The Annals of Statistics
1981, Vol. 9, No. 5, 1026–1034

SIMULTANEOUS CONFIDENCE INTERVALS FOR ALL DISTANCES FROM THE "BEST"

BY JASON C. HSU

The Ohio State University

In practice, comparisons with the "best" are often the ones of primary interest. In this paper, parametric and nonparametric simultaneous upper confidence intervals for all distances from the "best" are derived under the location model. Their improvement upon the results of Bechhofer (1954), Gupta (1956, 1965), Fabian (1962), and Desu (1970) in the parametric case is discussed. In the nonparametric case, no comparable confidence statements were available previously.

1. Introduction and statement of the problem. Consider a balanced one-way layout with $k \geq 2$ populations. A number of procedures for comparing all pairs of populations are being used today. Of these, only Tukey's and Scheffé's procedures give simultaneous confidence intervals. However, it often happens in practice that only comparisons with the "best" are of primary interest. In this paper we derive parametric and nonparametric simultaneous upper confidence intervals for all distances from the "best" under the location model. These confidence intervals are sharper than those that can be deduced from Tukey's or the nonparametric analogue of Tukey's simultaneous confidence intervals (p. 243, Lehmann, 1975) respectively.

In the parametric case these confidence intervals represent a substantial strengthening of a result of Fabian (1962) on the Indifference Zone selection procedure of Bechhofer (1954). They allow fuller assessment of the data than the selection procedure of Desu (1970). Our approach also enables us to significantly strengthen the basic probability statement associated with the Subset Selection procedure of Gupta (1956, 1965). In the nonparametric case no comparable confidence statement was available previously.

Let $\pi_1, \cdots, \pi_k$ be k independent populations. Assume that the distributions of the k populations differ in location only. For $i = 1, \cdots, k$, let $F(x - \theta_i)$ be the distribution of X in π_i and let $X_{i1}, \cdots, X_{in}$ be a random sample from π_i so that the joint distribution of $X_{11}, \cdots, X_{1n}, \cdots, X_{kl}, \cdots, X_{kn}$ is $\prod_{i=1}^{k} \prod_{\alpha=1}^{n} F(x_{i\alpha} - \theta_i)$. The "best" population is the population with the largest location parameter. Let $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k)$ and let

$$\theta_{[1]} \leq \cdots \leq \theta_{[k]}$$

be the ordered location parameters. We shall derive $100P^*\%$ simultaneous upper confidence intervals for

$$\theta_{[k]} - \theta_1, \cdots, \theta_{[k]} - \theta_k$$

where we assume $1/k < P^* < 1$. The problem of simultaneous lower confidence intervals for the same parameters is essentially different and is not discussed in this paper.

2. Confidence intervals for means of normal populations. Let $F(x) = \Phi(x/\sigma)$, the distribution function of a normal random variable with mean 0 and standard deviation σ. We consider two cases according as σ is known or unknown.

Received August 1979; revised September 1980.

AMS 1970 *subject classifications.* Primary 62F25; secondary 62G15.

Key words and phrases. Simultaneous confidence intervals, distances from the best, ranking and selection procedures.

2.1. *σ known.* Let $\bar{X}_1, \cdots, \bar{X}_k$ be the sample means of the k populations. For notation, let

$$\bar{X}_{[1]} \leq \cdots \leq \bar{X}_{[k]}$$

be the ordered sample means. Let $\pi_{(k)}$ be the unknown "best" population, i.e. $E(\bar{X}_{(k)}) = \theta_{[k]}$. In case more than one population has a θ-value which is tied for the largest, then exactly one of these tied populations is defined to be the "best" population $\pi_{(k)}$ according to some fixed rule.

Let d_{k,P^*} be the constant such that

$$P(Z_k \geq Z_i - d_{k,P^*} \quad \text{for} \quad i = 1, \cdots, k-1) = P^*,$$

where $Z_1, \cdots, Z_k$ are i.i.d. standard normal random variables. It is easy to see that d_{k,P^*} is the solution of

$$\int_{-\infty}^{\infty} \Phi^{k-1}(z + d_{k,P^*})\, d\Phi(z) = P^*,$$

where Φ is the standard normal distribution function. Tables of d_{k,P^*} can be found in Gupta, Nagel and Panchapakesan (1973), Gupta (1963), Milton (1963), and Bechhofer (1954).

