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CONSTRUCTION METHODS FOR *D*-OPTIMUM WEIGHING DESIGNS WHEN $n \equiv 3 (\text{mod } 4)$

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In the setting where the weights of k objects are to be determined in n weighings on a chemical balance (or equivalent problems), for $n \equiv 3 \text{ (mod 4)}$, Ehlich and others have characterized certain "block matrices" C such that, if $X'X = C$ where $X(n \times k)$ has entries ± 1, then X is an optimum design for the weighing problem. We give methods here for constructing X's for which $X'X$ is a block matrix, and show that it is the optimum C for infinitely many (n, k). A table of known constructibility results for $n < 100$ is given.

1. Introduction. Let k and n be positive integers with $k \le n$, and let $\mathscr{X} = \mathscr{X}(n, k)$ be the set consisting of every $n \times k$ matrix X whose entries are all ± 1. Our goal is to find a D-optimum X, i.e., one that maximizes $\det(X'X)$ over $\mathscr{X}$. A discussion of the settings (weighing, fractional factorial, first order regression) where this problem arises, some history, and statements of results obtained up to that time are contained in Galil and Kiefer (1980a, b). The most interesting unsolved cases noted there are ones for which $n \equiv 3 (\text{mod } 4)$.

Throughout this paper $n \equiv 3 (\text{mod } 4)$. In the present paper we give methods for construction of D-optimum X's for infinitely many pairs (n, k) for which an optimum X was not previously known.

Section 2 gives methods for constructing X's for which $X'X$ is a "block matrix" of a type we now describe. A *block* of size r is an $r \times r$ matrix with diagonal elements n and off-diagonal elements 3. A $k \times k$ *block matrix* with s blocks of sizes $r_1, r_2, \cdots, r_s$, satisfying $\sum_1^s r_i = k$, was diagonal blocks of those sizes and elements -1 everywhere else. If the blocks are of two neighboring sizes (one size R if $s \mid k$), the sizes are $R = \lceil k/s \rceil$ and $R - 1$, where $\lceil x \rceil$ is the least integer $\ge x$. (We shall also later use $\lfloor x \rfloor$ for the greatest integer $\le x$.) The X's constructed in Section 2, denoted $X(n, k, s, R)$ or $X(n, k, s)$, have $X'X$ of the last form for certain n, k, s, R, although (Remark 3) the method also yields block matrices $X'X$ with blocks of more than two sizes.

The reason for constructing X's of this form is that for each (n, k) there are particular values (usually one, sometimes two) s_{OPT} such that, if an $X(n, k, s_{\text{OPT}})$ exists, it is D-optimum. This is a consequence of a theory originated by Ehlich (1964) and further developed in Galil and Kiefer (1980a, 1981a), which shows that, for each pair (n, k),

$$\max_{X \in \mathscr{X}} \det(X'X) \le \max_s \det C_s \tag{1.1}$$

where C_s is a block matrix with s blocks of at most two neighboring sizes. Thus, if s_{OPT} is a value maximizing $\det C_s$, a sufficient condition for X to be D-optimum is that $X = X(n, k, s_{\text{OPT}})$. The condition is not necessary because X's of this form do not always exist.

In Section 3 we use our knowledge of s_{OPT} to give examples of pairs (n, k) for which the methods of Section 2 yield D-optimum X's. Although s_{OPT} is not known exactly for *every*

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(n, k), it is known for many (n, k). The following information is taken from the above references; other values of s_{OPT} are listed here, or are obtainable by the methods of those papers.

(A) If $2k - n < 5$, $s_{\text{OPT}} = k$ uniquely.

