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GENERAL ADMISSIBILITY AND INADMISSIBILITY RESULTS FOR ESTIMATION IN A CONTROL PROBLEM.

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Let $\mathbf{X} = (X_1, \cdots, X_p)^t$ be an observation from a p-variate normal distribution with unknown mean $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)^t$ and identity covariance matrix. We consider a control problem which, in canonical form, is the problem of estimating $\boldsymbol{\theta}$ under the loss $L(\boldsymbol{\theta}, \boldsymbol{\delta}) = (\boldsymbol{\theta}^t\boldsymbol{\delta} - 1)^2$, where $\boldsymbol{\delta}(\mathbf{x}) = (\delta_1(\mathbf{x}), \cdots, \delta_p(\mathbf{x}))^t$ is the estimate of $\boldsymbol{\theta}$ for a given $\mathbf{x}$. General theorems are given for establishing admissibility or inadmissibility of estimators in this problem. As an application, it is shown that estimators of the form $\boldsymbol{\delta}(\mathbf{x}) = (|\mathbf{x}|^2 + c)^{-1}\mathbf{x} + |\mathbf{x}|^{-4}w(|\mathbf{x}|)\mathbf{x}$, where $w(|\mathbf{x}|)$ tends to zero as $|\mathbf{x}| \to \infty$, are inadmissible if $c > 5 - p$, but are admissible if $c \le 5 - p$ and $\boldsymbol{\delta}$ is generalized Bayes for an appropriate prior measure. Also, an approximation to generalized Bayes estimators for large $|\mathbf{x}|$ is developed.

1. Introduction. The control problem deals with a situation in which it is desired to choose the levels of certain factors in a system so that the "output" of the system is at the desired control level. The system could be an economic system, a production system, or a biological system. (As an example of the latter, it might be desired to achieve and maintain certain normal levels of certain chemical concentrations in a patient.)

Zaman (1981) considers a standard normal model of the control problem, in which the output, z, occurs as

$$z = \boldsymbol{\theta}^t\mathbf{y} + \varepsilon, \tag{1.1}$$

where $\boldsymbol{\theta}$ is a p-vector of unknown coefficients of the system, ε is a normally distributed error, and $\mathbf{y}$ is a p-vector of nonstochastic control variables to be chosen so as to achieve some desired output z^*. Suppose that the loss in achieving output z is $(z - z^*)^2$, and that an estimate $\boldsymbol{\delta}(\mathbf{x}) = (\delta_1(\mathbf{x}), \cdots, \delta_p(\mathbf{x}))^t$ of $\boldsymbol{\theta}$ is available, from, say, past normal data, $\mathbf{x}$, on the system. Zaman (1981) (see also Basu, 1974 and Zellner, 1971) then shows that the problem can be reduced (with suitable redefinitions of variables) to the following problem. Suppose $\mathbf{X} = (X_1, \cdots, X_p)^t$ is a p-variate normal random variable with unknown mean $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)^t$ and identity covariance matrix, and that it is desired to estimate $\boldsymbol{\theta}$ under loss

$$L(\boldsymbol{\theta}, \boldsymbol{\delta}) = (\boldsymbol{\theta}^t\boldsymbol{\delta} - 1)^2.$$

The estimator $\boldsymbol{\delta}$ is allowed to assume any value in R^p, but the parameter space is restricted to be $\Theta = R^p - \{\mathbf{0}\}$. Zero is excluded from the parameter space, because $\boldsymbol{\theta} = \mathbf{0}$ corresponds to a control system in which the inputs have no effect. (From a decision theoretic viewpoint, it is necessary to exclude zero to prevent every estimator from being a Bayes estimator with respect to the prior distribution which puts mass one at zero.)

As usual, an estimator will be evaluated in terms of its risk function $R(\boldsymbol{\theta}, \boldsymbol{\delta})$, which is simply the expected loss $E_\theta[L(\boldsymbol{\theta}, \boldsymbol{\delta}(\mathbf{X}))]$. A basic decision theoretic goal in any problem is

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to classify the inadmissible estimators (those which can be improved upon in terms of risk) and the admissible estimators (those which cannot be improved upon). This is of particular interest in the above control problem because there is no "natural" estimator which can be recommended without significant reservations. The natural estimators (the MLE and uniform prior generalized Bayes rule) can be inadmissible. In such a situation it is desirable to carefully pinpoint the "boundary" between admissible and inadmissible estimators, since estimators on this boundary will typically be the most "objective" admissible rules available. To see this, note that at the one extreme lie the admissible but highly subjective proper Bayes rules, while at the other extreme lie the objective rules, which are typically generalized Bayes with respect to very flat prior densities and are often inadmissible. If one wishes to be as objective as possible (i.e., to use as little subjective prior information as possible) while maintaining admissibility, the "boundary" is the right place to look for an estimator. (This boundary is also of interest to Bayesians, in that estimators on this boundary can incorporate prior information, but tend to be robust with respect to possible misspecification of the prior distribution. Hence, to a robust Bayesian, this boundary is a natural place from which to select an estimator.)

A complete classification of estimators according to admissibility has previously been done only for the case of estimating a multivariate normal mean under quadratic loss (Brown, 1971). Recently, a number of powerful techniques have been developed, based on work in Stein (1974), Brown (1979), and Brown (1980), which give hope of obtaining such a classification in other problems. A major purpose in writing this paper is to provide a relatively simple and thorough example of the implementation of these techniques. As such, we will not consider in depth the practical issue of selecting an estimator for use in this control problem. This question is considered further in Berliner (1981). Also, it should be noted that the control problem treated here is probably too simplified for many applications. The major limitations of the model (1.1) are that the control variables are only allowed to enter linearly into the model, and that the model lacks a constant term. Nevertheless, the results obtained should provide guidance in selecting control procedures in more general situations.

In this paper, attention will be restricted to nonrandomized, spherically symmetric estimators. The restriction to nonrandomized estimators is made since an argument parallel to the proof of the Rao-Blackwell theorem shows that the nonrandomized estimators form a complete class (i.e., any randomized estimator can be improved upon by some nonrandomized estimator). The restriction to spherically symmetric estimators, though natural for an orthogonally invariant problem, may not always be appropriate. Unfortunately, results for nonsymmetric estimators appear to be very difficult to obtain. Note that since the problem is invariant under the orthogonal group (which is compact), admissibility within the class of spherically symmetric estimators implies overall admissibility.

It will prove convenient to write a spherically symmetric estimator $\boldsymbol{\delta}$ as

$$\boldsymbol{\delta}(\mathbf{x}) = \phi(|\mathbf{x}|)|\mathbf{x}|^{-1}\mathbf{x}, \tag{1.2}$$

where $|\mathbf{x}|^2 = \sum_{i=1}^{p} x_i^2$. Conditions will be derived under which $\boldsymbol{\delta}$ is admissible or inadmissible. For example, it will be shown that an estimator of the form

$$\boldsymbol{\delta}(\mathbf{x}) = (|\mathbf{x}|^2 + c)^{-1}\mathbf{x} + |\mathbf{x}|^{-4}w(|\mathbf{x}|)\mathbf{x}, \tag{1.3}$$

where $w(|\mathbf{x}|) = o(1)$ as $|\mathbf{x}| \to \infty$, is inadmissible if $c > 5 - p$. On the other hand, if $\boldsymbol{\delta}^{\pi}$ is generalized Bayes with respect to a generalized prior distribution $\pi(d\boldsymbol{\theta}) = g(|\boldsymbol{\theta}|^2)\,d\boldsymbol{\theta}$, for which $g(v) \le Kv^{(4-p)/2}$, then (under certain technical conditions) it will be shown that $\boldsymbol{\delta}^{\pi}$ is admissible. It will also be shown that

$$\boldsymbol{\delta}^{\pi}(\mathbf{x}) = (|\mathbf{x}|^2 + 1 + 2|\mathbf{x}|^2 g'(|\mathbf{x}|^2)/g(|\mathbf{x}|^2))^{-1}\mathbf{x} + |\mathbf{x}|^{-4}w(|\mathbf{x}|)\mathbf{x}, \tag{1.4}$$

(where w is as in (1.3) and $g'(v) = dg(v)/dv$), so that if $2vg'(v)/g(v) \to (c-1)$ as $v \to \infty$, then $\boldsymbol{\delta}^{\pi}$ will be as in (1.3) and will be admissible if $c \le 5 - p$. Hence, the "boundary of admissibility" can be alternatively viewed as being $c = 5 - p$ in (1.3) or as being those generalized Bayes rules for which

$$\lim_{v\to\infty} 2vg'(v)/g(v) = 4 - p.$$

Previous results for this control problem have dealt only with the case $w(|\mathbf{x}|) \equiv 0$ in (1.3). Takeuchi (unpublished) established the inadmissibility of $\boldsymbol{\delta}$ for $c > 0$ and $p \geq 6$. Stein and Zaman (1980) established inadmissibility of $\boldsymbol{\delta}$ for $\mathbf{c} = 1$ and $p = 5$. Zaman (1981) proved that $\boldsymbol{\delta}$ is admissible for $c = 1$ and $p \leq 3$, while Stein and Zaman (1980) proved admissibility for $c = 1$ and $p = 4$. Note that the estimator corresponding to $c = 1$, namely $\boldsymbol{\delta}(\mathbf{x}) = (|\mathbf{x}|^2 + 1)^{-1}\mathbf{x}$, is the uniform prior generalized Bayes rule (Zellner, 1971).