For $i = 1, \cdots, k$, define

$$D_i = \max(\max_{j \neq i} \bar{X}_j - \bar{X}_i + d_{k,P^*}\sigma/\sqrt{n}, 0).$$

THEOREM 2.1. *A set of* $100P^*\%$ *simultaneous confidence intervals for*

$$\theta_{[k]} - \theta_1, \cdots, \theta_{[k]} - \theta_k$$

is given by

$$[0, D_1], \cdots, [0, D_k]. \tag{2.1}$$

PROOF.

$$\begin{aligned} P^* &= P(Z_{(k)} \geq Z_i - d_{k,P^*} \quad \text{for} \quad i = 1, \cdots, k, \quad i \neq (k)) \\ &= P(\bar{X}_{(k)} - \theta_{[k]} \geq \bar{X}_i - \theta_i - d_{k,P^*}\sigma/\sqrt{n} \quad \text{for} \quad i = 1, \cdots, k, \quad i \neq (k)) \\ &= P(\bar{X}_{(k)} - \theta_{[k]} \geq \bar{X}_i - \theta_i - d_{k,P^*}\sigma/\sqrt{n} \quad \text{for} \quad i \neq (k), 0 \geq \theta_{[k]} - \theta_i, i = (k)) \\ &\leq P\{\max(\max_{j\neq i} \bar{X}_j - \bar{X}_i + d_{k,P^*}\sigma/\sqrt{n}, 0) \geq \theta_{[k]} - \theta_i \quad \text{for} \quad i = 1, \cdots, k\} \\ &= P(D_i \geq \theta_{[k]} - \theta_i \quad \text{for} \quad i = 1, \cdots, k). \end{aligned} \tag{2.2}$$

REMARK 2.1. The probability (2.2) depends on the true parameter $\boldsymbol{\theta}$. However, if $\theta_{[k]} - \theta_{[k-1]} \geq d_{k,P^*}\sigma/\sqrt{n}$ then the occurrence of the event $\{\bar{X}_{(k)} - \theta_{[k]} \geq \bar{X}_i - \theta_i - d_{k,P^*}\sigma/\sqrt{n}$ for $i = 1, \cdots, k, i \neq (k)\}$ implies $\bar{X}_{(k)} = \bar{X}_{[k]}$. It follows that the nominal confidence coefficient $100P^*\%$ is attained whenever $P^* \geq 1/k$ and $n \geq \{d_{k,P^*}\sigma/(\theta_{[k]} - \theta_{[k-1]})\}^2$.

2.1.1. *Relation to the Indifference Zone Selection Procedure of Bechhofer (1954).* Bechhofer (1954) considered the problem of selecting the "best" population when the "best" and the remaining populations are sufficiently apart, that is, $\theta_{[k]} - \theta_{[k-1]} \geq \delta^*$ where δ^* is a specified positive constant. Bechhofer's procedure, which shall be denoted by $R_B(P^*)$, is as follows:

$R_B(P^*)$: Select the population yielding $\bar{X}_{[k]}$.

A Correct Decision (CD) is the event that the selected population is the "best" population. Bechhofer (1954) showed that if, in *designing* the experiment, n is set so large that

$$d_{k,P^*}\sigma/\sqrt{n} \le \delta^*,$$

then

$$P\{CD \mid R_B(P^*)\} \ge P^* \text{ for all } \boldsymbol{\theta} \text{ such that } \theta_{[k]} - \theta_{[k-1]} \ge \delta^*. \tag{2.3}$$