(B) Two neighboring values of s_{OPT} exist iff

$$2k - n = 5,\ 7\ (\text{and } n \equiv 3 (\text{mod } 8)),\ 9,\ 11,\ 17\ (\text{and } n \equiv 3 (\text{mod } 12)).$$

(C) For $d = 2k - n = 5, 7, 9, 11, 13, 15$ and 17, s_{OPT} and $R_{\text{OPT}} = \lceil k/s_{\text{OPT}} \rceil$ are given by

$d = 5$: $(s_{\text{OPT}}, R_{\text{OPT}}) = (k, 1)$ or $(k - 1, 2)$;

$$d = 7:\quad (s_{\text{OPT}}, R_{\text{OPT}}) = \begin{cases} ((3k+1)/4, 2) \quad \text{or} \quad ((3k-3)/4, 2) & \text{if } n \equiv 3 (\text{mod } 8), \\ ((3k-1)/4, 2) & \text{if } n \equiv 7 (\text{mod } 8); \end{cases}$$

$d = 9$: $(s_{\text{OPT}}, R_{\text{OPT}}) = ((k + 2)/2, 2)$ or $(k/2, 2)$;

$d = 11$: $(s_{\text{OPT}}, \text{R}_{\text{OPT}}) = ((k + 1)/2, 2)$ or $((k - 1)/2, 3)$;

$d = 13$: $(s_{\text{OPT}}, R_{\text{OPT}}) = (\lfloor (4k + 3)/9 \rfloor, 3)$;

$d = 15$: $(s_{\text{OPT}}, R_{\text{OPT}}) = (\lfloor (13k + 25)/36 \rfloor, 3)$;

$$d = 17:\quad (s_{\text{OPT}}, R_{\text{OPT}}) = \begin{cases} ((k+2)/3, 3) \quad \text{or} \quad ((k-1)/3, 4) & \text{if } n \equiv 3 (\text{mod } 12), \\ (k/3, 3) & \text{if } n \equiv 7 (\text{mod } 12), \\ ((k+1)/3, 3) & \text{if } n \equiv 11 (\text{mod } 12). \end{cases}$$

We refer to $\{(n, k) : 2k - n = d\}$ as "the series $2k - n = d$".

(D) For all (n, k),

$$6k/(2k - n + 7) \le s_{\text{OPT}} \le 8k/(2k - n - 3),$$

$$(2k - n + 3)/8 \le R_{\text{OPT}} \le (2k - n + 7)/6.$$

(E) For $n \to \infty$ with $k/n \to 1 - \lambda$, we have

$$\lim s_{\text{OPT}} \begin{cases} = 7 & \text{if } 0 \le \lambda \le .08837, \\ = 8 & \text{if } .08838 \le \lambda \le .17027, \\ = 9 & \text{if } .17028 \le \lambda \le .22494, \\ > 9 & \text{if } .22495 \le \lambda. \end{cases}$$

In Table 1 we list the values of s_{OPT} for $n < 100$, omitting (A), and also summarize the known results, including those of this paper, on D-optimum designs X. These constructibility results, in the form of symbols x, e, # and * as described there, are based on the results of Section 3 herein and also the following facts from the earlier and other references.

(F) Whenever H_{n+1} (a Hadamard matrix of order $n + 1$) exists, normalize it by letting the first row consist of 1's, and delete that row. Any k columns of the resulting matrix yield an X with $X'X = C_k$, optimum for (A) above and for the first value, k, of s_{OPT} in (C) with $d = 5$. This exists for all "practical" n including all $n < 100$, and is omitted from Table 1 in the domain of (A). It is denoted in Table 1 as constructible by the methods of this paper for $d = 5$ because this simple and old construction scheme is a degenerate case of our methods. Even when existence of H_{n+1} is unknown, an X with $X'X = C_k$ can be constructed for sufficiently small k by adjoining several H_j's; see Galil and Kiefer (1980a, top of page 1297).

TABLE 1

Values of s_{OPT} *for* $(n+5)/2 \le k \le n < 100$, $n \equiv 3 \pmod 4$, *with some constructibility results*

KEY: $\bar{j}$ indicates a tie, $s_{\mathrm{OPT}} = j$ or $j+1$, which occurs when $2k - n = 5$, 7 (every other n), 9, 11, 17 (every third n); * a design attaining the bound (1.1) is constructible by the methods of the present paper; # a design is constructed in an earlier paper; see (G) and (H) of Section 1; x design is not constructible, as described in (I) and (J) of Section 1; e a design attaining the bound (1.1) is