Section 2 presents several needed preliminary results, including a crucial representation for the Bayes risk of a spherically symmetric estimator. Section 3 derives the approximation (1.4) for a generalized Bayes estimator, and Section 4 establishes the admissibility result. The analysis in these sections is based on the techniqpes discussed in Brown (1979). Section 5 presents the inadmissibility results. The method of proof here is based on the techniques in Brown (1980) and Berger (1980), and is one of the first examples of application of these techniques. Finally, some illustrations of the theory are given in Section 6.

2. Preliminaries. We will make considerable use of the complete class theorm in Zaman (1981). This theorem states that if an estimator of the form (1.2) is admissible, then, for some probability measure μ on $\Gamma = [0, \infty)$,

$$\phi(v) = \frac{\int_0^\infty \gamma^{-1}\sinh(\gamma v)\mu(d\gamma)}{\int_0^\infty \cosh(\gamma v)\mu(d\gamma)}, \tag{2.1}$$

where $\gamma^{-1}\sinh(\gamma v)$ is defined to be v when $\gamma = 0$. Since

$$y\sinh(y) \leq \cosh(y), \qquad \lim_{v\to 0}\cosh(\gamma v) = 1, \qquad \lim_{v\to 0}[\gamma^{-1}\sinh(\gamma v)]/v = 1,$$

it is easy to see from (2.1) that, if δ is admissible, then

$$0 \leq \phi(v)/v \leq 1 \quad \text{and} \quad \lim_{v\to 0}[\phi(v)/v] = 1. \tag{2.2}$$

It is convenient to define the functions

$$f(\boldsymbol{\theta}, r) = \exp\{-\tfrac{1}{2}(\theta_1 - r)^2\}\exp(-\tfrac{1}{2}|\boldsymbol{\theta}^*|^2) \tag{2.3}$$

where $\boldsymbol{\theta}^* = (\theta_2, \cdots, \theta_p)^t$, and, for a (generalized) spherically symmetric prior distriution π,

$$N(r) = \int_\Theta \theta_1 f(\boldsymbol{\theta}, r)\pi(d\boldsymbol{\theta}), \qquad D(r) = \int_\Theta \theta_1^2 f(\boldsymbol{\theta}, r)\pi(d\boldsymbol{\theta}). \tag{2.4}$$

Zaman (1981) has shown that if $\boldsymbol{\delta}$ is (generalized) Bayes with respect to π then $\boldsymbol{\delta}$ is unique and ϕ, as defined in (1.2), is given by

$$\phi(|\mathbf{x}|) = N(|\mathbf{x}|)/D(|\mathbf{x}|). \tag{2.5}$$

We conclude this section with the development of a representation for the Bayes risk

$$r(\pi, \boldsymbol{\delta}) = \int_\Theta R(\boldsymbol{\theta}, \boldsymbol{\delta})\pi(d\boldsymbol{\theta})$$

of an estimator of the form (1.2) with respect to a (generalized) prior $\pi \in \Theta^*$, where Θ^* is the set of all spherically symmetric, σ-finite measures on Θ. It is convenient to define the measure $\tilde{\pi}$ on $(-\infty, \infty)$ by

$$\tilde{\pi}(A) = \int_A \int_{R^{(p-1)}} S_p(2\pi)^{-p/2}\exp\left(-\frac{1}{2}|\boldsymbol{\theta}|^2\right)\pi(d\boldsymbol{\theta}), \tag{2.6}$$

where S_p is the surface area of the unit p-sphere.

THEOREM 2.1. *Suppose $\pi \in \Theta^*$ and $\boldsymbol{\delta}$ is an estimator of the form* (1.2). *Then*

$$r(\pi, \boldsymbol{\delta}) = (2\pi)^{-p/2} \int_\Theta \int_0^\infty r^{(p-1)} \{\phi(r)\theta_1 - 1\}^2 f(\boldsymbol{\theta}, r)\, dr\pi(d\boldsymbol{\theta}) \tag{2.7}$$
$$= \int_{-\infty}^\infty \int_0^\infty r^{(p-1)} \{\phi(r)\theta_1 - 1\}^2 \exp\left(-\frac{1}{2} r^2\right) \exp(r\theta_1)\, dr\tilde{\pi}(d\theta_1).$$

Furthermore, if π is a finite measure and ϕ is continuous and piecewise differentiable on $[0, \infty)$ *and satisfies*

(i) $0 \le \phi(r)/r \le K_0 < \infty$,

(ii) $\lim_{r\to 0}[r^{(p-1)}\{\phi(r)\}^2] = 0$, *and*

(iii) $\displaystyle\int_0^\infty r^p \exp\left(-\frac{1}{2} r^2\right) \exp(r\theta_1)\phi'(r)\, dr < \infty$,

where $\phi'(r) = d\phi(r)/dr$, then

$$r(\pi, \boldsymbol{\delta}) = \pi(\Theta) + 2 \int_0^\infty \int_0^\infty g(r)\gamma \sinh(\gamma r)\, dr\tilde{\pi}(d\gamma). \tag{2.8}$$

where

$$g(r) = r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\}\phi(r)[-2 + \{r - (p-1)r^{-1}\}\phi(r) - 2\phi'(r)]. \tag{2.9}$$

PROOF. Clearly

$$r(\pi, \boldsymbol{\delta}) = \int_\Theta \int_{\mathscr{X}} \{\boldsymbol{\theta}'\boldsymbol{\delta}(\mathbf{x}) - 1\}^2 (2\pi)^{-p/2} \exp\left(-\frac{1}{2}|\mathbf{x} - \boldsymbol{\theta}|^2\right) d\mathbf{x}\pi(d\boldsymbol{\theta})$$
$$= \int_{\mathscr{X}} \int_\Theta \{\phi(|\mathbf{x}|)|\mathbf{x}|^{-1}\boldsymbol{\theta}'\mathbf{x} - 1\}^2 (2\pi)^{-p/2} \exp\left(-\frac{1}{2}|\mathbf{x} - \boldsymbol{\theta}|^2\right) \pi(d\boldsymbol{\theta})\, d\mathbf{x}.$$

Transforming to polar coordinates and using spherical symmetry easily yields (2.7). We next need the following lemma.

LEMMA 2.1. *Suppose $h(r)$ is a continuous, piecewise differentiable function on* $(0, \infty)$ *which satisfies, for all $s \in R^1$ (letting $h'(r) = dh(r)/dr$),*

(i) $\displaystyle\int_0^\infty |h'(r)| \exp(rs) dr < \infty$,

(ii) $\lim_{r\to 0} h(r) = 0$, *and*

(iii) $\lim_{r\to\infty}[h(r)\exp(rs)] = 0$.

Then

$$\int_0^\infty h(r)s \exp(rs)\, dr = -\int_0^\infty h'(r)\exp(rs)\, dr. \tag{2.10}$$

The proof of this lemma is a simple integration by parts, and will be omitted. Setting

$$h(r) = r^{(p-1)}\{\phi(r)\}^2\exp(-\tfrac{1}{2}r^2),$$

it is easy to check that the conditions of Theorem 2.1 imply that $h(r)$ satisfies the conditions of Lemma 2.1. Hence, (2.10) implies that

$$\int_0^\infty r^{(p-1)}\{\phi(r)\}^2\exp\left(-\frac{1}{2}r^2\right)\theta_1\exp(r\theta_1)\,dr$$
$$= \int_0^\infty r^{(p-1)}\phi(r)\exp\left(-\frac{1}{2}r^2\right)[-(p-1)r^{-1}\phi(r) - 2\phi'(r) + r\phi(r)]\exp(r\theta_1)\,dr.$$

Expanding $\{\phi(r)\theta_1 - 1\}^2$ in (2.7), and using the above result, shows that

$$\begin{aligned} r(\pi, \boldsymbol{\delta}) = \int_{-\infty}^\infty \int_0^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right) \\ \cdot[1 - 2\theta_1\phi(r) + \theta_1\phi(r)\{-(p-1)r^{-1}\phi(r) - 2\phi'(r) + r\phi(r)\}]\exp(r\theta_1)\,dr\tilde{\pi}(d\theta_1). \end{aligned} \tag{2.11}$$