If we define s to be the index of the selected population (i.e. π_s is the selected population), then it is known that one can give the confidence statement concerning the single parameter $\theta_{[k]} - \theta_s$

$$P(\theta_{[k]} - \theta_s \le \delta^*) \ge P^*. \tag{2.4}$$

Fabian (1962) showed that (2.4) can be strengthened to

$$P(\theta_{[k]} - \theta_s \le D_s) \ge P^*. \tag{2.5}$$

Our result is

$$P(\theta_{[k]} - \theta_i \le D_i \quad \text{for} \quad i = 1, \cdots, k) \ge P^*.$$

In other words, we have shown that without decreasing P^*, (2.5) can be strengthened to include the $k - 1$ additional confidence intervals $[0, D_i]$, $i = 1, \cdots, k$, $i \ne s$, for $\theta_{[k]} - \theta_i$, $i = 1, \cdots, k$, $i \ne s$.

2.1.2. *Relation to the Selection Procedure of Desu (1970).*

Desu (1970) considered the problem of selecting a subset of the populations so that none of the selected populations is "bad." More precisely, his formulation is as follows:

A population π_i is said to be

$$\text{"good" if } \theta_{[k]} - \theta_i \le \delta_1^*$$

$$\text{"bad" if } \theta_{[k]} - \theta_i \ge \delta_2^*$$

where δ_1^*, δ_2^* are specified constants such that $0 < \delta_1^* < \delta_2^*$. If S denotes the set of selected populations, then Correct Decision (CD) is the event $\{\theta_{[k]} - \theta_i < \delta_2^* \text{ for all } \pi_i \in S\}$.

For selecting normal populations, Desu's procedure which shall be denoted by $R_D(P^*)$ is as follows:

$$R_D(P^*)\text{: Select } \pi_i \text{ iff } \bar{X}_{[k]} - \bar{X}_i + d_{k,P^*}\sigma/\sqrt{n} \le \delta_2^*.$$

To ensure that a nonempty set is selected, in *designing* the experiment, n must be set so large that

$$d_{k,P^*}\sigma/\sqrt{n} \le \delta_2^*. \tag{2.6}$$

Desu (1970) showed

$$P_{\boldsymbol{\theta}}\{CD \mid R_D(P^*)\} \ge P^* \text{ for all } \boldsymbol{\theta} \in \mathbb{R}^k \tag{2.7}$$

so that after selection one can give the confidence statement

$$P(\theta_{[k]} - \theta_i < \delta_2^* \quad \text{for all} \quad \pi_i \in S) \ge P^*.$$

His proof of (2.7) involves first finding the least favorable configuration for fixed t_1 and t_2, where t_1 and t_2 are the number of "good" and "bad" populations respectively, and then minimizing over t_1, t_2. It can be readily seen that (2.7) follows immediately from the confidence statement (2.2) of Theorem 2.1 and the above steps are unnecessary. In fact, it is plain for the same data set, the simultaneous confidence intervals given by Theorem 2.1 allow fuller assessment of the data than Desu's procedure. While Desu's procedure asserts only that all those selected are within δ_2^* of the "best," the simultaneous confidence intervals not only give bounds which are always sharper for the same populations, but also bounds for the populations not selected by Desu's procedure.

2.1.3. *Relation to the Subset Selection Procedure of Gupta* (1956, 1965).
Gupta (1956, 1965) considered the problem of selecting a subset of the populations to contain the "best" population. A Correct Selection (CS) is the event that the selected subset contains $\pi_{(k)}$, the population associated with $\theta_{[k]}$. (Recall that in case more than one population has a θ-value which is tied for the largest, then exactly one of these tied populations is defined to be the "best" population $\pi_{(k)}$ according to some fixed rule.) Gupta's procedure which shall be denoted by $R_G(P^*)$ is as follows:

$$R_G(P^*)\text{: Select } \pi_i \text{ iff } \bar{X}_i \geq \bar{X}_{[k]} - d_{k,P^*}\sigma/\sqrt{n}.$$

Gupta (1956, 1965) showed

$$P_\theta\{CS \mid R_G(P^*)\} \geq P^* \quad \text{for all} \quad \boldsymbol{\theta} \in \mathbb{R}^k. \tag{2.8}$$