k \ n	7	11	15	19	23	27	31	35	39	43	47	51
6	$\overline{5}$#*											
7	5e											
8		$\overline{7}$#*										
9		$\overline{6}$#x										
10		$\overline{5}$##	$\overline{9}$#*									
11		$\overline{5}$ee	8#									
12			$\overline{6}$##	$\overline{10}$#*								
13			$\overline{6}$xx	$\overline{9}$								
14			6x	$\overline{7}$**	$\overline{13}$#*							
15			6x	$\overline{7}$*	11							
16				7	$\overline{8}$	$\overline{15}$#*						
17				6	$\overline{8}$	$\overline{12}$*						
18				6	8	$\overline{9}$**	$\overline{17}$#*					
19				6x	7	$\overline{9}$*	14					
20					7	9	$\overline{10}$	$\overline{19}$#*				
21					7	8	$\overline{10}$	$\overline{15}$**				
22					6	$\overline{7}$	10	$\overline{11}$**	$\overline{21}$#*			
23					6x	7	9	$\overline{11}$*	17			
24						7	8	11	$\overline{12}$	$\overline{23}$#*		
25						7	8	9	$\overline{12}$	$\overline{18}$**		
26						6	7	9	11	$\overline{13}$**	$\overline{25}$#*	
27						6x	7	9	10	$\overline{13}$*	20	
28							7	8*	$\overline{9}$	12	$\overline{14}$	$\overline{27}$#*
29							7	8*	9	11	$\overline{14}$	$\overline{21}$**
30							6	7	8	10	13	$\overline{15}$**
31							6x	7	8	10	11	$\overline{15}$*
32								7	8	9	11	14
33								7	8	9	10	12*
34								6	7	8	10	$\overline{11}$**
35								6x	7	8	9	11*
36									7	8	9	10*
37									7	7	8	10*
38									6	7	8	9
39									6x	7	8	9
40										7	8	8
41										7	7	8
42										6	7	8
43										6x	7	8
44											7	7
45											7	7
46											7	7
47a											6x	7
48												7
49												7
50												7
51												6x
52												
53												
54												
55												
56												

TABLE 1 continued

not constructible but the optimum design is known; see (I) of Section 1. The pair of constructibility symbols after a tie $\overline{j}$ refer in order to j and $j+1$. If only the symbol * is given, it refers to j, while $\overline{*}$ alone refers to $j+1$. Absence of a constructibility symbol denotes ignorance.

55	59	63	67	71	75	79	83	87	91	95	99	n / k
	7	7	8*	8	9	10	11	13	15	18	21*	57
	7	7	8*	8	9	10	11	12	14	16	$\overline{19}$**	58
	$6x$	7	7	8	9	10	10	12	13	15	19*	59
		7	7	8	8	9	10	11	12	15	17*	60
		7	7	8	8	9	10	11	12	14	16*	61
		7	7	8	8	9	9	10	12	13	15	62
		$7x$	7	7	8	9	9	10	11	13	14	63
			7	7	8	8	9	10	11	12	13	64
			7	7	8	8	9	10	11	11	13	65
			7	7	7	8	9	9	10	11	13	66
			$7x$	7	7	8	8	9	10	11	12*	67
				7	7	8	8	9	10	10	11*	68
				7	7	8	8	9	9	10	11*	69
				7	7	7	8	8	9	10	11*	70
				$7x$	7	7	8	8	9	10	10*	71
					7	7	8	8	9	9	10*	72
					7	7	8	8	9	9	10*	73
					7	7	7	8	8	9	10*	74
					$7x$	7	7	8	8	9	9	75
						7	7	8	8	9	9	76
						7	7	7	8	8	9	77
						7	7	7	8	8	9	78
						$7x$	7	7	8	8	9	79
							7	7	8	8	8*	80
$\overline{29}$#*							7	7	7	8	8*	81
23							7	7	7	8	8*	82
$\overline{16}$	$\overline{31}$#*						$7x$	7	7	8	8*	83
$\overline{16}$	$\overline{24}$**							7	7	7	8*	84
15	$\overline{17}$**	$\overline{33}$#*						7	7	7	8*	85
13	$\overline{17*}$	26						7	7	7	8*	86
12	16	$\overline{18}$	$\overline{35}$#*					$7x$	7	7	8*	87
12	14	$\overline{18}$	$\overline{27}$**						7	7	7	88
11	13	17	$\overline{19}$**	$\overline{37}$#*					7	7	7	89
10	12	14	$\overline{19*}$	29					7	7	7	90
10	11	$\overline{13}$	18	$\overline{20}$	$\overline{39}$#*				7	7	7	91
9	11	13	15*	$\overline{20}$	$\overline{30}$**					7	7	92
9	10	12	14*	19	$\overline{21}$**	$\overline{41}$#*				7	7	93
9	10	11	14*	16	$\overline{21*}$	32				7	7	94
8	9	11	12*	15	19	$\overline{22}$	$\overline{43}$#*			$7x$	7	95
8	9	10	12*	14	16	$\overline{22}$	$\overline{33}$**				7	96
8	9	10	11	13	$\overline{15}$	20	$\overline{23}$**	$\overline{45}$#*			7	97
8	8	9	11	12	15	17	$\overline{23*}$	35			7	98
7	8	9	10	12	13	16	21	$\overline{24}$	$\overline{47}$#*		$7x$	99
7	8	9	10	11	13	16	18*	$\overline{24}$	$\overline{36}$**			
7	8	8	10	10	12	14	17*	22	$\overline{25}$**	$\overline{49}$#*		
7	8	8	9	10	12	13	16*	19	$\overline{25*}$	38		
7	7	8	9	10	11	13	15*	$\overline{17}$	23	$\overline{26}$	$\overline{51}$#*	
7	7	8	9	9	11	12	14*	17	19	$\overline{26}$	$\overline{39}$**	
7	7	8	8*	9	10	11	13	15	18	24	$\overline{27}$**	
$6x$	7	7	8*	9	10	11	12	14	18	20	$\overline{27*}$	
	7	7	8*	9	9	11	12	14	16	19	25	

