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SPHERICALLY SYMMETRIC PROBABILITY ORDERINGS USEFUL IN MULTIPLE COMPARISONS[1]

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The statement that one region has more probability content than another with respect to all spherically symmetric (rotation invariant) distributions is a partial ordering among such regions. A simple geometric characterization of this ordering is given for star-shaped regions containing the origin. This characterization has several different interpretations. New techniques, inequalities, and examples are produced from this geometrical approach. The results have particular application to simultaneous confidence levels.

1. Introduction.

1.1 *Background.* Test sizes and confidence levels for simultaneous inference about many parameters can often be expressed in terms of the probability content of certain regions with respect to known multivariate distributions such as the normal or Student t. Thus F and Chi squared tests give rise to spherical regions. For simultaneous tests and confidence intervals concerning several individual parameters or for comparisons with a control, hypercubes or rhombi may be needed. More complex regions may arise, for example, from tests based on Studentized range procedures. Details may be found in Scheffé (1959), Miller (1966), Wynn and Bohrer (1978), or in review articles such as O'Neill and Wetherill (1971), Spjøtvoll (1974), and Miller (1977). A number of useful inequalities have been derived for comparing the probability contents of such regions. Thus, the papers of Dunn (1958), Slepian (1962), Sidak (1967), Jogdeo (1970), Das Gupta *et al.* (1972), and Wynn (1977) discuss the probability content of some standard figures or their generalizations with respect to particular or arbitrary spherically symmetric distributions.

In this paper we give a simple geometric approach leading to a rather different style of proof. The techniques are simplified and exemplified in the case of Scheffé sets (Section 2.1), in a partial ordering for correlation matrices (Section 2.1), in new perspectives on the work of Wynn (1977) (Section 2.2), in the use of facial distributions to compare measures of solids with congruent faces (Section 3.1), and in two methods for ordering the three-dimensional Platonic and Studentized range solids with some statistical interpretations (Sections 3.2 and 3.3).

1.2 *Characterization.* Let $\mathbf{X} = (X_1, X_2, \cdots, X_n)'$ be a random vector in $\mathscr{R}^n$ with a spherically symmetric distribution. Thus, if μ is the probability measure associated with X, then for any rotation, P, about the origin, $\mathbf{O}$, and for any measurable set, A, we have $\mu[P(A)] = \mu(A)$. It is convenient to represent the random vector $\mathbf{X}$ in $\mathscr{R}^n$ by putting $\mathbf{X} = R\mathbf{S}$, where $R = \|\mathbf{X}\|$ and $\mathbf{S} = \mathbf{X}/\|\mathbf{X}\| (\mathbf{X} \neq \mathbf{0})$ are the length of $\mathbf{X}$ and its direction, interpreted as a point on $\mathscr{S}_n$, the surface of the unit sphere in $\mathscr{R}^n$. With this representation in terms of R and $\mathbf{S}$, the measure μ can be rewritten as the product of a probability measure, μ_R, on $[0, \infty)$ and uniform measure, $\mu_\mathbf{S}$, on $\mathscr{S}_n$. Thus, $\mu = \mu_R \times \mu_\mathbf{S}$. The random variables associated with μ_R and $\mu_\mathbf{S}$ are denoted R and $\mathbf{S}$.

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A star-shaped region, $\mathscr{H}$, in $\mathscr{R}^n$ is a region such that for every $\mathbf{s}$ in $\mathscr{S}_n$ the set of c for which $c\mathbf{s}$ is in $\mathscr{H}$ is a closed interval, $[0, H(\mathbf{s})]$ (say), or $[0, \infty]$, in which we interpret $H(\mathbf{s}) = \infty$. Thus, for any $\mathbf{X}$ in a star-shaped set H, $c\mathbf{X}$ is in H for $0 \le c \le 1$, and H includes its boundary, although this may, in some directions, be at ∞.