One criticism of the subset selection formulation has been that while one asserts that the selected subset contains the "best" population with a probability of at least P^*, no statement is made concerning the *individual* populations in the selected set. The following derivation shows, however, that (2.8) can be strengthened.

$$\begin{aligned} P^* &= P\{Z_{(k)} \geq Z_i - d_{k,P^*} \quad \text{for} \quad i = 1, \cdots, k, \quad i \neq (k)\} \\ &= P\{\bar{X}_{(k)} - \theta_{[k]} \geq \bar{X}_i - \theta_i - d_{k,P^*}\,\sigma/\sqrt{n} \quad \text{for} \quad i = 1, \cdots, k, \quad i \neq (k)\} \\ &\leq P\{\bar{X}_{(k)} \geq \bar{X}_{[k]} - d_{k,P^*}\,\sigma/\sqrt{n} \quad \text{and} \quad \theta_{[k]} - \theta_i \leq D_i \quad \text{for} \quad i = 1, \cdots, k\}. \end{aligned} \tag{2.9}$$

Since $\{CS\} = \{\bar{X}_{(k)} \geq \bar{X}_{[k]} - d_{k,P^*}\sigma/\sqrt{n}\}$ for $R_G(P^*)$, we have shown that for Gupta's procedure $R_G(P^*)$, not only can one assert (2.8), but without any decrease in P^* one can also give the simultaneous upper confidence intervals (2.1) for distances from the "best" for all the populations.

2.2 *σ unknown.* We use the same notation as before. In addition, since σ is unknown, we estimate σ by

$$W = \{\textstyle\sum_{i=1}^k \sum_{\alpha=1}^n (X_{i\alpha} - \bar{X}_i)^2/k(n-1)\}^{1/2},$$

the usual pooled estimator of σ with $v = k(n-1)$ degrees of freedom. Let $d_{k,P^*,v}$ be the constant such that

$$P(Z_k \geq Z_i - d_{k,P^*,v}W \quad \text{for} \quad i = 1, \cdots, k-1) = P^*,$$

where again $Z_1, \cdots, Z_k$ are i.i.d. standard normal random variables and W is a $\chi_v/\sqrt{v}$ random variable independent of $Z_1, \cdots, Z_k$. It is easy to see that $d_{k,P^*,v}$ is the solution of

$$\int_0^\infty \int_{-\infty}^\infty \Phi^{k-1}(z + d_{k,P^*,v}\, w)\, d\Phi(z)\, dQ_v(w) = P^*,$$

where again Φ is the standard normal distribution function and Q_v is the distribution function of a $\chi_v/\sqrt{v}$ random variable. Tables of $d_{k,P^*,v}$ can be found in Gupta and Sobel (1957), Dunnett (1955), and Krishnaiah and Armitage (1966).

For $i = 1, \cdots, k$, define

$$D_i' = \max(\max_{j \neq i} \bar{X}_j - \bar{X}_i + d_{k,P^*,v}W/\sqrt{n}, 0).$$

THEOREM 2.2. *A set of* $100P^*\%$ *simultaneous confidence intervals for*

$$\theta_{[k]} - \theta_1, \cdots, \theta_{[k]} - \theta_k$$

is given by

$$[0, D_1'], \cdots, [0, D_k']. \tag{2.10}$$

PROOF. Similar to Theorem 2.1.

REMARK 2.2.1. Under the formulation of either Bechhofer (1954) or Desu (1970), when σ is unknown, multi-stage (such as two-stage or sequential) sampling is required. In contrast, the simultaneous confidence intervals (2.10) given by Theorem 2.2 enable the researcher to assess the data for single-stage designs.