(G) Assuming H_{n+1} exists, a construction of D-optimum X with $X'X = C_{k-1}$ when $d = 5$, the other optimum structure for that case of (C) is given in Galil and Kiefer (1980b, page 185). (For $n = 7, 11, 15$, these X had been found earlier by computer search.) That case $s_{\text{OPT}} = k - 1$ is not covered by the methods of the present paper.

(H) D-optimum X's with $X'X = C_{s_{\text{OPT}}}$, obtained by computer search, are given in Mitchell (1974) for $(n, k, s_{\text{OPT}}) = (11, 10, 5)$; in Galil and Kiefer (1980a) for (11, 9, 6), (11, 10, 6), (15, 11, 8), (15, 12, 6); and in Galil and Kiefer (1980b) for (15, 12, 7). These are not obtainable by the methods of the present paper.

(I) For the saturated cases, $k = n$, $\det(X'X)$ is a square, and hence equality in (1.1) is rarely achieved. Thus, one can easily check that with the possible exception of $k = n = 91$, an X with $X'X = C_{s_{\text{OPT}}}$ does not exist in all saturated cases in Table 1. When $k = n = 7$, a D-optimum X was first found and proved optimum by Williamson (1946), and when $k = n = 11$, three X's yielding the three possible nonisomorphic matrices $X'X$ that are D-optimum were first found and proved optimum in unpublished work of Ehlich, and are given in Mitchell (1974) (who found one by computer search) and in Galil and Kiefer (1980a).

(J) Nonattainability of (1.1) for $(n, k, s_{\text{OPT}}) = (11, 9, 7)$, (15, 13, 6), (15, 13, 7), (15, 14, 6) was shown by Galil and Kiefer (1980b) using a computer search of the tree of all possibilities, reduced somewhat by taking account of certain symmetries. For $(n, k) = (11, 9)$, we know a D-optimum design from (H); for (15, 13) and (15, 14), optimum designs are still unknown at this writing.

In the cases mentioned in (B) and (I) in which the D-optimum $X'X$ may not be unique to within obvious isomorphisms, these designs may be compared according to other criteria. For example, in the cases listed in (B) in which $\det C_s = \det C_{s+1}$ for some s, it is always true that C_s is better than C_{s+1} in terms of giving a smaller value of the Φ_p-criterion for $0 < p \le \infty$, where $\Phi_p(C) = (k^{-1}\text{tr}(C^{-p}))^{1/p}$ for $0 < p < \infty$, and $\Phi_\infty(C) =$ maximum eigenvalue of C^{-1}. See Galil and Kiefer (1980b, 1981b).