Now, if $\phi(r) \equiv 0$, then clearly

$$r(\pi, \boldsymbol{\delta}) = \int_\Theta (1)\pi(d\boldsymbol{\theta}) = \pi(\Theta).$$

Setting $\phi(r) \equiv 0$ in (2.11), this gives that

$$\int_{-\infty}^\infty \int_0^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\exp(r\theta_1)\,dr\tilde{\pi}(d\theta_1) = \pi(\Theta).$$

Using this and (2.9) in (2.11) shows that

$$r(\pi, \boldsymbol{\delta}) = \pi(\Theta) + \int_{-\infty}^\infty \int_0^\infty \theta_1 g(r)\exp(r\theta_1)\,dr\tilde{\pi}(d\theta_1). \tag{2.12}$$

Now, since zero was excluded from the parameter space and π is symmetric, $\tilde{\pi}$ cannot give positive mass to $\theta_1 = 0$. Also, $\tilde{\pi}$ is symmetric, so that (2.12) can be written

$$r(\pi, \boldsymbol{\delta}) = \pi(\Theta) + \int_{-\infty}^0 \int_0^\infty \theta_1 g(r)\exp(r\theta_1)\,dr\tilde{\pi}(d\theta_1) + \int_0^\infty \int_0^\infty \theta_1 g(r)\exp(r\theta_1)\,dr\tilde{\pi}(d\theta_1)$$
$$= \pi(\Theta) + 2\int_0^\infty \int_0^\infty g(r)\gamma\sinh(\gamma r)\,dr\tilde{\pi}(d\gamma),$$

completing the proof. □

3. Asymptotic approximation of generalized Bayes rules. In this section the asymptotic approximation of generalized Bayes rules discussed in Section 1 is derived. Let $\boldsymbol{\delta}$ denote the generalized Bayes rule with respect to the prior measure π defined by

$$\pi(d\boldsymbol{\theta}) = g(|\boldsymbol{\theta}|^2)\,d\boldsymbol{\theta}. \tag{3.1}$$

The approximation of $\boldsymbol{\delta}(\mathbf{x})$ given below is valid (locally at $\mathbf{x}$) for $|\mathbf{x}|$ large.

For notational convenience, define

$$g'(r^2) = dg(r^2)/dr^2$$

throughout this section and the next.

ASSUMPTION 1. (i) There exist positive constants $K_0 \le 1$, B, and ε such that

(a) $g(r^2) \le r^{(\varepsilon-p)}$ for $r^2 \le K_0$,

(b) $g(r^2) \le B$ for $r^2 > K_0$.

(ii) There exists a constant $T > 0$ such that if $r^2 \ge T$, the following conditions hold:

(a) g has a continuous second derivative;

(b) There exists positive constants c_1 and c_2 such that

$$(1)\ |g'(r^2)| \le c_1 r^{-2} g(r^2), \qquad (2)\ |g''(r^2)| \le c_2 r^{-4} g(r^2);$$

(c) There exists positive constants c_3 and q such that

$$g(r^2) \ge c_3 r^{(1-q)};$$

(d) For some positive constant c_4,

$$\sup_{\{y:|y|\le r^2/2\}} g(y + r^2) \le c_4 g(r^2).$$

For technical reasons it is convenient to consider the function ϕ^* defined by $\phi^*(r) = r\phi(r) - 1$ where ϕ is defined in (1.2).

THEOREM 3.1. *Under Assumption* 1, *if* $r^2 > T$ *then*

$$\phi^*(r) = -r^{-2}\{1 + 2r^2 g'(r^2)/g(r^2) + O(r^{-1})\}. \tag{3.2}$$

Before proceeding with the proof of this result, some comments are in order. First, the implied approximation of $\boldsymbol{\delta}$, for $|\mathbf{x}|$ sufficiently large, is

$$\boldsymbol{\delta}(\mathbf{x}) = [1 - \{|\mathbf{x}|^{-2} + 2g'(|\mathbf{x}|^2)/g(|\mathbf{x}|^2)\} + o(|\mathbf{x}|^{-2})]|\mathbf{x}|^{-2}\mathbf{x}.$$

(This is clearly equivalent to (1.4).) Second, the proof is based on a Taylor's series approximation of g. This accounts for the flatness and smoothness requirements (specifically, Conditions (ii)(a) and (b)) on $g(r^2)$ for large r^2. Sharp tailed priors, such as exponentially decreasing priors, are eliminated from consideration. However, such priors are proper and therefore yield admissible rules. Conditions (ii)(c) and (d) are technical, but not very restrictive. Condition (i)(a) guarantees the existence of $\boldsymbol{\delta}$.

The proof requires additional notation and some preliminary lemmas. Let $K_1 = (2\pi)^{p/2}$; throughout this discussion, K (or K', etc.) denotes a generic constant. Regions of integrations for θ are Θ, unless otherwise indicated.

LEMMA 3.1.

(i) $$\int \theta_1 f(\boldsymbol{\theta}, r)\, d\boldsymbol{\theta} = K_1 r,$$

(ii) $$\int \theta_1 \{|\boldsymbol{\theta}|^2 - r^2\} f(\boldsymbol{\theta}, r)\, d\boldsymbol{\theta} = K_1(p + 2)r,$$

(iii) $$\int \theta_1^2 f(\boldsymbol{\theta}, r)\, d\boldsymbol{\theta} = K_1(1 + r^2),$$

(iv) $$\int \theta_1^2 \{|\boldsymbol{\theta}|^2 - r^2\} f(\boldsymbol{\theta}, r)\, d\boldsymbol{\theta} = K_1(p + 4)r^2 + K.$$

PROOF. Simple calculation. □

Next, define the quantity N^* by

$$N^*(r) = \int (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}.$$

(Hence, $\phi^* = N^*D^{-1}$.) Also, define the set $\mathscr{A} = \{\boldsymbol{\theta}: | |\boldsymbol{\theta}|^2 - r^2 | \le r^2/2\}$.

LEMMA 3.2. *For any positive constant M,*

$$\int_{\mathscr{A}^c} (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} \le Kr^{-2M}(1 + r^{(M+1)}), \tag{3.3}$$

$$\int_{\mathscr{A}^c} \theta_1^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} \le Kr^{-2M}(1 + r^{(M+2)}). \tag{3.4}$$

PROOF. Let $\mathscr{A}_1^c = \mathscr{A}^c \cap \{\boldsymbol{\theta}: | |\boldsymbol{\theta}|^2 \le K_0\}$ and $\mathscr{A}_2^c = \mathscr{A}^c \cap \{\boldsymbol{\theta}: | |\boldsymbol{\theta}|^2 > K_0\}$. First, it is easy to show that, by Assumption 1 (i)(a),

$$\int_{\mathscr{A}_1^c} (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} \le K(r+1)\exp\left\{-\frac{1}{2}(r - K_0)^2\right\} \le K' r^{-2M}(1 + r^{(M+1)})$$

for all $M \ge 0$. Next, since $\boldsymbol{\theta} \in \mathscr{A}^c$ implies $| |\boldsymbol{\theta}|^2 - r^2|^M \ge Kr^{2M}$ and g is bounded for $\boldsymbol{\theta}c \in \mathscr{A}_2^c$, a simple Chebyshev argument yields

$$\int_{\mathscr{A}_2^c} (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} \le KBr^{-2M} \int |r\theta_1 - \theta_1^2| \, | |\boldsymbol{\theta}|^2 - r^2|^M f(\boldsymbol{\theta}, r)\, d\boldsymbol{\theta}.$$

Simple computation then gives the result. (3.4) is established similarly. □

PROOF OF THEOREM 3.1. The first step is to approximate N^*. Recall that

$$N^* = \int_{\mathscr{A}} (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} + \int_{\mathscr{A}^c} (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}.$$

Call these two integrals I_1 and I_2 respectively. I_2 can be bounded by applying Lemma 3.2. In particular, choose M to be equal to $q + 1$. Then for r large (i.e., $r > 1$), it is clear that

$$I_2 \le Kr^{-q}. \tag{3.5}$$

Next, consider I_1. By Taylor's Theorem, g can be written as

$$g(|\boldsymbol{\theta}|^2) = g(r^2) + (|\boldsymbol{\theta}|^2 - r^2) g'(r^2) + \tfrac{1}{2}(|\boldsymbol{\theta}|^2 - r^2)^2 g''(r_0^2),$$

where r_0^2 is some point contained in the interval $[\tfrac{1}{2}r^2, \tfrac{3}{2}r^2]$. Now, substitute this expression for g into I_1 and integrate term by term. Denote the resulting three integrals I_a, I_b, and I_c, respectively. Rewrite I_a as

$$I_a = g(r^2)\left\{\int (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r)\, d\boldsymbol{\theta} - \int_{\mathscr{A}^c} (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r)\, d\boldsymbol{\theta}\right\}.$$