REMARK 2.2.2. For σ unknown, the Subset Selection procedure proposed by Gupta (1956, 1965) which we shall denote by $R_G'(P^*)$ is as follows:

$$R_G'(P^*)\text{: Select } \pi_i \text{ iff } \bar{X}_i \geq \bar{X}_{[k]} - d_{k,P^*,\nu} W/\sqrt{n}.$$

Gupta (1956, 1965) showed

$$P_{\theta,\sigma}\{CS \mid R_G'(P^*)\} \geq P^* \text{ for all } \boldsymbol{\theta} \in \mathbb{R}^k, \sigma > 0. \tag{2.11}$$

Analogous to 2.1.3, it can be shown that

$$P^* \leq P\{\bar{X}_{(k)} \geq \bar{X}_{[k]} - d_{k,p^*,\nu} W/\sqrt{n} \quad \text{and} \quad \theta_{[k]} - \theta_i \leq D_i' \quad \text{for} \quad i = 1, \cdots, k\}. \tag{2.12}$$

Since $\{CS\} = \{\bar{X}_{(k)} \geq \bar{X}_{[k]} - d_{k,P^*,\nu} W/\sqrt{n}\}$ for $R_G'(P^*)$, again the comment at the end of Section 2.1.3 applies.

3. Nonparametric confidence intervals. Suppose F is absolutely continuous with density f but otherwise unknown. For notation, let $R_i^{j\alpha}(\Delta)$ denote the rank of $X_{j\alpha} - \Delta$ in the combined sample

$$X_{i1}, \cdots, X_{in}, X_{j1} - \Delta, \cdots, X_{jn} - \Delta.$$

For $1 \leq i, j \leq k$, $i \neq j$, let

$$S_i^j(\Delta) = (2/n)\{\textstyle\sum_{\alpha=1}^n a_{2n}(R_i^{j\alpha}(\Delta)) - n\bar{a}_{2n}\}, \tag{3.1}$$

where $a_m(\cdot)$ is a given score function converging in quadratic mean to a square integrable function $\phi(\cdot)$:

$$\lim_{m\to\infty} \int_0^1 \{a_m(1 + [um]) - \phi(u)\}^2 \, du = 0.$$

Here

$$\bar{a} = m^{-1} \textstyle\sum_{\alpha=1}^m a_m(\alpha) \to \displaystyle\int_0^1 \phi(u) \, du \equiv \bar{\phi}$$

and $[um]$ denotes the largest integer contained in um. In case of ties, average scores are to be used. For notation, let $S_i^j = S_i^j(0)$.

We assume $a_m(\cdot)$ is a nondecreasing function and ϕ is non-constant.

As in hypothesis testing, we are particularly interested in the scores

$$a_m(\alpha, f_0) = E\{\phi(U_m^{(\alpha)}, f_0)\}, \qquad 1 \leq \alpha \leq m \tag{3.2}$$

and the approximate scores

$$a_m(\alpha, f_0) = \phi(\alpha/(m+1), f_0), \qquad 1 \leq \alpha \leq m \tag{3.3}$$

where

$$\phi(u, f_0) = -\frac{f_0'(F_0^{-1}(u))}{f_0(F_0^{-1}(u))}, \qquad 0 \leq u \leq 1.$$

Here $U_m^{(\alpha)}$ is the αth order statistic of a sample of size m from the uniform distribution.

As before, let

$$\theta_{[1]} \leq \cdots \leq \theta_{[k]}$$

be the ordered unknown parameters. Let $\pi_{(k)}$ be the "best" population, i.e. the population associated with $\theta_{[k]}$, resolving ties as in the previous section.

Let $c_{k,P^*,n}$ be the smallest number such that

$$P_0(S_i^k \geq -c_{k,P^*,n} \quad \text{for} \quad i = 1, \cdots, k-1) \geq P^*,$$

where P_0 indicates that the probability is computed under $\theta_1 = \cdots = \theta_k$.