2. Construction methods. We again write H_q for a Hadamard matrix of order q. Let H_J have first row $e_J = (1, 1, \cdots, 1)$, and write G_J for the $(J - 1) \times J$ submatrix of H_J consisting of the last $J - 1$ rows. Let $\bar{H}_{M+4}$ be an $(M + 4) \times M$ matrix of ±1's with orthogonal columns and first row e_M. Our basic construction is the $(JM + 4) \times JM$ matrix

$$Z \equiv Z(J, M) = \begin{bmatrix} e_J \otimes \bar{H}_{M+4} \\ G_J \otimes H_M \end{bmatrix}. \tag{2.1}$$

Here $\otimes$ denotes the Kronecker (tensor) product.

If we denote the ith column of H_J by $\begin{bmatrix} 1 \\ g_i \end{bmatrix}$, we have

$$\begin{bmatrix} \bar{H}_{M+4} \\ g_i \otimes H_M \end{bmatrix}' \begin{bmatrix} \bar{H}_{M+4} \\ g_{i'} \otimes H_M \end{bmatrix} = \begin{cases} (JM + 4)I_M & \text{if } i = i', \\ 4I_M & \text{if } i \ne i'. \end{cases} \tag{2.2}$$

We hereafter write $L = JM + 4$. Hence, if P is the $J \times J$ matrix with diagonal entries L and off-diagonal entries 4, we have

$$Z'Z = P \otimes I_M. \tag{2.3}$$

Let Y be obtained from Z by deleting the first row e_{JM} of the latter. Then $Y'Y = Z'Z - E_{JM}$, where $E_m = e'_m e_m$ is a matrix of 1's; $Y'Y$ has diagonal entires $L - 1$ and off-diagonal entries 3 or -1. We permute the columns of Y to form an $(L - 1) \times JM$ matrix $\bar{X} \equiv \bar{X}(J, M)$, as follows: for i and h integers, $0 \le i \le J - 1$, $1 \le h \le M$, the $((h - 1)J + (i + 1))$th column of $\bar{X}$ is the $(iM + h)$th column of Y. Then, denoting by $B(\rho, n)$ the $\rho \times \rho$ "block matrix" with diagonal entries $n = L - 1$ and off-diagonal entries 3, we obtain

(2.4)
$$\bar{X}'\bar{X} = \begin{bmatrix} B(J,n) - E_J & \cdots - E_J \\ -E_J & B(J,n) & \\ \vdots & \ddots & \vdots \\ -E_J & \cdots & B(J,n) \end{bmatrix},$$

a block matrix with M blocks of size J.

Finally, for $s \le M$, $R \le J$, and $1 \le v \le s$, write $k = sR - s + v$, and let $X \equiv X(n, k, s, R)$ be the $n \times k$ matrix obtained by selecting R columns from the ith set of J contiguous columns of $\bar{X}$, $1 \le i \le v$, and $R - 1$ columns from the ith set for $v < i \le s$ (there are none of the latter, if $v = s$). We have obtained

(2.5) $X(n, k, s, R)$, $n \times k$, with $X'X$ having s blocks, maximum block size R, all blocks of size R or $R - 1$.

Of course, n, k, s determine R, but n, k, R do not determine s. Given n, k, s, R, and $n \equiv 3 (\text{mod } 4)$, write $n' = n + 1$. The above method of construction yields an $X(n, k, s, R)$ if and only if, for some J and M for which H_J, H_M, $\bar{H}_{M+4}$ exist,

(2.6)
$$n' = JM + 4, \quad k \le JM, \quad s \le M, \quad \lceil k/s \rceil = R \le J.$$

Suppose (2.6) is satisfied for $(J, M) = (J_1, M_1)$ and that $J_2M_2 = J_1M_1$ and $J_2 < J_1$ and the required H's and $\bar{H}$ exist. If $R \le J_2$, then since $s \le M_1 < M_2$ we see that (2.6) is satisfied for (J_2, M_2). Similarly, if $J_3M_3 = J_1M_1$ and $s \le M_3 < M_1$, we have (2.6) satisfied for (J_3, M_3). In summary,

(2.7) If the above construction works to yield an $X(n, k, s, R)$, then it works for the smallest J for which the required H's and $\bar{H}$ exist and for which $R \le J$, and it also works for the smallest M for which the required H's and $\bar{H}$ exist and for which $s \le M$.