For two regions, H and H^*, in $\mathscr{R}^n$, we denote $H > H^*$ if and only if $\mu(H) > \mu(H^*)$ for all spherically symmetric μ. The purpose of the present paper is to investigate this partial ordering ">". Note that we write $H = H^*$ if $\mu(H) = \mu(H^*)$ for every spherically symmetric μ. This holds trivially if $H = P(H^*)$ for some rotation P. We sometimes use this fact implicitly.

For any star-shaped region, H, and any direction, $\mathbf{s}$, we have $H(\mathbf{s}) = \sup\{c: c\mathbf{s} \in H\}$, so that $H(\mathbf{s})$ is the distance to the boundary of H from the origin, $\mathbf{0}$, in the direction $\mathbf{s}$. Now if $\mathbf{S}$ is uniformly distributed on $\mathscr{S}_n$, then $H(\mathbf{S})$ is a random variable. For r positive, define the distance distribution of H to be $F_H(r) = \mu_{\mathbf{S}}[H(\mathbf{S}) < R]$. Note the use of $<$, rather than $\le$, in this definition. With these definitions, we can give a simple geometric characterization of the partial ordering $>$.

THEOREM 1. *For two star-shaped regions, H and H^*, in $\mathscr{R}^n$,*

(i) $$H > H^*$$

if and only if

(ii) $$F_H(r) \le F_{H^*}(r) \text{ for all } r > 0.$$

PROOF. For any spherically symmetric $\mu = \mu_R \times \mu_{\mathbf{S}}$ and for any star-shaped set G, $\mu(G) = \int_0^\infty \mu_{\mathbf{S}}\{\mathbf{s}: G(\mathbf{s}) > r\}\, d\mu_R(r) = \int_0^\infty [1 - F_G(r)]\, d\mu_R(r)$. From this, if (ii) holds, then (i) follows.

Now assume (i), i.e., that $\mu(H) > \mu(H^*)$ for all spherically symmetric measures μ. Consider $\mu = \mu_R \times \mu_{\mathbf{S}}$, where μ_R puts unit mass at $R = r^*$. Then the distribution $F_R(r)$ takes a single, unit step at r^*, so that for any star-shaped set G, we have

$$\mu(G) = \int_{S_n} F_R(G(\mathbf{s}))\, d\mu_{\mathbf{S}}(\mathbf{s}) = \int_{G^*} d\mu_{\mathbf{S}}(\mathbf{s}) = 1 - \mathrm{F}_G(r^*),$$

where $G^* = \{\mathbf{s}: G(\mathbf{s}) > r^*\}$. Applying this with $G = H$ and $G = H^*$ and using (i) gives $F_H(r^*) \le F_{H^*}(r^*)$ for arbitrary r^*, which completes the proof.

The distance distribution, F_H, is a purely geometrical quantity. Analytically, it can be easier to deal with than the partial ordering $>$, to which it is so closely related by Theorem 1.

2. Partial Orderings of Scheffé Sets.

2.1 *Scheffé sets.* One type of convex, star-shaped region arises naturally from the theory of simultaneous confidence intervals. These "Scheffé sets" originated in the work of Working and Hotelling (1929), were generalized by Scheffé (1953), and appear in the constrained extensions studied by Bohrer (1967, 1969), Bohrer and Francis (1972a, b), Wynn and Bloomfield (1971), and Wynn (1975, 1977).