For $1 \leq i, j \leq k$ $(i \neq j)$, let

$$D_i^j = \inf\{\Delta: S_i^j(\Delta) < -c_{k,P^*,n}\}$$

and for $i = 1, \cdots, k$, let

$$D_i^* = \max(\max_{j\neq i} D_i^j, 0).$$

THEOREM 3.1. *A set of* $100P^*\%$ *simultaneous confidence intervals for*

$$\theta_{[k]} - \theta_1, \cdots, \theta_{[k]} - \theta_k$$

is given by

$$[0, D_1^*], \cdots, [0, D_k^*]. \tag{3.4}$$

PROOF.

$$\begin{aligned} P^* &\leq P\{S_i^{(k)}(\theta_{[k]} - \theta_i) \geq -c_{k,P^*,n} \quad \text{for} \quad i = 1, \cdots, k, \quad i \neq (k)\} \\ &\leq P\{D_i^{(k)} \geq \theta_{[k]} - \theta_i \quad \text{for} \quad i = 1, \cdots, k, \quad i \neq (k)\} \\ &= P\{D_i^{(k)} \geq \theta_{[k]} - \theta_i \quad \text{for} \quad i = 1, \cdots, k, i \neq (k), \quad 0 \geq \theta_{[k]} - \theta_i, \quad i = (k)\} \\ &\leq P\{\max(\max_{j\neq i} D_i^j, 0) \geq \theta_{[k]} - \theta_i \quad \text{for} \quad i = 1, \cdots, k\} \\ &= P(D_i^* \geq \theta_{[k]} - \theta_i \quad \text{for} \quad i = 1, \cdots, k). \end{aligned}$$

EXAMPLE 3.1 *Confidence intervals corresponding to the Wilcoxon statistic.*

Consider the case when the scores are defined by (3.2) where f_0 is the logistic density so that S_i^j are essentially the two-sample Wilcoxon statistics. For $1 \leq i, j \leq k (i \neq j)$, let

$$D_{i[1]}^j \leq \cdots \leq D_{i[n^2]}^j$$

denote the n^2 ordered differences $X_{j\alpha} - X_{i\beta}$, $1 \leq \alpha, \beta \leq n$. Let $r_{k,P^*,n} = \{n(2n+1)/4\}c_{k,P^*,n} + 1$. Then it can be shown that

$$D_i^j = D_{i[n^2/2+r_{k,P^*,n}]}^j$$

so that

$$D_i^* = \max(\max_{j\neq i} D_{i[n^2/2+r_{k,P^*,n}]}^j, 0). \tag{3.5}$$

Tables of $r_{k,P^*,n}$(more precisely, of $n(2n+1)/2 + r_{k,P^*,n}$) can be found in Miller (1966) and Steel (1959).

EXAMPLE 3.2. *Confidence intervals corresponding to the median statistic.*

Consider the case when the scores are the approximate scores defined by (3.3) with f_0 being the double exponential density so that

$$S_i^j(\Delta) = (2/n) \sum_{\alpha=1}^n \text{sign}(R_i^{j\alpha}(\Delta) - (2n+1)/2),$$

where sign $(x) = 1, 0$, or -1 as $x >, =,$ or < 0. For $1 \leq i, j \leq k$, let

$$X_{i[1]} < \cdots < X_{i[n]}$$
$$X_{j[1]} < \cdots < X_{j[n]}$$

be the ordered samples from the ith and jth populations. Let $m_{k,P^*,n} = (n/2)c_{k,P^*,n}$. Then it can be shown that

$$D_i^j = X_{j[(n+m_{k,P^*,n})/2+1]} - X_{i[(n-m_{k,P^*,n})/2]}$$

so that

$$D_i^* = \max(\max_{j\neq i} X_{j[(n+m_{k,P^*,n})/2+1]} - X_{i[(n-m_{k,P^*,n})/2]}, 0). \tag{3.6}$$

There does not appear to be any published table of $m_{k,P^*,n}$ at the present tim

REMARK 3.1. Lehmann (1963a) considered selection procedures that have the same form as the procedure of Bechhofer (1954) except that k-sample statistics based on joint ranks were used in place of the sample means. Randles (1970) and Ghosh (1973) considered selection procedures that have the same form as the procedure of Bechhofer (1954) except that point estimators based on ranks were used in place of the sample means. However, in contrast to the situation in 2.1.1, these procedures cannot give confidence statements of the kind given by Theorem 3.1.