We now turn to the role of existence of H's and $\bar{H}$. The trivial case $J = 1$ produces blocks of size 1, the design of (F) of Section 1. We hereafter assume $R \ge 2$ so $J \ge 2$. If $\bar{H}_{M+4}$ is obtained as M columns of an H_{M+4}, then $4 \mid M$, say $M = 4m$.

In the present and next paragraph we treat the construction with $R = 2$. When $R = 2$, we obtain $X(n, k, s, 2)$ by the above construction with $J = 2$, provided that, for some positive integer m, H_{4m} and H_{4m+4} exist and

(2.8)
$$n = 8m + 3, \quad k \le 8m, \quad s \le 4m, \quad R = 2.$$

Since $s \le k - 1$ for $R = 2$, a sufficient condition for (2.8) when $R = 2$ is $n = 8m + 3$, $k \le 4m + 1$.

Because of (2.7), when $R = 2$, one cannot do better by our method of construction than with $J = 2$. When $J = 2$ there remains the single additional case $M = 2$: the possibility that H_{4m} and $\bar{H}_{4m+4}$ but not H_{4m+4} exist is eliminated by a result of Vijayan (1976); but for $M = 2$ we know that H_2 exists and H_6 does not, but $\bar{H}_6$ does (a column of 1's, a column of three 1's and three -1's). This yields a construction of $X(7, k, s, 2)$ for $k \le 4$; $s \le 2$.

For $R > 2$, we need $J > 2$ and hence $4 \mid J$, say $J = 4j$. We obtain an $X(n, k, s, R)$ by our method if, for some positive integers m and j, H_{4j}, H_{4m}, and H_{4m+4} exist and

(2.9)
$$n = 16mj + 3, \quad k \le 16mj, \quad s \le 4m, \quad R \le 4j.$$

For given $n \equiv 3 (\text{mod } 16)$, write $n^* = n - 3$. If j_R is the smallest divisor of $n^*/16$ which is $\ge R/4$, we see by (2.7) that X can be constructed (for some j and m) by this method if and only if (assuming all necessary H's exist)

(2.10)
$$k \le n^*, \quad s \le n^*/4j_R.$$

Since $s \le (k - 1)/(R - 1)$, a sufficient condition for (2.10) is $k \le (R - 1)n^*/4j_R + 1$. Additionally, corresponding to the special case described in the previous paragraph when $R = 2$ there, we now obtain, for $M = 2$, the designs $X(8j + 3, k, s, R)$ for $k \le 8j$, $s \le 2$, $R \le 4j$.

REMARK 1. We have phrased the results, using (2.6), in terms of given R. Similarly, one can work in terms of s, an m_s, and the second inequality of (2.10) replaced by $R \leq n^*/4m_s$.

REMARK 2. The H_M's (resp., $\bar{H}_{M+4}$'s) are the same across each set of M contiguous rows of Z, but can vary from one set of M rows to the next.

REMARK 3. It is clear that submatrices X of $\bar{X}$ can also be selected to yield block matrices $X'X$ with blocks of more than two sizes. This may be of interest, e.g., for optimality criteria other than D-optimality.

3. Construction of optimum designs. We now apply the methods of the previous section to construct optimum designs by implementing (2.6), usually in the form (2.8) or (2.9), for value(s) of s_{OPT} listed in Section 1 (A) through (E) for various (n, k).

I. *Series* $2k - n = d$, $5 \leq d \leq 17$; see (C).

(a) $d = 5$: The "very regular" case $s_{\text{OPT}} = k$ is a trivial construction, but as noted earlier, $R = J = 1$ in (2.6) formally includes it. The construction for $s_{\text{OPT}} = k - 1$ with $R = 2$ requires $J \geq 2$ and thus $s_{\text{OPT}} \leq k/2$, which is false. However, a design for $s_{\text{OPT}} = k - 1$ is listed in (G) of Section 1.