The first integral is computed using Lemma 4.1; the second is bounded as above. Therefore, I_a is given by

$$I_a = -K_1 g(r^2) + BKr^{-q}. \tag{3.6}$$

Essentially the same argument, ignoring lower order terms, implies that

$$I_b = -K_1 2r^2 g'(r^2). \tag{3.7}$$

Finally, consider I_c. Clearly,

$$I_c \le \frac{1}{2}\int_{\mathscr{A}} |r\theta_1 - \theta_1^2| (|\boldsymbol{\theta}|^2 - r^2)^2 |g''(r_0^2)| f(\boldsymbol{\theta}, r)\, d\boldsymbol{\theta}.$$

Applying Conditions (ii)(b), (d) and Lemma 3.2, and performing a calculation as in Lemma 3.1, we have

$$I_c \le K'O(r^{-1})g(r^2). \tag{3.8}$$

Condition (ii)(c) and (3.5), (3.6), (3.7), and (3.8) yield

$$N^*(r) = -K_1 g(r^2)\{1 + 2r^2 g'(r^2)/g(r^2) + O(r^{-1})\}.$$

By similar arguments it can be shown that

$$D(r) = K_1 r^2 g(r^2)\{1 + O(r^{-1})\}.$$

The desired result follows. □

4. Admissibility. The proof of admissibility employed is basically a verification of Stein's sufficient condition for admissibility (Stein, 1955). Brown (1979) has presented a general methodology for carrying out such proofs; see also Berger (1976b).

The admissibility proof given below employs the results of Section 3 and, hence, applies only to generalized priors which satisfy the technical conditions of Assumption 1. The following additional requirement identifies the boundary of admissibility discussed in Section 1.

ASSUMPTION 2. There exists a constant c such that

$$g(r^2) \le c r^{(4-p)} \tag{4.1}$$

for all $r^2 \ge T$.

THEOREM 4.1. *Let $\boldsymbol{\delta}$ be the generalized Bayes rule with respect to the prior measure π defined in* (3.1). *Under Assumptions* 1 *and* 2, *$\boldsymbol{\delta}$ is admissible.*

The proof is an application of Stein's condition in the following form, essentially due to Farrell (1968b).

PROPOSITION 4.1. *Let $\boldsymbol{\delta}$ be defined as in Theorem* 4.1. *Suppose there exists a sequence of finite, non-negative functions $h_n(|\boldsymbol{\theta}|^2)$ such that*

(i) $$\int g(|\boldsymbol{\theta}|^2)h_n(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} < \infty, \qquad \forall n = 1, 2, \cdots;$$

(ii) $$\lim_{n\to\infty} h_n(|\boldsymbol{\theta}|^2) = 1; \quad \text{and}$$

(iii) $$\lim_{n\to\infty} \int \{R(\boldsymbol{\theta}, \boldsymbol{\delta}) - R(\boldsymbol{\theta}, \boldsymbol{\delta}_n)\} g(|\boldsymbol{\theta}|^2)h_n(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} = 0,$$

where $\boldsymbol{\delta}_n$ is the Bayes rule with respect to the prior defined by $\pi_n(d\boldsymbol{\theta}) = g(|\boldsymbol{\theta}|^2)h_n(|\boldsymbol{\theta}|^2)$ $d\boldsymbol{\theta}$. Then $\boldsymbol{\delta}$ is admissible.

The heuristic proof of admissibility given in Brown (1979) suggests choices for the functions $h_n(|\boldsymbol{\theta}|^2)$ and proposes methods for approximating the integrals appearing in Condition (iii) of Proposition 4.1. The choice of $h_n(|\boldsymbol{\theta}|^2)$ used below is

$$h_n(|\boldsymbol{\theta}|^2) = \begin{cases} 1 & \text{if } 0 \le |\boldsymbol{\theta}|^2 \le 1 \\ H_n(|\boldsymbol{\theta}|^2) & \text{if } 1 \le |\boldsymbol{\theta}|^2 \le n^2 \\ 0 & \text{if } |\boldsymbol{\theta}|^2 \ge n^2 \end{cases}$$

where

$$H_n(|\boldsymbol{\theta}|^2) = \{1 - (\ln|\boldsymbol{\theta}|^2)/(\ln n^2)\}^{17}.$$

The key to the analysis below is the approximation of both $\boldsymbol{\delta}$ and $\boldsymbol{\delta}_n$. As in Section 3, Taylor series approximations for both $g(|\boldsymbol{\theta}|^2)$ and $g(|\boldsymbol{\theta}|^2)H_n(|\boldsymbol{\theta}|^2)$ (at appropriate values of $|\boldsymbol{\theta}|$) are used.

Recall the definitions of ϕ, ϕ^*, N, etc. Let the analogous quantities corresponding to $\boldsymbol{\delta}_n$ be denoted ϕ_n, ϕ_n^*, N_n, etc.

PROOF OF THEOREM 4.1. Clearly, the functions h_n defined above satisfy Conditions (i) and (ii) of Proposition 4.1. To verify Condition (iii), define

$$\mathscr{E}_n = \int \{R(\boldsymbol{\theta}, \boldsymbol{\delta}) - R(\boldsymbol{\theta}, \boldsymbol{\delta}_n)\}\pi_n(d\boldsymbol{\theta}).$$

By formula (2.7), $\mathscr{E}_n$ can be written as

$$\mathscr{E}_n = (2\pi)^{-p/2} \int \left[\int_0^\infty r^{(p-1)}\{(\theta_1\phi(r) - 1)^2 - (\theta_1\phi_n(r) - 1)^2\} f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2) h_n(|\boldsymbol{\theta}|^2)\, dr \right] d\boldsymbol{\theta}.$$

The interchange of order of integration (by Fubini's Theorem, since the Bayes risk is finite) and simplification yield

$$\mathscr{E}_n = (2\pi)^{-p/2} \int_0^\infty r^{(p-1)}\{\phi(r) - \phi_n(r)\}^2 D_n(r)\, dr.$$

Next, $\mathscr{E}_n$ is partitioned into the following three integrals (ignoring the constant):

$$\mathscr{E}_n^1 = \int_0^{(\ln\ln n^2)^{1/2}} r^{(p-1)}\{\phi(r) - \phi_n(r)\}^2 D_n(r)\, dr,$$

$$\mathscr{E}_n^2 = \int_{(\ln\ln n^2)^{1/2}}^{n-n^{8/9}} r^{(p-1)}\{\phi(r) - \phi_n(r)\}^2 D_n(r)\, dr,$$

and

$$\mathscr{E}_n^3 = \int_{n-n^{8/9}}^\infty r^{(p-1)}\left(\int [\{\theta_1\phi(r) - 1\}^2 - \{\theta_1\phi_n(r) - 1\}^2] f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2) h_n(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} \right) dr.$$

The proof is completed by showing that these quantities vanish as $n \to \infty$.

(*a*) $\mathscr{E}_n^1 \to 0$. Define the sets: $A = \{\boldsymbol{\theta} : |\boldsymbol{\theta}|^2 \le 1\}$, $B = \{\boldsymbol{\theta} : 1 \le |\boldsymbol{\theta}|^2 \le n^2\}$, and $C = \{\boldsymbol{\theta} : |\boldsymbol{\theta}|^2 > n^2\}$. Also, let

$$\lambda_n = \int_B \theta_1 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\{1 - H_n(|\boldsymbol{\theta}|^2)\}\, d\boldsymbol{\theta} + \int_C \theta_1 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}$$

and

$$\gamma_n = \int_B \theta_1^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\{1 - H_n(|\boldsymbol{\theta}|^2)\}\, d\boldsymbol{\theta} + \int_C \theta_1^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}.$$

Simple algebra then implies that $\phi - \phi_n = D^{-1}(\lambda_n - \phi_n\gamma_n)$. Hence, $\mathscr{E}_n^1$ can be written as

$$\mathscr{E}_n^1 = \int_0^{(\ln\ln n^2)^{1/2}} r^{(p-1)}(\lambda_n^2 + \phi_n^2\gamma_n^2 - 2\lambda_n\phi_n\gamma_n) D^{-2} D_n\, dr.$$

Since $D_n D^{-1} \le 1$, it is clearly sufficient to show that

$$(4.2) \qquad \int_0^{(\ln\ln n^2)^{1/2}} r^{(p-1)}\lambda_n^2 D^{-1}\, dr + \int_0^{(\ln\ln n^2)^{1/2}} r^{(p-1)}(\phi_n\gamma_n)^2 D^{-1}\, dr$$

goes to zero. Let I_n denote the second integral of (4.2). We next show that $I_n \to 0$; the proof for the first integral is essentially the same.