REMARK 3.2. A class of nonparametric subset selection procedures based on the statistics (3.1) has been proposed by Hsu (1980). Analogous to 2.1.3, the basic probability statement associated with these procedures can be strengthened to include the simultaneous upper confidence intervals (3.4) for all distances from the "best".

The relative efficiencies of confidence intervals can be measured in terms of their abilities to exclude false parameter values. We define the asymptotic relative efficiency (ARE) of a set of $100P^*\%$ confidence bounds $\mathbf{D}^*$ relative to another set of $100P^*\%$ confidence bound $\mathbf{D}$ to be the reciprocal of the limiting ratio of sample sizes needed so that they have the same limiting probability of excluding parameter values $\psi^{(n)} = (\psi_1^{(n)}, \cdots, \psi_k^{(n)})$ where

$$\psi_i^{(n)} = \theta_{[k]} - \theta_i + \delta_i/n^{1/2} + o(n^{1/2}). \qquad (\delta_i > 0)$$

In the sequel we assume F has a finite variance of σ^2.

THEOREM 3.2. *Suppose f is bounded, then the* ARE *of the confidence intervals* (3.5) *relative to the confidence intervals* (2.10) *is* $12\sigma^2 \int_{-\infty}^{\infty} f^2(x)\,dx$.

PROOF. The asymptotic distribution of $\{n^{1/2}(D_i^j - (\theta_j - \theta_i)), i \neq j\}$ can be obtained by getting the asymptotic distribution of $\{n^{1/2}(D_i^j - (\theta_j - \theta_i)), i < j\}$ along the lines of Lehmann (1963b), and then using Theorem 1 of Sen (1966). A comparison with the asymptotic distribution of $\{n^{1/2}(\bar{X}_j - \bar{X}_i - (\theta_j - \theta_i)) + d_{k,P^*,\nu}W, i \neq j\}$ yields the result.

REMARK 3.3. AREs for other confidence bounds $\mathbf{D}^*$ with smooth ϕ satisfying the regularity conditions of Koziol and Reid (1977) and Theorem 1 of Sen (1966) can be obtained as outlined above using the results of those two papers.

THEOREM 3.3. *Let $\xi_{1/2} = \inf\{x: F(x) \geq 1/2\}$. Suppose, in a neighborhood of $\xi_{1/2}$, f is positive and has a bounded derivative, then the* ARE *of the confidence intervals* (3.6) *relative to the confidence intervals* (2.10) *is* $4\sigma^2 f^2(\xi_{1/2})$.

PROOF. Here we utilize the well known asymptotic distribution of sample medians and the results of Bahadur (1966).

TABLE 1
$P^* = .95$

$v \backslash k$	2	5	10	20	30	50
15	2.48	3.34	3.78	4.17	4.37	4.62
	3.01	4.37	5.20	5.96	6.38	6.89
30	2.40	3.19	3.59	3.92	4.10	4.31
	2.89	4.10	4.82	5.48	5.83	6.27
60	2.36	3.12	3.50	3.82	3.99	4.19
	2.83	3.98	4.65	5.24	5.57	5.96
∞	2.33	3.06	3.42	3.72	3.88	4.07
	2.77	3.86	4.74	5.01	5.30	5.65

Comparisons of $d_{k,P^*,v}$ (upper entry) and $q_{k,P^*,v}$ (lower entry)

TABLE 2
$P^* = .99$

$v \backslash k$	2	5	10	20	30	50
15	3.68	4.54	4.99	5.38	5.58	5.82
	4.17	5.56	6.44	7.26	7.73	8.30
30	3.47	4.20	4.59	4.92	5.10	5.30
	3.89	5.05	5.76	6.41	6.77	7.22
60	3.38	4.06	4.41	4.72	4.88	5.07
	3.76	4.82	5.45	6.02	6.33	6.71
∞	3.29	3.92	4.25	4.52	4.67	4.85
	3.64	4.60	5.16	5.65	5.91	6.23