(b) $d = 7$: Here $R = 2$, and it turns out here and in the next two series that only (2.8), with $J = 2$, can be used; the inequality $s_{\text{OPT}} \leq (n - 3)/4$ required for (2.9) is always false. We find $(3k - 3)/4 = 3(n + 5)/8 \leq (n - 3)/2 \Leftrightarrow n \geq 27$ and $3(n + 5)/8 + 1 \leq (n - 3)/2 \Leftrightarrow n \geq 35$. Thus, for $n \equiv 3 \pmod 8$ we obtain designs for both s_{OPT} values if $n \geq 35$, but only for the smaller s_{OPT} value when $n = 27$. We note that in this and the next two series, the method gives a construction once it starts, only for every other value of $n \equiv 3 \pmod 4$, since it requires $n \equiv 3 \pmod 8$.

(c) $d = 9$: Again $R = 2$ and we find both inequalities $s_{\text{OPT}} \leq (n - 3)/2$ are satisfied for $n \equiv 3 \pmod 8$ when $n \geq 19$. Thus, we get both designs.

(d) $d = 11$: For $s_{\text{OPT}} = (k + 1/2, R = 2$, the method of (2.8) works for $n \geq 19$, $n \equiv 3 \pmod 8$. However, for $s_{\text{OPT}} = (k - 1)/2$, we have $R = 3$, and the method does not work since we never have $s_{\text{OPT}} \leq (n - 3)/4$.

(e) $d = 13$: We use (2.9) with $j = 1$, with $n \equiv 3 \pmod{16}$. We have $s_{\text{OPT}} = \lfloor (2n + 29)/9 \rfloor$, which is $\leq (n - 3)/4$ if $n \geq 115$ and $n \equiv 3 \pmod{16}$, and thus we obtain an optimum design for these values, all of which fall outside Table 1.

(f) $d = 15$: We again use (2.9) with $j = 1$. Now $s_{\text{OPT}} = \lfloor (13n + 245)/72 \rfloor$, which is $\leq (n - 3)/4$ if $n \geq 51$ and $n \equiv 3 \pmod{16}$, so we obtain optimum designs in all these cases.

(g) $d = 17$: Once more we use (2.9) with $j = 1$ and $n \equiv 3 \pmod{16}$. From (C) we obtain the four expressions for s_{OPT} in the three cases (mod 12), three with $R = 3$ and one with $R = 4$. We use (2.9) with $j = 1$ and check $s_{\text{OPT}} \leq (n - 3)/4$, and find that both possible values of s_{OPT} satisfy this condition when $n \geq 51$ for $n \equiv 3 \pmod{48}$, i.e., $n \equiv 3 \pmod{16}$ and $n \equiv 3 \pmod{12}$; and find that the condition is satisfied when $n \geq 67$ for $n \equiv 19 \pmod{48}$ and when $n \geq 83$ for $n \equiv 35 \pmod{48}$. In summary, then, when $n \equiv 3 \pmod{16}$, designs for all values of s_{OPT} are constructible if $n \geq 51$.

II. Other constructions when $n < 100$. In addition to the parameter values that fall into the series of part I, we obtain constructions in the following cases, using (2.9):

$$n = 35, \quad k = 28, 29 \quad (m = 2, j = 1).$$

$$n = 51, \quad k = 35, 36, 37 \quad (m = 3, j = 1).$$

$$n = 67, \quad k = 43, 44, 45 \quad (m = 4, j = 1);$$

$$k = 54(1)58 \quad (m = 2, j = 2).$$

$$n = 83, \quad k = 51, 52, 53 \quad (m = 5, j = 1).$$

$$n = 99, \quad k = 59, 60, 61 \quad (m = 6, j = 1);$$
$$k = 67(1)74 \quad (m = 3, j = 2);$$
$$k = 80(1)87 \quad (m = 2, j = 3).$$

The above designs and those for the series of part I above for $n < 100$ are designated by an asterisk (*) in Table 1.