First, since $1 - H_n(|\boldsymbol{\theta}|^2)$ is non-decreasing in $|\boldsymbol{\theta}|^2$, γ_n is bounded by

$$\gamma_n \leq (1 - H_n(\ln n^2)) \int_{1 \leq |\theta|^2 \leq \ln n^2} \theta_1^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} + \int_{\ln n^2 \leq |\theta|^2} \theta_1^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}.$$

Applying Lemma 3.1 and the bound (for large n)

$$1 - H_n(\ln n^2) \leq K(\ln n^2)^{-3/4}$$

on the first term above, together with a Chebyshev argument on the second term, it follows that

$$\gamma_n \leq K(1 + r^4)(\ln n^2)^{-3/4}.$$

Next, since $\exp(-\frac{1}{2}r^2) \geq (\ln n^2)^{-1/2}$ on the region of integration of I_n, it follows that $D^{-1} \leq K(\ln n^2)^{1/2}$. Finally, since the δ_n are admissible (π_n is proper), (2.2) implies $\phi_n^2 \leq r^2$. Combining the above bounds then yields

$$\begin{aligned} I_n &\leq K \int_0^{(\ln \ln n^2)^{1/2}} r^{(p+1)}(\ln n^2)^{1/2}\{(1 + r^4)(\ln n^2)^{-3/4}\}^2\, dr \\ &\leq K'(\ln n^2)^{-1}\{1 + (\ln \ln n^2)^{(p+10)/2}\} \to 0. \end{aligned}$$

(b) $\mathscr{E}_n^2 \to 0$. The heart of the proof here is the approximation of both ϕ^* and ϕ_n^*. The arguments used for ϕ_n^* are essentially the same, though more delicate, as those used in Theorem 3.1. The key result is given in the following proposition. First, however, define the function s by

$$s(y^2) = \ln n^2 - \ln y^2,$$

so that $H_n(y^2) = \{s(y^2)/\ln n^2\}^{17}$.

PROPOSITION 4.2. *Assume* $(\ln \ln n^2)^{1/2} \leq r \leq n - n^{8/9}$. *For n sufficiently large*

$$|\phi^*(r) - \phi_n^*(r)| \leq K[r^{-3} + \{r^2 s(r^2)\}^{-1}]. \tag{4.3}$$

The proof of Proposition 4.2 requires the following lemmas.

LEMMA 4.1. *If* $1 < y^2 < n^2$, *then*

(i) $|H_n'(y^2)| = 17\{s(y^2)\}^{16}/\{y^2(\ln n^2)^{17}\}$ *and*

(ii) $|H_n''(y^2)| \leq K\{s(y^2)\}^{15}\{1 + s(y^2)\}/\{y^4(\ln n^2)^{17}\}$.

PROOF. Simple computation. □

LEMMA 4.2. *Assume that* $(\ln \ln n^2)^{1/2} \leq r \leq n - n^{8/9}$. *Then uniformly in r,*

(i) $\lim_{n\to\infty}\{H_n(r^2)r^2\}^{-1} = 0$ *and*

(ii) $\lim_{n\to\infty}\{s(r^2)r\}^{-1} = 0$.

PROOF. The proof is identical to that given in Berger (1976b), Lemma 3.2.8. □

PROOF OF PROPOSITION 4.2. Define the sets $\mathscr{A}_1$ and $\mathscr{A}_2$ by

$$\mathscr{A}_1 = \{\boldsymbol{\theta} : \||\boldsymbol{\theta}|^2 - r^2| \leq \tfrac{1}{2}r^2 \quad \text{and} \quad |\boldsymbol{\theta}|^2 \leq n^2\}$$

and

$$\mathscr{A}_2 = \{\boldsymbol{\theta} : \||\boldsymbol{\theta}|^2 - r^2| > \tfrac{1}{2}r^2 \quad \text{and} \quad |\boldsymbol{\theta}|^2 \leq n^2\}.$$

Since $h_n(|\boldsymbol{\theta}|^2) = 0$ if $|\boldsymbol{\theta}|^2 > n^2$, it is clear that

$$N_n^* = \int_{\mathscr{A}_1} (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2) h_n(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta} + \int_{\mathscr{A}_2} (r\theta_1 - \theta_1^2) f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2) h_n(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}.$$

Call these two integrals I_1 and I_2, respectively. By noting that $h_n(|\boldsymbol{\theta}|^2) \le 1$ and then applying Lemmas 3.2 and 4.2, I_2 can be bounded by

$$I_2 \le K r^{-q} H_n(r^2) \tag{4.4}$$

for sufficiently large n.

The analysis of I_1 is a Taylor's series argument (expanding $g(|\boldsymbol{\theta}|^2) H_n(|\boldsymbol{\theta}|^2)$) similar to that of Theorem 3.1. This argument and (4.4) imply that

$$N_n^* = -K_1 g(r^2) H_n(r^2)\left[\left\{1 + 2r^2 \frac{g'(r^2)}{g(r^2)} + O(r^{-1})\right\} + K\{s(r^2)\}^{-1}\{1 + o(1)\}\right]. \tag{4.5}$$

It can also be shown that

$$D_n = K_1 r^2 g(r^2) H_n(r^2)[1 + \{rs(r^2)\}^{-1} + O(r^{-1})]. \tag{4.6}$$

Recalling that $\phi_n^* = N_n^*/D_n$ and using (4.5), (4.6), Lemma 4.2, and Theorem 3.1, yields the desired result. □

To complete the argument that $\mathscr{E}_n^2 \to 0$, note that

$$\phi(r) - \phi_n(r) = \{\phi^*(r) - \phi_n^*(r)\}/r.$$

Applying Proposition 4.2 and the approximation of D_n given in (4.6), it is clearly sufficient to show that

$$\mathscr{E}_n^4 = \int_{(\ln\ln n^2)^{1/2}}^{n-n^{8/9}} r^{(p-1)} (r^{-3}[r^{-1} + \{s(r^2)\}^{-1}])^2 r^2 g(r^2) H_n(r^2)\, dr$$

vanishes as $n \to \infty$. Computation and Assumption 2 imply that

$$\begin{aligned}\mathscr{E}_n^4 &\le K \int_{(\ln\ln n^2)^{1/2}}^{n-n^{8/9}} [r^{-3} + r^{-1}\{s(r^2)\}^{-2} H_n(r^2)]\, dr \\ &\le K'\{(\ln\ln n^2)^{-1} + (\ln n)^{-2} \int_{(\ln\ln n^2)^{1/2}}^{n} r^{-1}\, dr\} \\ &= K'\{(\ln\ln n^2)^{-1} + (\ln n)^{-2}(\ln n - \tfrac{1}{2}\ln\ln\ln n^2)\} \to 0.\end{aligned}$$

(c) $\mathscr{E}_n^3 \to 0$. Clearly

$$|\mathscr{E}_n^3| \le \int_{n-n^{8/9}}^{\infty} r^{(p-1)} \left(\int [\{\theta_1\phi(r) - 1\}^2 + \{\theta_1\phi_n(r) - 1\}^2] f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2) h_n(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}\right) dr.$$

Since $\boldsymbol{\delta}_n$ is Bayes with respect to $\pi_n = gh_n$, the inner integral above can only be increased if ϕ_n is replaced by r^{-1}. Also, Theorem 3.1 implies that ϕ is of the form $\phi(r) = r^{-1}\{1 + o(r^{-1})\}$. It can be shown that the $o(r^{-1})$ term is negligible in the analysis, so that the problem reduces to showing that

$$\mathscr{E}_n^5 = \int_{n-n^{8/9}}^{\infty} r^{(p-1)} \left\{\int (r^{-1}\theta_1 - 1)^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2) h_n(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}\right\} dr \to 0.$$

Define the sets $\Gamma_1 = \{\boldsymbol{\theta} : |\boldsymbol{\theta}| \le n - 2n^{8/9}\}$ and $\Gamma_2 = \{\boldsymbol{\theta} : n - 2n^{8/9} < |\boldsymbol{\theta}| < n\}$. Since $h_n(|\boldsymbol{\theta}|^2) = 0$ for $|\boldsymbol{\theta}| \ge n$, it is clear that

$$(4.7)\quad \mathscr{E}_n^5 \le \int_{n-n^{8/9}}^{\infty} r^{(p-3)}\left\{\int_{\Gamma_1} (\theta_1 - r)^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2) h_n(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}\right\} dr$$

$$+ \int_{n-n^{8/9}}^{\infty} r^{(p-3)}\left\{\int_{\Gamma_2} (\theta_1 - r)^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2) h_n(|\boldsymbol{\theta}|^2) d\boldsymbol{\theta}\right\} dr.$$

Noting that since $r \ge n - n^{8/9}$, $\boldsymbol{\theta} \in \Gamma_1$ implies that $r - \theta_1 \ge n^{8/9}$, Chebyshev arguments can be used to show that the first integral in (4.7) tends to zero as $n \to \infty$.