Let $\mathbf{X}$ have a spherically symmetric distribution in $\mathscr{R}^n$, and let σ^2 denote the variance of X_i for $i = 1, \cdots, n$. Typical of the type of set measured in simultaneous inference is the set of all $\mathbf{X}$ such that

(1) $$\mathbf{a}'\mathbf{X} < K[\mathrm{Var}(\mathbf{a}'\mathbf{X})]^{1/2} \quad \text{for all } \mathbf{a} \text{ in } A,$$

where A is a given set of n-vectors. Since the X_i are uncorrelated, (1) can be rewritten as

$$\mathbf{a}'\mathbf{X} \le K\sigma \| \mathbf{a} \| \quad \text{for all } \mathbf{a} \text{ in } A \text{ or } \mathbf{s}'\mathbf{X} \le c \quad \text{for all } \mathbf{s} \text{ in } S,$$

where $S^* = \{\mathbf{s}: \mathbf{s} = \mathbf{a}/\|\mathbf{a}\|,\, a \text{ in } A\}$ and where $c = K\sigma$.

With this motivation and for given $c > 0$ and S any set of unit vectors, we define the Scheffé set $H = \{\mathbf{X}: \mathbf{s}'\mathbf{X} < c \text{ for all } s \text{ in } S\}$. This is merely the intersection of all the half-spaces on the origin sides of the hyperplanes $\{\mathbf{X}: \mathbf{s}'\mathbf{X} = c\}$, which are tangent to the sphere of radius c and perpendicular to $\mathbf{s}$, for all $\mathbf{s}$ in S. Such regions are convex, contain the origin, and, hence, are star-shaped.

For Scheffé sets, H, the distance to the boundary, $H(\mathbf{s})$, is monotonically related to two other quantities, which are of interest to investigate. Suppose that $S = \{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_m\}$ is finite, so that $c\mathbf{s}_i$, $i = 1, \cdots, \mu$, are the tangent points of the faces of H to the sphere $c\mathscr{S}_n$. Suppose that the points s_i are distinct and that H is bounded. For a given point $H(\mathbf{s})\mathbf{s}$ of the boundary of H, suppose that it lies in the hyperplane $\{\mathbf{X}: \mathbf{s}_J'X = c\}$. Thus $H(\mathbf{s})\ \mathbf{s}_J = c$, or $H(\mathbf{s}) = c/(\mathbf{s}'\mathbf{s}_J)$ where $\mathbf{s}'\mathbf{s}_J$ is the maximum over $i = 1, \cdots, m$ of $\mathbf{s}'\mathbf{s}_i$. We may interpret $\theta(\mathbf{s}) = \arccos(\mathbf{s}_J'\mathbf{s}_J)$, where $0 \le \theta \le \pi/2$, as the angle between $\mathbf{s}$ and $\mathbf{s}_J$ and $c\theta(\mathbf{s})$ as the geodesic distance in the c-sphere between $c\mathbf{s}$ and the nearest tangent point, $c\mathbf{s}_J$. Thus, $H(\mathbf{s})$ is a monotonic function of this geodesic distance. We cite the consequence of this relationship as a corollary to Theorem 1.

COROLLARY 1. *Let* $\mathbf{X}$ *have a spherically symmetric distribution in* $\mathscr{R}^n$. *For c positive and finite, two sets, S and* S^*, *of unit vectors define two Scheffé sets, H and* H^*, *for which* $H > H^*$ *if and only if, for* $\mathbf{S}$ *distributed uniformly in* $\mathscr{S}_n$,

$$\mu_{\mathbf{S}}(\sup_S \mathbf{s}'\mathbf{S} < r) \ge \mu_{\mathbf{S}}(\sup_{S^*} \mathbf{s}'\mathbf{S} < r)$$

for all positive r.

The statement in Corollary 1 imposes a partial ordering on correlation matrices in the following sense. Let S denote a set of m^* unit vectors and let S^* denote a set of m^* unit vectors, all in $\mathscr{R}^n$. Let Z $(n \times m)$ and $Z^*(n \times m^*)$ denote the matrices with these vectors as the respective columns. Then $Z'Z$ and $Z^{*\prime}Z^*$ are correlation matrices. The element in row i and column j is the correlation of $\mathbf{s}_i'\mathbf{X}$ with $\mathbf{s}_j'\mathbf{X}$, where $\mathbf{s}_k$ is the kth unit vector from S^* or S^{**}, respectively. Corollary 1 can be interpreted to say that $Z^{*\prime}Z^* > Z^{*\prime}Z^*$ in the sense that the maximum correlation between a randomly chosen $\mathbf{S}'\mathbf{X}$ and its "nearest" $\mathbf{S}_J'\mathbf{X}$ is stochastically larger for the set S than for S^*. The authors consider that this rather strong partial ordering will have application elsewhere in multivariate analysis. The Platonic figures of Section 3.2 provide examples.