In the above listing, the large number of cases covered when $n = 67$ and 99 is a reflection of divisibility properties of $n - 3$. In the actual construction, one really needs to use only a subset of the columns of H_M and $\bar{H}_{M+4}$ in some cases; for example, we need use only 16 of the 24 columns for $n = 99$, $k = 61$, $s = 16$, $R = 4$. We can only use (2.9), never (2.8), for cases outside the series of Part I, since we never have $R = 2$ for $d > 11$.

III. Constructibility without knowing s_{OPT}. Suppose we use only the simple inequalities (D) which do not determine R_{OPT} or s_{OPT}. Sometimes these inequalities are nevertheless sufficient, in that the conditions (2.6) are sufficiently weak that we can conclude constructibility of an optimum design by our methods *whatever* s_{OPT} turns out to be. We give only one example, since the development is straightforward, and since one would be unable to use it without knowing s_{OPT} exactly. Nevertheless, the technique could be useful for showing, for a given (k, n), that it is worthwhile working out s_{OPT} exactly because it will definitely be possible to construct an optimum design.

Suppose $d = 41$. From (D), $R_{\text{OPT}} \leq 8$, so we can use (2.9) with $j = 2$. Using our method, we see we must have $s_{\text{OPT}} \leq (n - 3)/8$. But (D) gives $s_{\text{OPT}} \leq 8k/(d - 3)$, so we can use (2.9) if $k/(n - 3) \leq 19/32$. We conclude that, for $n \equiv 3 (\text{mod } 32)$ and $n \geq 259$, with $k = (n + 41)/2$, the method works with $j = 2$ and $m = (n - 3)/32$ in (2.9).

IV. Asymptotics. As $n \to \infty$ with $k/n \to 1 - \lambda$, considerations like those of III above can be obtained from (D), but we now consider the more precise results obtainable from (E).

If $s_{\text{OPT}} = 7$, using (2.9) we need $R \leq 4j$ and $m \geq 2$, so that $k/n \leq 7R/(16mj + 3) \leq 7/8$. From (E), $s_{\text{OPT}} = 7$ for n large only if $k/n > .91$, so we conclude that our construction method can never work in that domain for n large. In fact, from looking at the results for $n < 100$, we doubt that the method applies to any cases for which $s_{\text{OPT}} = 7$, except for the single case $(n, k) = (19, 14)$.

If $s_{\text{OPT}} = 8$ and $n \equiv 3 (\text{mod } 32)$, for $m = 2$ we obtain, if $.83 < k/n < .91$ (see (E)) with $j = (n - 3)/32$, that $R/4j \approx (k/8)/4j \approx k/(n - 3) < 1$ and $n \to \infty$, so the method works as $n \to \infty$. In fact, Table 1 (for $n = 35, 67, 99$) indicates that the method might *always* work for $s_{\text{OPT}} = 8$ and $n \equiv 3 (\text{mod } 32)$. For $m \geq 3$, one finds that the construction only works if $k/n < .8$, so this cannot work asymptotically. Again, examination of part II above indicates that this may be the case for all n of the right congruence, when $s_{\text{OPT}} = 8$.

When $s_{\text{OPT}} = 9$, we need $m \geq 3$ in (2.9) and hence $k/n \approx 9R/n \leq 36j/16mj \leq .75$, whereas asymptotically, from (E), $k/n > .78$ for s_{OPT} to be 9. So the method does not work for large n. Again, from Table 1 (for $n = 51$ or 99) it seems likely that the method never works when $s_{\text{OPT}} = 9$.

When $s_{\text{OPT}} = 10$, 11, or 12, for $n \equiv 3 (\text{mod } 32)$, $m = 3$, and $j = (n - 3)/48$, we obtain $R/4j \approx k/4js_{\text{OPT}} \approx (k/n)(12/s_{\text{OPT}}) < 1$ asymptotically, since from (E) $k/n < .78$ asymptotically in order that $s_{\text{OPT}} > 9$. Thus the method works for large $n \equiv 3 (\text{mod } 48)$ when $s_{\text{OPT}} = 10$, 11, or 12, and Table 1 for $n = 51, 99$ indicates that it might work for all n. We omit discussion of larger s_{OPT} values, which are handled similarly.

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