Let J denote the second term in (4.7). For n large enough and $\boldsymbol{\theta} \in \Gamma_2$, we have that

$$h_n(|\boldsymbol{\theta}|^2) = H_n(|\boldsymbol{\theta}|^2) \le H_n((n - 2n^{8/9})^2) \le K(n^{1/9} \ln n)^{-17}.$$

Hence, it can be concluded that

$$J \le K(n^{1/9} \ln n)^{-17} \int_{n-n^{8/9}}^{\infty} r^{(p-3)}\left\{\int_{\Gamma_2} (\theta_1 - r)^2 f(\boldsymbol{\theta}, r) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}\right\} dr.$$

Case 1. $p < 4$. Since $r \ge n - n^{8/9}$ implies that $r^{(p-3)} \le 1$ (n large), interchange of order of integration yields

$$J \le K(n^{1/9} \ln n)^{-17} \int_{\Gamma_2} \exp\left(-\frac{1}{2}|\boldsymbol{\theta}^*|^2\right) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta},$$

and so, since g is bounded on Γ_2 for n large,

$$J \le K'(n^{1/9} \ln n)^{-17} n \to 0.$$

Case 2. $p \ge 4$. Clearly, we have

$$J \le K(n^{1/9} \ln n)^{-17} \int_{\Gamma 2} \left\{\int_{-\infty}^{\infty} |r|^{(p-3)} (\theta_1 - r)^2 f(\boldsymbol{\theta}, r)\, dr\right\} g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}$$

$$\le K'(n^{1/9} \ln n)^{-17} \int_{\Gamma_2} (1 + |\theta_1|^{(p-3)}) \exp\left(-\frac{1}{2}|\boldsymbol{\theta}^*|^2\right) g(|\boldsymbol{\theta}|^2)\, d\boldsymbol{\theta}.$$

Then, using Assumption 2, it follows that

$$J \le K'(n^{1/9} \ln n)^{-17} \int_{\Gamma_2} (1 + |\theta_1|) \exp\left(-\frac{1}{2}|\boldsymbol{\theta}^*|^2\right) d\boldsymbol{\theta}$$

$$\le K''(n^{1/9} \ln n)^{-17}\left[n + \int \{n^2 - (n - n^{8/9})^2\} \exp\left(-\frac{1}{2}|\boldsymbol{\theta}^*|^2\right) d\boldsymbol{\theta}^*\right]$$

$$\le K'''(\ln n)^{-17} \to 0. \ \square$$

5. Inadmissibility. To prove that an estimator $\boldsymbol{\delta}^0$ is inadmissible, we will make use of the technique developed in Brown (1980) and Berger (1980). The heuristic basis of this technique was given in Brown (1979). The technique is to find an estimator $\boldsymbol{\delta}^*$ which has smaller risk than $\boldsymbol{\delta}^0$ for large $|\boldsymbol{\theta}|$, and then to argue that this leads to a violation of Stein's necessary condition for admissibility (Stein, 1955).

For use in the analysis, define, for any estimator $\boldsymbol{\delta}$ which satisfies the conditions of Theorem 2.1, the function, on $\Gamma = [0, \infty)$,

$$R^*(\gamma, \delta) = 2 \int_0^{\infty} g(r) \gamma \sinh(\gamma r)\, dr,$$

where $g(r)$ is defined in (2.9). Furthermore, if $\boldsymbol{\delta}^0$ and $\boldsymbol{\delta}^*$ both satisfy the conditions of

Theorem 2.1, define

$$\Delta(\gamma) = R^*(\gamma, \boldsymbol{\delta}^0) - R^*(\gamma, \boldsymbol{\delta}^*) = 2\int_0^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\gamma\sinh(\gamma r)\,dr, \tag{5.1}$$

where

$$\Delta^*(r) = -2\{\phi^0(r) - \phi^*(r)\} + \{r - (p-1)r^{-1}\}\{\phi^0(r)^2 - \phi^*(r)^2\} + 2\{\phi^*(r)\phi^{*\prime}(r) - \phi^0(r)\phi^{0\prime}(r)\}. \tag{5.2}$$

Note that, for any finite measure $\pi \in \Theta^*$ for which either $r(\pi, \boldsymbol{\delta}^0)$ or $r(\pi, \boldsymbol{\delta}^*)$ is finite, it follows from Theorem 2.1 that

$$\begin{aligned} r(\pi, \boldsymbol{\delta}^0) - r(\pi, \boldsymbol{\delta}^*) &= \int_0^\infty \Delta(\gamma)\tilde{\pi}(d\gamma) \\ &= 2\int_0^\infty\int_0^\infty r^{(p-1)}\exp(-\frac{1}{2}r^2)\Delta^*(r)\gamma\sinh(\gamma r)\,dr\tilde{\pi}(d\gamma). \end{aligned} \tag{5.3}$$

THEOREM 5.1. *Suppose that $\boldsymbol{\delta}^0$ is of the form* (1.2), *with $\phi^0(r)$ continuous and piecewise differentiable, and that $\boldsymbol{\delta}^*$ is another estimator of the form* (1.2) *with $\phi^*(r)$ satisfying the conditions of Theorem* 2.1 *and the further conditions*

(i) *there exists a constant $K_1 \geq 0$ such that $\phi^*(r) = \phi^0(r)$ for $r \leq K_1$;*
(ii) *there exist $\varepsilon > 0$, $\alpha > 0$, and $0 < K_2 < \infty$ such that $\Delta^*(r) \geq \varepsilon r^{-\alpha}$ for $r > K_2$; and*
(iii) $\int_{K_1}^\infty r^{(p-1)}\exp(-\frac{1}{2}r^2)\Delta^*(r)\exp\{\psi(r)\}\,dr > 0$, *where* $\psi(r) = \int_{K_1/2}^r \{\phi^0(v)\}^{-1}\,dv.$

Then δ^0 is inadmissible.

PROOF. The proof will be by contradiction. Assume that $\boldsymbol{\delta}^0$ is admissible, and is hence of the form (2.1), with ϕ^0 satisfying (2.2). It is easy to check that the conditions of Theorem 2.1 are satisfied by $\boldsymbol{\delta}^0$.

Now let $\mathscr{D}$ be the class of all spherically symmetric estimators such that $0 \leq \phi(|\mathbf{x}|)/|\mathbf{x}| \leq 1$. As stated in Section 2, $\mathscr{D}$ is a complete class of estimators. The problem has now been put in the framework of Berger (1980), and so, to prove inadmissibility of $\boldsymbol{\delta}^0$, it is only necessary to verify that the conditions of the theorem in Berger (1980) are satisfied. These conditions are that there exists a sequence $\{\pi_n\}$ of finite measures in Θ^*, with corresponding Bayes rules $\boldsymbol{\delta}^n$ such that $r(\pi_n, \boldsymbol{\delta}^n) < \infty$, and a nonnegative function $h(\gamma)$, which is strictly positive on the interior of Γ, such that

(a) $\tilde{\pi}_n(C) \geq 1$ $(n = 1, 2, \cdots)$, for some compact set C in the interior of Γ;
(b) $\lim_{n\to\infty} \int_\Theta\{R(\boldsymbol{\theta}, \boldsymbol{\delta}^0) - R(\boldsymbol{\theta}, \boldsymbol{\delta}^n)\}\pi_n(d\boldsymbol{\theta}) = 0$;
(c) the measures $\mu_n(d\gamma) = h(\gamma)\tilde{\pi}_n(d\gamma)/\int_\Gamma h(\gamma)\tilde{\pi}_n(d\gamma)$ converge weakly to a probability measure μ on Γ; and
(d) the function $g(\gamma) = \{h(\gamma)\}^{-1}\Delta(\gamma)$ (see (3.1)) is continuous on Γ and is positive outside some compact set $B \subset \Gamma$.

The theorem in Berger (1980) states that, if these conditions hold, then

$$\int_\Gamma g(\gamma)\mu(d\gamma) \leq 0. \tag{5.4}$$

The existence of finite measures $\pi_n \in \Theta^*$ which satisfy conditions (a) and (b) follows from Stein's necessary condition for admissibility (Stein, 1955). It can be verified that Stein's necessary condition applies here, using Theorem 3.5 of Farrell (1968a). The verification that the risk set is weakly subcompact can be carried out by checking the conditions in LeCam (1955).