The second quantity related monotonically to $H(\mathbf{s})$ is the distance in the face of H from the boundary point $H(\mathbf{s})\mathbf{s}$ to the tangent point, $c\mathbf{s}_J$ (say), viz.,

$$D(\mathbf{s})^2 = \| H(\mathbf{s})\mathbf{s} - c\mathbf{s}_J\|^2 = H(\mathbf{s})^2 - c^2.$$

We shall return to the use of faces in Section 3.

2.2 *The Scheffé sets for* $n = 2$. As an application of Section 2.1, we rework the main result of Wynn (1977). Suppose that $n = 2$ and that $S^* = \{\mathbf{s}_i : i = 1, \cdots, m\}$ consist of m distinct vectors defining, for $c > 0$, a bounded Scheffé set, H. Suppose the points $\mathbf{s}_i$ are labeled in order around the unit circle, S_2. Define $\theta_i = \arccos(\mathbf{s}_i'\mathbf{s}_{i+1})$ for $i = 1, \cdots, m - 1$ and $\theta_m = \arccos(s_m'\mathbf{s}_1)$. These θ_i are thus the angles subtended at the origin by the neighboring tangent points. Since the $\mathbf{s}_i$ are distinct and since H is bounded, it follows that $0 < \theta_i < \pi$ for all i and that $\Sigma\theta_i = 2\pi$. Without loss of generality, order the angles so that $\theta_{i+1} \ge \theta_i$. Now take a random $\mathbf{S}$ on $\mathscr{S}_2$, and write $\theta(\mathbf{S}) = \arccos(\mathbf{S}'\mathbf{s}_J)$, where $\mathbf{S}'\mathbf{s}_J = \sup(\mathbf{S}'\mathbf{s}_i) \ge 0$. Now the distance distribution is

$$F_H(r) = \mu_{\mathbf{S}}\{\mathbf{s} : H(\mathbf{s}) < r\} = \mu_{\mathbf{S}}\{\mathbf{s} : c/(\mathbf{s}'\mathbf{s}_J) < r\} = \mu_{\mathbf{S}}\{0 < \theta(\mathbf{S}) < \psi\},$$

where $\psi = \arccos(c/r)$. Thus

$$F_H(r) = 1 - \mu_{\mathbf{S}}(\theta(\mathbf{S}) \ge \psi) = 1 - \Sigma(\theta_i - \psi)^+/(2\pi).$$

Thus, for two such figures, H and H^*, we have $H > H^*$ if and only if

$$\Sigma(\theta_i - \psi)^+ > \Sigma(\theta_i^* - \psi)^+ \tag{2}$$

for all ψ in $[0, \pi]$.

Now note that the case where $\theta_m \geq \pi$ can be dealt with by allowing a mass of probability at $\theta = \pi$, which does not change the condition.

In Theorem 2 of Wynn (1977), a sufficient condition also was given. Set $m = m^*$ by including "dummy" values of $\theta = 0$ for the figure with fewer angles. Then (2) is equivalent to

$$\textstyle\sum_{i=k}^{m} \theta_i \geq \sum_{i=k}^{m} \theta_i^*, \qquad k = 2, \cdots, m.$$

This condition, together with the condition that the angles sum to 2π, is exactly the condition that the vector $\boldsymbol{\theta}$ majorizes the vector $\boldsymbol{\theta}^*$, and the sufficient conditions follow immediately. This fact is a consequence of Theorem 8 in Hardy, Littlewood, and Polya (1929), as noted by Marshall and Olkin (1979) (Proposition E' *et seq.*). It follows that $m(H)$ is a Schur convex function of $\boldsymbol{\theta}$. There seems to be no simple extension of this result to higher dimensions solely in terms of angles, although the methods of the next section give a partial solution.