Comparisons of $d_{k,P^*,v}$ (upper entry) and $q_{k,P^*,v}$ (lower entry)

4. Concluding remarks. If comparisons with the "best" are the ones of interest, then the results of this paper are sharper than those that can be deduced from an all pairwise comparisons procedure. For example, in the setting of 2.2, if one were to deduce from Tukey's procedure upper bounds for $\theta_{[k]} - \theta_i$, $i = 1, \cdots, k$, one would obtain (2.10) with $q_{k,P^*,v}$, the upper P^* quantile of the Studentized range distribution, substituting for $d_{k,P^*,v}$. The difference is substantial, as can be seen from Tables 1 and 2. Comparisons with Scheffé's procedure are of course even more favorable.

Acknowledgment. The author wishes to thank the referees for their helpful comments.

REFERENCES

BAHADUR, R. R. (1966). A note on quantiles in large samples. *Ann. Math. Statist.* **37** 577–580.

BECHHOFER, R. E. (1954). A single-sample multiple decision procedure for ranking means of normal populations with known variances. *Ann. Math. Statist.* **25** 16–39.

DESU, M. M. (1970). A selection problem. *Ann. Math. Statist.* **41** 1596–1603.

DUNNETT, C. W. (1955). A multiple comparison procedure for comparing several treatments with a control. *J. Amer. Statist. Assoc.* **50** 1096–1121.

FABIAN, V. (1962). On multiple decision methods for ranking population means. *Ann. Math. Statist.* **33** 248–254.

GHOSH, M. (1973). Nonparametric selection procedures for symmetric location parameter populations. *Ann. Statist.* **1** 773–779.

GUPTA, S. S. (1956). On a decision rule for a problem in ranking means. *Inst. of Statist. Mimeo Series* No. 150, Univ. of North Carolina.

GUPTA, S. S. (1963). Probability integrals of the multivariate normal and multivariate *t*. *Ann. Math. Statist.* **34** 792–828.

GUPTA, S. S. (1965). On some multiple decision (selection and ranking) rules. *Technometrics* **7** 225–245.

GUPTA, S. S., NAGEL, K. and PANCHAPAKESAN, S. (1973). On the order statistics from equally correlated normal random variables. *Biometrika* **60** 403–413.

GUPTA, S. S. and SOBEL, M. (1957). On a statistic which arises in selection and ranking problems. *Ann. Math. Statist.* **28** 957–967.

HSU, J. C. (1980). A class of nonparametric subset selection procedures. *Sankhyā Ser.* **B** 42. To appear.

KOZIOL, J. and REID, N. (1977). On the asymptotic equivalence of two ranking methods for k-sample linear rank statistics. *Ann. Statist.* **5** 1099–1106.

KRISHNAIAH, P. R. and ARMITAGE, J. V. (1966). Tables for multivariate t-distribution. *Sankhyā Ser. B* **28** 31–56.

LEHMANN, E. L. (1963a). A class of selection procedures based on ranks. *Math. Ann.* **150** 268–275.

LEHMANN, E. L. (1963b). Robust estimation in analysis of variance. *Ann. Math. Statist.* **34** 957–966.

LEHMANN, E. L. (1975). *Nonparametrics: Statistical Methods Based on Ranks.* Holden-Day, San Francisco.

MILLER, R. G. (1966). *Simultaneous Statistical Inference.* McGraw Hill, New York.

MILTON, R. C. (1963). Tables of equally correlated multivariate normal probability integral. Tech. Report No. 27, Dept. of Statistics, Univ. of Minnesota, Minneapolis, Minnesota.

RANDLES, R. H. (1970). Some robust selection procedures. *Ann. Math. Statist.* **41** 1640–1645.

SEN, P. K. (1966), On a distribution-free method of estimating asymptotic efficiency of a class of nonparametric tests. *Ann. Math. Statist.* **37** 1759–1770.

STEEL, R. G. D. (1959). A multiple comparison rank sum test: Treatment versus control. *J. Am. Statist. Assoc.* **54** 767–775.

DEPARTMENT OF STATISTICS
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210