Zaman (1981) showed that if $h(\gamma) = \gamma^2$, then the measures

$$\mu_n(d\gamma) = \frac{\gamma^2\tilde{\pi}_n(d\gamma)}{\int_0^\infty \gamma^2\tilde{\pi}_n(d\gamma)}$$

are probability measures on Γ, and that a subsequence of these measures converges weakly to a probability measure μ on Γ. Since this subsequence still satisfies conditions (a) and (b), we can assume that (a), (b), and (c) all hold.

Verification of condition (d). It is clear from (5.1) that $\Delta(\gamma)$, and hence $g(\gamma) = \gamma^{-2}\Delta(\gamma)$, is continuous on $(0, \infty)$. Also, it is possible to show that

$$\lim_{\gamma\to 0}\gamma^{-2}\Delta(\gamma) = \int_0^\infty 2r^p\exp(-\tfrac{1}{2}r^2)\Delta^*(r)\,dr \equiv \beta,$$

which is easily seen to be finite for $\boldsymbol{\delta}^*$ and $\boldsymbol{\delta}^0$ satisfying the conditions of Theorem 2.1. Hence, $g(\gamma)$ is continuous at $\gamma = 0$ if we define $g(0) = \beta$.

The remaining part of condition (d) is verified in the following lemma.

LEMMA 5.1. *There exists a $K_3 < \infty$ such that $g(\gamma) > 0$ for $\gamma > K_3$. Thus $g(\gamma)$ is positive outside the compact set $B = [0, K_3]$.*

PROOF. From (5.1), it clearly suffices to prove that, for $\gamma > K_3$,

$$I \equiv \int_0^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\gamma\sinh(\gamma r)\,dr > 0. \tag{5.5}$$

Note first that, from Condition (ii) of the theorem,

$$I_1 \equiv \int_{K_2}^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\gamma\sinh(\gamma r)\,dr \ge \varepsilon\int_{K_2}^\infty r^{(p-1-\alpha)}\exp\left(-\frac{1}{2}r^2\right)\gamma\sinh(\gamma r)\,dr.$$

Hence, for large enough γ, say $\gamma > K_4$,

$$\begin{aligned} I_1 &\ge \frac{\varepsilon}{3}\int_{K_2}^\infty r^{(p-1-\alpha)}\exp\left(-\frac{1}{2}r^2\right)\gamma\exp(\gamma r)\,dr \\ &\ge \frac{\varepsilon}{4}\gamma\exp\left(\frac{1}{2}\gamma^2\right)\gamma^{(p-1-\alpha)} = \frac{\varepsilon}{4}\exp\left(\frac{1}{2}\gamma^2\right)\gamma^{p-\alpha}, \end{aligned} \tag{5.6}$$

the last inequality following from a standard Taylors series argument.

An integration by parts, as in the proof of Theorem 2.1, shows that

$$\begin{aligned} I_2 &= \int_0^{K_2} r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\gamma\sinh(\gamma r)\,dr \\ &= -K_2^{(p-1)}\{\phi^0(K_2)^2 - \phi^*(K_2)^2\}\exp\left(-\frac{1}{2}K_2^2\right)\gamma\sinh(\gamma K_2) \\ &\quad + \int_0^{K_2} r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\{\phi^0(r)^2 - \phi^*(r)^2\}\gamma^2\cosh(\gamma r)\,dr \\ &\quad - \int_0^{K_2} 2\{\phi^0(r) - \phi^*(r)\}r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\gamma\sinh(\gamma r)\,dr. \end{aligned} \tag{5.7}$$

Since $0 \le \phi^*(r)/r \le K_0 < \infty$ and $0 \le \phi^0(r)/r \le 1$, it follows that

$$
\begin{aligned}
|I_2| &\le (K_0^2+1)K_2^{(p+1)}\exp\left(-\frac{1}{2}K_2^2\right)\gamma\exp(K_2\gamma) \\
&\quad + (K_0^2+1)\int_0^{K_2} r^{(p+1)}\exp\left(-\frac{1}{2}r^2\right)\gamma^2\exp(\gamma r)\,dr \\
&\quad + (K_0+1)\int_0^{K_2} r^p\exp\left(-\frac{1}{2}r^2\right)\gamma\exp(\gamma r)\,dr \\
&\le (K_0^2+2)(K_2+1)^{p+1}\exp\left(\frac{1}{2}\gamma^2\right)\gamma\left[\exp\left\{-\frac{1}{2}(K_2-\gamma)^2\right\}\right. \\
&\quad \left. + (\gamma+1)\int_0^{K_2}\exp\left\{-\frac{1}{2}(r-\gamma)^2\right\}dr\right].
\end{aligned}
$$

Since, for $\gamma > K_2$,

$$\int_0^{K_2}\exp\left\{-\frac{1}{2}(r-\gamma)^2\right\}dr \le K_2\exp\left\{-\frac{1}{2}(K_2-\gamma)^2\right\},$$

it can be concluded that, for $\gamma > K_2$,

$$|I_2| \le (K_0^2+2)(K_2+1)^{p+2}\exp\left(\frac{1}{2}\gamma^2\right)\gamma(\gamma+1)\exp\left\{-\frac{1}{2}(K_2-\gamma)^2\right\}. \tag{5.8}$$

Choosing $K_3 > \max(K_2, K_4)$ so that

$$\exp\left\{-\frac{1}{2}(K_2-K_3)^2\right\} \le \frac{\varepsilon}{8}\gamma^{(p-\alpha-1)}(\gamma+1)^{-1}(K_0^2+2)^{-1}(K_2+1)^{-(p+2)},$$

it follows from (5.5), (5.6), and (5.8) that, for $\gamma > K_3$,

$$I = I_1 + I_2 \ge \frac{\varepsilon}{8}\exp\left(\frac{1}{2}\gamma^2\right)\gamma^{(p-\alpha)} > 0,$$

completing the proof of the lemma. □

We have thus verified all the conditions of the theorem in Berger (1980), and so can conclude that (5.4) must be satisfied. Note, however, from (5.1), that

$$
\begin{aligned}
\int_\Gamma g(\gamma)\mu(d\gamma) &= \int_0^\infty \gamma^{-2}\Delta(\gamma)\mu(d\gamma) \\
&= \int_0^\infty\int_0^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\gamma^{-1}\sinh(\gamma r)\,dr\,d\mu(\gamma).
\end{aligned}
$$

Since $\phi^*(r) = \phi^0(r)$ for $r \le K_1$ (Condition (i) of the theorem), it is clear from (5.2) that

$$\int_\Gamma g(\gamma)\mu(d\gamma) = 2\int_0^\infty\int_{K_1}^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\gamma^{-1}\sinh(\gamma r)\,dr\,d\mu(\gamma).$$

Because $\Delta^*(\mathrm{r})$ is positive for $r > K_2$, orders of integration can be interchanged above to give

$$\int_\Gamma g(\gamma)\mu(d\gamma) = 2\int_{K_1}^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\int_0^\infty \gamma^{-1}\sinh(\gamma r)\,d\mu(\gamma)\,dr. \tag{5.9}$$

Since we are assuming that $\boldsymbol{\delta}^0$ is admissible, ϕ^0 must satisfy (2.1), which can be rewritten

$$\phi^0(v) = \left\{\frac{d}{dv}\log\int_0^\infty \gamma^{-1}\sinh(\gamma v)\,d\mu(\gamma)\right\}^{-1}.$$

Hence, for $r > K_1/2$,

$$\psi(r) = \int_{K_1/2}^{r} \{\phi^0(v)\}^{-1}\,dv = \log\left\{\int_0^\infty \gamma^{-1}\sinh(\gamma r)\,d\mu(\gamma)\right\} - \rho,$$

where

$$\rho = \int_0^\infty \gamma^{-1}\sinh(K_1\gamma/2)\,d\mu(\gamma).$$

It follows that

$$\int_0^\infty \gamma^{-1}\sinh(\gamma r)\,d\mu(\gamma) = \exp\{\psi(r) + \rho\},$$

which, when used in (5.9), gives

$$\int_\Gamma g(\gamma)\mu(d\gamma) = 2e^\rho\int_{K_1}^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\exp\{\psi(r)\}\,dr.$$

By Condition (iii) of the theorem, this is positive, contradicting (5.4). The conclusion is that $\boldsymbol{\delta}^0$ cannot be admissible, completing the proof of the theorem. □

As mentioned in the Introduction, virtually all estimators studied have been of the form (1.3). We can obtain the following inadmissibility result for estimators of this form.

THEOREM 5.2. *Assume that $\boldsymbol{\delta}^0$ is of the form* (1.2) *with*

$$\phi^0(r) = \frac{r}{r^2 + c} - \frac{w(r)}{r^3}, \tag{5.10}$$

where $w(r) = o(1)$ (as $r \to \infty$). Then $\boldsymbol{\delta}^0$ is inadmissible if $c > 5 - p$.