3. Facial distributions.

3.1 *Using faces.* As mentioned in Section 2.1,

$$D(\mathbf{S})^2 = H(\mathbf{S})^2 - c^2, \tag{3}$$

where $D(\mathbf{S})$ is the distance to the nearest tangent point, $c\mathbf{s}_J$. In comparing two polyhedral, Scheffé sets, it is natural to consider the $(n - 1)$-dimensional faces. Let H be a closed Scheffé region defined by $c > 0$ and finitely many distinct unit vectors, $S = \{\mathbf{s}_i : i = 1, \cdots, m\}$. The uniform distribution on $\mathscr{S}$ induces a spherically symmetric distribution on the "facial" planes $\{\mathbf{x} : \mathbf{s}'\mathbf{x} = c\}$ for $i = 1, \cdots, m$. The facial probability distribution analogous to $F(r)$ we denote by

$$F_D(d) = \mu_{\mathbf{S}}\{\mathbf{s} : D(\mathbf{s}) \leq d\} = \textstyle\sum_i \mu_{\mathbf{S}}(\mathbf{s}_J = \mathbf{s}_i)\mu_{\mathbf{S}}\{\mathbf{s} : D(s) \leq d \mid \mathbf{s}_J = \mathbf{s}_i\}$$

where, as in Section 2.1, $\mathbf{s}_J$ is the S-vector which is closest to the random direction $\mathbf{S}$.

One difficulty in measuring general polyhedral regions, H, is that the faces are often very different in shape. Suppose, however, that H is regular, in the sense that all its faces are congruent, as, for example, obtains when H is one of the Platonic solids or any other of the classically regular polyhedra. That is, for every pair of faces, there is a rotation of the one onto the other. Then

$$F_D(d) = \mu_{\mathbf{S}}\{D(\mathbf{S}) \leq d \mid \mathbf{s}_J = \mathbf{s}_1\},$$

so that we need to consider only one face. If that one face were the entire hyperplane, $H_1 = \{\mathbf{X} : \mathbf{s}_1'\mathbf{X} = c\}$, then

$$\mu_{\mathbf{S}}\{D(\mathbf{S}) \leq d \mid H_1\} = \mu_{\mathbf{S}}\{\theta(S) \leq \arccos[c^2/(c^2 + d^2)]^{1/2} \mid H_1\}$$

$$= k \int_0^t \sin^{n-2}(u)\, du,$$

where $t = \arccos[c^2/(c^2 + d^2)]^{1/2}$ and $k \int_0^{2\pi} \sin^{n-2}(u)\, du = 1$. In this case, the density for $D(\mathbf{S})$ is then

$$f_D(d \mid H_1) = kcd^{n-2}/(c^2 + d^2)^n.$$

Now the face perpendicular to $\mathbf{s}_1$ is a proper subset of the hyperplane H_1, so at a distance, d, the previous density is multiplied by m, to account for the m faces, and reduced by

$$(4) \qquad A(d) = \text{the proportion of the content of the surface of } S(d) \text{ which is contained in the Scheffe set face,}$$

where $S(d)$ is the $(n-1)$-dimensional sphere of radius d, lying in the Scheffe set face and with center at the tangent point, $\mathbf{s}_1$. Thus,

$$f_D(d) = mkcdA(d)/(c^2 + d^2)^n.$$

The dependence of this density on angles is thus through $A(d)$, which is the solid angle cut by the face in the surface of $\mathbf{S}(d)$. Since the faces are convex, this solid angle decreases from unity when $d = 0^+$, i.e., in a neighborhood of the tangent point $\mathbf{s}_1$, to 0 when the face is wholly within the sphere $S(d)$.

