(t)$. Since $F_{T_{k+1}}(t)$ is the distribution of the maximum of more variables that $F_{T_k}(t)$, necessarily $F_{T_{k+1}}(t) \le F_{T_k}(t)$, $\forall t$. By monotone convergence, F_T must exist, $F_{T_k}(t)$ being bounded below by 0. Therefore, we can write

$$(13)\quad F_T(t) = \frac{1}{1+X} - \lim_{k\to\infty}\sum_{j=1}^{k}\frac{(2j)!}{j!(j+1)!}\frac{X^{j+1}}{(1+X)^{2j+1}}.$$

Note that, since F_T is the distribution of the maximum of an infinite number of symmetric random variables, $F_T(t) = 0$, $\forall t \le 0$ (that is, the right-hand side of (13) is zero for all $X \ge 1$).

Finally we show that the limiting distribution of T_k as $k \to \infty$ is an exponential distribution.

THEOREM 2.3.2.

$$F_T(t) = \begin{cases} 0, & t \le 0 \\ 1 - e^{-2t}, & t > 0. \end{cases}$$

PROOF. The proof follows easily from the identity

$$\sum_{j=0}^{\infty}\frac{(2j)!}{j!(j+1)!}u^{j+1} = \frac{1}{2}(1 - \sqrt{1-4u}), \quad |u| \le \frac{1}{4},$$

and recalling that X is our stand-in for e^{-2t}.

3. The range of the midranges. In Section 1, we considered the set

$$B(G) = \{\hat{\mu} \mid \min_{1\le i\le \bar{n}} M_i \le \hat{\mu} \le \max_{l\le i\le \bar{n}} M_i\}.$$

We are interested in whether the range of this set tends to zero in probability when the underlying distribution is in the family of symmetric, unimodal distributions defined in the introduction. It is straightforward to show that this class of distributions satisfies Assumption 2.1, so that the theorems derived in Section 2 are applicable to the problem at hand.

Since the distributions we are considering are symmetric, we know that

$$\lim_{n\to\infty} P(\max_{1\le i\le \bar{n}} M_i < x) = 0, \quad \forall x \le 0,$$

and

$$\lim_{n\to\infty}P(\min_{l\leq i\leq \bar{n}}M_i < x) = 1, \quad \forall x \geq 0.$$

Therefore, if $\max_{l\leq i\leq \bar{n}}M_i$ has a nondegenerate limiting distribution, then the range of $B(G)$ cannot tend to zero in probability. In the previous section we have shown that

$$\lim_{k\to\infty}P\left(\max_{1\leq i\leq k}\lim \frac{M_i}{b_n} < x\right) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-2x}, & x > 0. \end{cases}$$

Whether the asymptotic distribution of $T_{\bar{n}} = \max_{1\leq i\leq \bar{n}}M_i$, which is greater than or equal to $T_k = \max_{1\leq i\leq k}M_i$ for all k, is nondegenerate clearly depends on the normalizing sequence b_n. Therefore, we will investigate this sequence below and give an expression for both a_n and b_n in Theorem 3.1.

In the remainder of this section, we will further discuss the asymptotic distribution of the maximum of the midranges. Finally, we will consider the implications of these theorems with respect to the test of symmetry referred to in the introduction.

THEOREM 3.1. *Let*

$$F_g(x) = \int_{-\infty}^{x} c_g e^{-g(t)}\, dt, \quad -\infty < x < \infty,$$

where $g \in G$. Then, with a_n and $b_n > 0$ as given below, the maximum of a sample of size n drawn independently from the distribution $F_g(x)$ has a nondegenerate limiting distribution

$$a_n = g^{-1}(\log n) - \frac{\log[g'(g^{-1}(\log n))] - \log c_g}{g'(g^{-1}(\log n))} \tag{14}$$

and

$$b_n = \frac{1}{g'(g^{-1}(\log n))}. \tag{15}$$

The proof makes use of the specific properties of $g \in G$ but otherwise follows similarly to the derivation given in Galambos (1978) page 66–67, and is therefore omitted.

From the form of b_n given above, we can immediately see that if $\lim_{x\to\infty}g'(x) = c$, then the limiting distribution of $\max_{1\leq i\leq \bar{n}}M_i$ is not degenerate at zero and the range of $B(G)$ does not tend to zero in probability. For the other possible case when $\lim_{x\to\infty}g'(x) = \infty$, since the limiting distribution of $\max_{1<i<\bar{n}}M_i \geq \max_{1<i<k}M_i$, the fact that $\lim g'(x) = \infty$ does not necessarily imply that $\max_{1\leq i\leq \bar{n}}M_i \to 0$ in probability. Let

$$F_{T_n}(t) = P\left(\max_{1\leq i\leq \bar{n}} \frac{M_i}{b_n} < t\right)$$

and

$$F_{T_k}(t) = P\left(\max_{1\leq i\leq k}\lim_{n\to\infty} \frac{M_i}{b_n} < t\right).$$

We make the following conjecture: $\lim_{n\to\infty} F_{T_n} = \lim_{k\to\infty}F_{T_k}$. The proof is difficult because it is hard to estimate $F_{T_{\bar{n}}}$ for finite n and arbitrary $g \in G$.

There is one case for which the above conjecture can be proved directly. This is the case of a sample drawn from a double exponential distribution. By a technique analogous to the proof of Theorems 2.2 and 2.3.1, it is possible to calculate

$$\lim_{n\to\infty}P(\max_{1\leq i\leq \alpha n}M_i < x), \quad 0 < \alpha < 1/4.$$

The details are given in Gilstein (1980). However, for this case, $\lim_{x\to\infty}g'(x)$ equals 1, so that this case does not shed any light on the more interesting case in which $\lim_{x\to\infty} g'(x)$ is infinite.

As a final consideration we apply the above results to the test of symmetry proposed by Wilk and Gnanadesikan (1968). Wilk and Gnanadesikan suggested plotting pairs of symmetrically placed order statistics and looking for a linear relationship with a slope of negative one. Consider the points (Z_i, Z_{n+1-i}) and the line with slope equal to negative one which passes through the point (x_0, y_0). The deviation of the point (Z_i, Z_{n+1-i}) from this line is equal to $x_0 + y_0 - Z_i - Z_{n+1-i} = x_0 + y_0 - 2M_i$. If the distribution is symmetric, then the best line would tend to pass through the point (med X_i, med X_i), since then the expected value of the deviations would be zero. However, the above theorems imply that even if the sample is drawn from a symmetric distribution, if the tails of the distribution do not die more quickly than an exponential distribution, then the maximum deviation of the points (Z_i, Z_{n+1-i}) from this line does not tend to zero in probability as the number of data points tend to infinity. This suggests that we should first trim some proportion, say 5 percent, of the extreme sample values. Since all the midranges would then be formed from quantiles, the maximum midrange among the trimmed set would tend to zero in probability. This final result can be shown by using the bounds given in Gilstein (1981) to show that the expected value of the maximum midrange among the trimmed set tends to zero in probability.

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