PROOF. For convenience, define $\varepsilon = c - (5 - p)$ and $\beta = -\varepsilon^2/8$. Assume that $c > 5 - p$, so that $\varepsilon > 0$. Define, for $K_1 > 0$ to be chosen later,

$$\phi^*(r) = \begin{cases} \phi^0(r) & \text{if } r \le K_1, \\ \dfrac{r}{r^2 + 5 - p} - \dfrac{w(r)}{r^3}\left\{1 - \dfrac{\varepsilon}{r^2 + c} + \dfrac{\beta}{r^2(r^2 + c)}\right\} & \text{if } r \ge 2K_1, \\ \left(2 - \dfrac{r}{K_1}\right)\phi^0(K_1) + \left(\dfrac{r}{K_1} - 1\right)\phi^*(2K_1) & \text{if } K_1 < r < 2K_1. \end{cases}$$

Assuming ϕ^0 is as in (2.1) and (2.2) (which, if not true, makes $\boldsymbol{\delta}^0$ trivally inadmissible), it is easy to see that ϕ^* satisfies the conditions of Theorem 2.1 and Condition (i) of Theorem 5.1. (In particular, $w'(r) = dw(r)/dr$ must be well behaved.)

To verify Condition (ii) of Theorem 5.1, a lengthy calculation for $r \ge 2K_1$, ignoring terms of order $o(r^{-5})$ and recalling that $w(r) = o(1)$, yields

$$\Delta^*(r) = \frac{\varepsilon^2}{r^5} + \frac{w'(r)}{r^8}\left\{\frac{\varepsilon^2}{4} + o(1)\right\} + o(r^{-5}). \tag{5.11}$$

Solving for w in (5.10) and differentiating gives

$$w'(r) = \frac{4r^3}{r^2+c} - \frac{2r^5}{(r^2+c)^2} - 3r^2\phi^0(r) - r^3\phi^{0\prime}(r). \tag{5.12}$$

Defining

$$h(r) = \int_0^\infty \gamma^{-1}\sinh(\gamma r)\, d\mu(\gamma),$$

it follows from (2.1) that, if ϕ^0 is admissible, then $\phi^0(r) = h(r)/h'(r)$. This implies that

$$\phi^{0\prime}(r) = 1 - h(r)h''(r)/\{h'(r)\}^2. \tag{5.13}$$

Observing that h and h'' are positive, it follows from (5.12) and (5.13) that

$$w'(r) \ge -3r^2\phi^0(r) - r^3.$$

From (5.10), it is clear that if K_1 is chosen large enough, then $\phi^0(r) \le 2/r$ for $r \ge 2K_1$. Thus

$$w'(r) \ge -6r - r^3$$

for $r \ge 2K_1$. From this and (5.11) it can be concluded that, for large enough K_1 and $r \ge 2K_1$,

$$\Delta^*(r) \ge \frac{\varepsilon^2}{r^5} - \frac{\varepsilon^2/4}{r^5} + o(r^{-5}) \ge \frac{\varepsilon^2}{2r^5}. \tag{5.14}$$

Thus Condition (ii) of Theorem 5.1 is satisfied.

To verify Condition (iii) of Theorem 5.1, note that for large enough K_1 and $r \ge K_1/2$,

$$\phi^0(r)^{-1} = \left\{\frac{r}{r^2+c} - \frac{w(r)}{r^3}\right\}^{-1} = r + \frac{c}{r} + \frac{h(r)}{r},$$

where $|h(r)| \le \varepsilon/2$. Hence, for $r \ge K_1/2$,

$$\phi^0(r)^{-1} \ge r + \frac{c - \varepsilon/2}{r},$$

so that

$$\psi(r) = \int_{K_1/2}^{r} \phi^0(v)^{-1}\, dv \ge \frac{1}{2}\left(r^2 - \frac{K_1^2}{4}\right) + \left(c - \frac{\varepsilon}{2}\right)\log\left(\frac{r}{K_1/2}\right).$$

It follows that, for $r \ge K_1/2$,

$$\exp\{\psi(r)\} \ge K_2\exp(\tfrac{1}{2}r^2)r^{c-\varepsilon/2},$$

where the constant K_2 depends on K_1 and is positive. Together with (5.14), this implies that

$$\begin{aligned}\int_{2K_1}^\infty r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\exp\{\psi(r)\}\, dr &\ge \frac{1}{2}K_2\varepsilon^2\int_{2K_1}^\infty r^{(p-1-5+c-\varepsilon/2)}\, dr \\ &= \frac{1}{2}K_2\varepsilon^2\int_{2K_1}^\infty r^{-1+\varepsilon/2}\, dr = \infty.\end{aligned}$$

It is easy to check that

$$\int_{K_1}^{2K_1} r^{(p-1)}\exp\left(-\frac{1}{2}r^2\right)\Delta^*(r)\exp\{\psi(r)\}\, dr$$

is finite, so that Condition (iii) of the theorem is clearly satisfied. Hence δ^0 is inadmissible. □

6. Applications and examples. In Section 1 we noted that the main goal of the theoretical analysis above is the identification of the boundary of admissibility. To that end the results of Sections 3 and 5 can be applied to obtain the following theorem concerning the inadmissibility of generalized Bayes rules.

THEOREM 6.1. *Suppose $\boldsymbol{\delta}$ is generalized Bayes with respect to the prior measure π defined in* (3.1). *Furthermore, assume that the prior kernel g satisfies the conditions of Assumption* 1 (*Section* 3) *and that*

$$\lim_{r^2\to\infty} 2r^2 g'(r^2)/g(r^2) = c - 1, \tag{6.1}$$

where $g'(r^2) = dg(r^2)/dr^2$. If $c > 5 - p$, then $\boldsymbol{\delta}$ is inadmissible.

PROOF. The approximation result of Theorem 3.1 together with (6.1) imply that $\boldsymbol{\delta}$ is of the form (1.2) with $\phi(r)$ of the form (5.10). Hence, the proof is an immediate consequence of Theorem 5.2. □

A natural class of prior measures for the control problem is the class of priors of the form

$$\pi(d\boldsymbol{\theta}) = |\boldsymbol{\theta}|^{c-1} d\boldsymbol{\theta}.$$

(To guarantee the existence of the generalized Bayes rule $\boldsymbol{\delta}^\pi$, assume $c > 1 - p$.) This class includes the uniform prior, for which the generalized Bayes estimator is $\boldsymbol{\delta}_u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}\mathbf{x}$ $(c = 1)$. For $c \neq 1$ the generalized Bayes rules are difficult to explicitly calculate, at least in closed form. For example, Zaman (1977) considered $\pi_0(d\boldsymbol{\theta}) = |\boldsymbol{\theta}|^{4-p} d\boldsymbol{\theta}$ $(c = 5 - p)$ and found the corresponding rules $\boldsymbol{\delta}^{\pi_0}$ to be

$$\boldsymbol{\delta}^{\pi_0}(\mathbf{x}) = \left\{ 1 + |\mathbf{x}|^2 \frac{\sum_{j=2}^{\infty} \frac{j(j+1)(j-1)}{\Gamma(p/2+j)} (|\mathbf{x}|^2/2)^{j-2}}{\sum_{j=1}^{\infty} \frac{j(j+1)}{\Gamma(p/2+j)} (|\mathbf{x}|^2/2)^{j-1}} \right\}^{-1} \mathbf{x}.$$

(Zaman's derivation could be reproduced for arbitrary c, leading to analogous results.)

To apply our results to $\boldsymbol{\delta}^\pi$, it is required that $c \le 1$, thus insuring that $|\boldsymbol{\theta}|^{c-1}$ is bounded for $|\boldsymbol{\theta}|$ large, as required in Condition (i)(b) of Assumption 1. Note that since

$$g(r^2) = (r^2)^{(c-1)/2} \quad \text{and} \quad g'(r^2) = \tfrac{1}{2}(c-1)(r^2)^{(c-3)/2},$$

we have

$$2r^2 g'(r^2)/g(r^2) = c - 1. \tag{6.2}$$

Therefore, by Theorem 3.1, $\boldsymbol{\delta}^\pi$ is given by

$$\boldsymbol{\delta}^\pi(\mathbf{x}) = (|\mathbf{x}|^2 + c)^{-1}\mathbf{x} + |\mathbf{x}|^{-4} w(|\mathbf{x}|)\mathbf{x},$$

where w is as in (1.3). Theorem 6.1 implies that $\boldsymbol{\delta}^\pi$ is inadmissible if $c > 5 - p$. On the other hand, Theorem 4.1 implies that $\boldsymbol{\delta}^\pi$ is admissible if $c \le 5 - p$. For example, $\boldsymbol{\delta}^{\pi_0}$ is admissible when $p \ge 4$. Also, $\boldsymbol{\delta}_u$ is admissible when $p \le 4$.

Another application of our theory can be found in Berliner (1981).

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