Now a necessary and sufficient condition for $H > H^*$ in comparing two Scheffé sets defined with the same $c > 0$ is that $F_D(d) < F_{D^*}(d)$ for all $d > 0$. If each of the two figures has (separately) congruent faces, then $H > H^*$ is equivalent to

$$\mu_{\mathbf{S}}\{D(\mathbf{S}) < d \mid \mathbf{s}_J = \mathbf{s}_1\} \le \mu_{\mathbf{S}}\{D^*(\mathbf{S}) < d \mid \mathbf{s}_J^* = \mathbf{s}_1^*\} \quad \text{for all} \quad d > 0,$$

in an obvious notation. We now use these ideas to give a useful sufficient condition.

THEOREM 2. *Let H and H^* be two Scheffé sets tangent to the same sphere of radius c, having m and m^* (separately) congruent faces, respectively. Suppose that* (i) $m \le m^*$, (ii) $\max H(\mathbf{s}) \ge \max H^*(\mathbf{s})$, *where the maximum is over all* $\mathbf{s}$ *in* $\mathscr{S}_n$, *and* (iii) $mA(d) - m^*A^*(d)$ *has at most one sign change from + to − as d goes from 0^+ to ∞. Here, A and A^* are the solid angles defined, as in* (4), *in the definitions of f_D and f_{D^*}.* Then $H > H^*$.

PROOF. Since $m \le m^*$ and $A(0^+) = A^*(0^+) = 1$, it follows that

$$f_D(0^+) - f_{D^*}(0^+) \le 0.$$

Moreover, there is only one sign change in $f_D(d) - f_{D^*}(d)$. Thus, $F_{D^*}(d) - F_D(d)$ is nonnegative and increasing at 0^+, increases to a maximum as d increases, and then decreases. By virtue of (ii), F_D then attains the maximum value, unity, at a smaller d-value than F_{D_*}. Thus, $F_{D^*}(d) \ge F_D(d)$ for all d, and the proof is complete.

3.2 *Ordering the Platonic solids.* To illustrate the method of Section 3.1, a complete ordering of the Platonic solids in three dimensions is given. These solids are: $H(4)$, the tetrahedron (4 triangular faces); $H(6)$, the cube (6 square faces); $H(8)$, the octahedron (8 triangular faces); $H(12)$, the (pentagonal) dodecahedron (12 pentagonal faces); and $H(20)$, the icosahedron (20 triangular faces).

The intuition suggests that if all these solids are scaled so that their faces are tangent to the same (inscribed) sphere then they are ordered so that their probability content, with respect to any spherically symmetric distribution, decreases as their number of faces increases. As a special case, it is easy to show, using Table I in Coxeter (1948), that their volumes are so ordered. The present result thus generalizes the volume comparison by showing that ordering persists for the area of intersection with every spherical shell.

Two of these Platonic solids arise as the regions measured to set critical points for confidence region or testing multiple comparison procedures. The cube arises in any procedure which requires the (possibly Studentized) maximum modulus distribution. See Miller (1966) and Bohrer (1979).

The other figure with an interesting statistical intepretation is the tetrahedron, which arises from a slippage test as follows. Let X_i, $i = 1, \cdots, 4$, be four independent normal, $N(\theta_i, \nu)$, random variables. For example, these might be means of n-samples from a "usual" one-way layout, as in Scheffé (1959), Section 2.1 and Chapter 3. Consider the uniformly most powerful unbiased test of $H_0: \theta_1 = \theta_2 = \theta_3 = \theta_4$ against alternative $H_i: \theta_i = \theta_j + \delta$ for $\delta > 0$ and for all $j \ne i$. Namely, this test rejects H_0 if $X_i - X.$ is too large, where $X.$ is the sample average of the X_i. If all four of these tests, all having the same size, are performed simultaneously, then

(i) the simultaneous acceptance region is a cylinder whose base is $H(4)$, and

(ii) the multiple test is optimal in the sense of Spjøtvoll (1972), i.e., it maximizes the minimum power among unbiased tests having the same (or smaller) expected number of Type I errors.

The technique for establishing the claimed ordering is first to orient the solids so that all are centered at the origin with faces tangent to the sphere of radius c. Denoting the facial distribution for the m-faced Platonic solid by $F_{D(m)}(d)$, we then show that these distributions are ordered. Since all the faces are congruent and since rotation of faces does not alter their spherically symmetric probabilities, we can "stack up" faces, with common center, as convenient.

(i) (*tetrahedron versus cube*). The tetrahedron has four triangular faces with side $c2\sqrt{6}$, and the cube has six square faces of side 2c. A square face is seen to be a subset of a triangular face when a side of one is rotated parallel to a side of the other. It is clear that, for $0 < d < c$, $F_{D(6)}(d) = 1.5F_{D(4)}(d)$ and that, for $d > c\sqrt{2}$, $F_{D(6)}(d) = 1$, its maximum value. For $c < d < c\sqrt{2}$, the rate of increase of $F_{D(6)}(d) - F_{D(4)}(d)$ decreases continuously, since more of the circle of radius d falls outside the square face than the triangular face.

(ii) (*cube versus octahedron*). The octahedron has eight triangular faces of side $c\sqrt{6}$. Picture all the faces with common center and the following orientation. For each of the two solids, rotate its "stacked up" faces so that they form a symmetric, regular, 24-pointed star, neighboring faces oriented at 15° from each other. Both stars have their points at distance $c\sqrt{2}$ from the center, and the two stars can be rotated so that their points coincide. In this position, the angles at the points of the octagon-star are contained in the corresponding angles of the cube-star. It is clear then that, for $c < d < c\sqrt{2}$, $F_{D(8)}(d) > F_{D(6)}(d)$, with both values being unity for $d > c\sqrt{2}$. For $0 < d < c$, the quantity $6\,A_6(d) = 6$ (cf. (4)), while $8\,A_8(d) = 8$ for $0 < d < c/\sqrt{2}$ and then decreases for $c/\sqrt{2} < d < c$. Thus the condition of Theorem 2 obtains.

(iii) (*octahedron versus dodecahedron*). The twelve pentagonal faces of the dodecahedron, $H(12)$, have sides $4c(10 + 22/\sqrt{5})^{1/2} \doteq 4.403c$, which are not "subsets" of the triangular faces of $H(8)$. The radii of the circles inscribed in the faces of $H(12)$ and $H(8)$ are given, respectively, by $d_1(12) = .5c(6 - 2\sqrt{5})^{1/2} \doteq .618c$ and $d_1(8) = c\sqrt{2} \doteq .707c$, and the circumscribing circle radii are $d_2(12) = c(14 - 6\sqrt{5})^{1/2} \doteq .764c$ and $d_2(8) = c\sqrt{2} \doteq 1.414c$. Since $d_1(12) < d_1(8) < d_2(12) < d_2(8)$, Theorem 2 says that it suffices to verify that $8\,A_8(d_1(12)) - 12\,A_{12}(d_2(8)) > 0$. Indeed, this difference, approximately 2.725, is positive.

(iv) (*dodecahedron versus icosahedron*). The faces of $H(12)$ and $H(20)$ have the same total number of vertices (12×5 = 20×3); and their circumcircle radii are identical, viz., $c(14 - 6\sqrt{5})^{1/2} \doteq .764c$. The vertex angles of the $H(12)$ faces is larger than for $H(20)$ (72° > 60°). Thus an argument completely analogous to that comparing the cube and octahedron proves that the dodecahedron dominates the icosahedron.

3.3 *Placing the rhombic dodecahedron*. For statistical and geometrical reasons, it is of interest to consider the semi-regular, three-dimensional dodecahedron with rhombic faces, $H'(12)$; cf. Coxeter (1963). This solid has facial semi-axes of c and $c/\sqrt{2}$ when its inscribed sphere has radius c. This rhombic dodecahedron is the figure whose uncorrelated trivariate normal (or corresponding Student t) probability is measured in determining $P(R \le c)$, where R is a (Studentized) range random variable with four (numerator) degrees of freedom.

Theorem 2 can be used to place $H'(12)$ into the ordering of the Platonic solids, between $H(8)$ and $H(12)$. Note how this result, saying that the less regular dodecahedron has more probability, and its derivation, through comparison of angles, are three-dimensional extensions in the spirit of Wynn (1977).

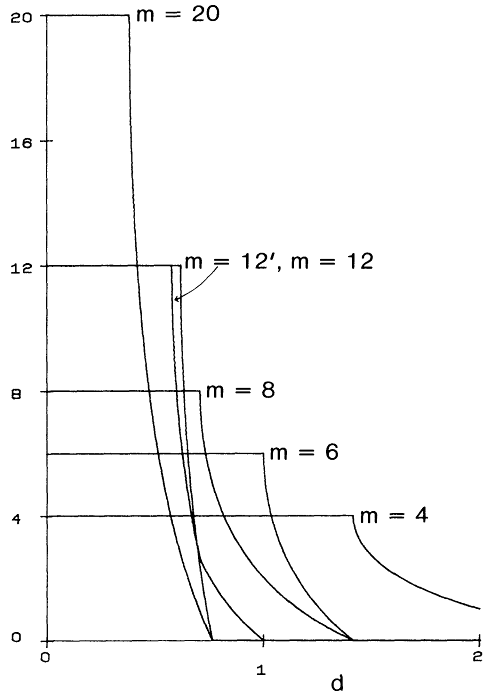

FIG. 1. *The angle functions*, $mA_m(d)$, *for* $m = 4, 6, 8, 12, 20$ *and* $12A'(d)$ $(m = 12')$.

The angle functions to be compared for the m-faced Platonic solids, $H(m)$, are

$$mA_m(d) = \begin{cases} m & \text{if } 0 < d < d_1(m), \\ m - mE(m)\arccos(d_1(m)/d)/\pi & \text{if } d_1(m) < d < d_2(m) \\ 0 & \text{otherwise,} \end{cases}$$

where $E(m)$ is the number of edges of each face and $d_1(m)$ and $d_2(m)$ are, respectively, the incircle and circumcircle radii of the $H(m)$ faces. With analogous trigonometry and notation, the angle function for $H'(12)$ is seen to be

$$12A'_{12}(d) = \begin{cases} 12 & \text{if } 0 < d < c/\sqrt{3} \\ 12 - 48\arccos(1/(d\sqrt{3}))/\pi & \text{if } c/\sqrt{3} < d < c/\sqrt{2} \\ 12 - 24(\arccos(\sqrt{2}/\sqrt{3}) + \arccos(1/(d\sqrt{3}))/\pi & \text{if } c/\sqrt{2} < d < c \\ 0 & \text{otherwise.} \end{cases}$$

This angle function for $H'(12)$ is seen, by using some elementary, if tedious, analysis and computation, to cross those for $H(8)$ and $H(12)$ exactly once, as required for Theorem 2 to give the ordering claimed. A more delicate geometrical proof can be given in the spirit of Section 3.2. The complementary methods of the present and previous sections are both presented, since either may be more useful in other applications.

Figure 1 illustrates all six of the angle functions, $mA_m(d)$, for H being the five Platonic solids as well as $H'(12)$. Inspection verifies the Theorem 2, one-crossing property, away from the crossing-point, while some elementary analysis shows that only one crossing occurs near the crossing of each pair of curves.

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