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LINEAR TRANSFORMATIONS PRESERVING BEST LINEAR UNBIASED ESTIMATORS IN A GENERAL GAUSS-MARKOFF MODEL[1]

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Under a general Gauss-Markoff model $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$, a necessary and sufficient condition is established for a linear transformation, $\mathbf{F}$, of the observable random vector $\mathbf{y}$ to have the property that there exists a linear function of $\mathbf{Fy}$ which is a BLUE of $\mathbf{X}\boldsymbol{\beta}$. A method for deriving a required BLUE from the transformed model $\{\mathbf{Fy}, \mathbf{FX}\boldsymbol{\beta}, \mathbf{FVF}'\}$ is also indicated.

1. Statement of the problem. The following notation will be used throughout this paper. Given a real matrix $\mathbf{A}$, the symbols $\mathbf{A}'$, $\mathbf{A}^{-1}$, $r(\mathbf{A})$, and $\mathscr{C}(\mathbf{A})$ will denote the transpose, inverse, rank, and column space, respectively, of $\mathbf{A}$. Further, $\mathbf{A}^-$ will stand for a g-inverse of $\mathbf{A}$, that is, for any matrix satisfying the equation $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$. Moreover, the triplet

$$\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\} \tag{1}$$

will denote a general Gauss-Markoff model in which $\mathbf{y}$ is an $n \times 1$ observable random vector with expectation $E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$ and with dispersion matrix $D(\mathbf{y}) = \mathbf{V}$, where $\mathbf{X}$ is a known $n \times p$ matrix of arbitrary rank, and $\mathbf{V}$ is an $n \times n$ nonnegative definite symmetric matrix, known entirely or except for a positive scalar multiplier.

Before stating precisely the problem considered in the paper, let us make the following observation. If the vector $\mathbf{y}$ subject to model (1) with $\mathbf{V} = \mathbf{I}$, the identity matrix, were transformed into the vector $\mathbf{w} = \mathbf{X}'\mathbf{y}$, then the best linear unbiased estimator (BLUE) of $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, $\bar{\boldsymbol{\mu}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'\mathbf{y}$, would also be obtainable as a linear function of $\mathbf{w}$, namely as $\bar{\boldsymbol{\mu}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{w}$. If, however, the same transformation were adopted in the case of $\mathbf{V}$ in (1) being a positive definite matrix different from $\mathbf{I}$, then the BLUE of $\boldsymbol{\mu}$, now having the form $\tilde{\boldsymbol{\mu}} = \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^-\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$, would no longer be obtainable as a linear function of $\mathbf{w} = \mathbf{X}'\mathbf{y}$ unless $\mathscr{C}(\mathbf{V}^{-1}\mathbf{X}) \subset \mathscr{C}(\mathbf{X})$. This exception might in fact be expected as the inclusion is a necessary and sufficient condition for the estimators $\tilde{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}$ to be identical (Haberman, 1975).

In view of the above example, it seems justified to look for a general criterion that would be useful in deciding whether or not a proposed linear transformation of $\mathbf{y}$ preserves the information indispensable for obtaining a BLUE of $\mathbf{X}\boldsymbol{\beta}$. More precisely, given model (1), the problem is to establish a necessary and sufficient condition for a $k \times n$ matrix $\mathbf{F}$ to have the property that there exists such a linear function of $\mathbf{Fy}$ which is a BLUE of $\mathbf{X}\boldsymbol{\beta}$. A further problem of interest is to indicate a method for deriving a required BLUE from the transformed model

$$\{\mathbf{Fy}, \mathbf{FX}\boldsymbol{\beta}, \mathbf{FVF}'\}. \tag{2}$$

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2. Results. To establish the main result of this paper, which is given in the theorem below, we use the following lemma.

LEMMA. *Let* $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ *be a general Gauss-Markoff model, and let* $\mathbf{C}\boldsymbol{\beta}$ *be a set of parametric functions estimable therein. Then* $\mathbf{Ay}$ *is a* BLUE *of* $\mathbf{C}\boldsymbol{\beta}$ *if and only if*

$$\mathbf{AT} = \mathbf{C}(\mathbf{X}'\mathbf{T}^{-}\mathbf{X})^{-}\mathbf{X}', \tag{3}$$

where

$$\mathbf{T} = \mathbf{V} + \mathbf{XUX}', \tag{4}$$

with $\mathbf{U}$ *being any* $p \times p$ *nonnegative definite symmetric matrix for which* $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{T})$.

PROOF. Arguing similarly as did Rao (1978) in the proof of his Theorem 1, it can be shown that $\mathbf{Ay}$ is a BLUE of $\mathbf{C}\boldsymbol{\beta}$ if and only if $\mathbf{A}$ admits a representation

$$\mathbf{A} = \mathbf{C}(\mathbf{X}'\mathbf{T}^{-}\mathbf{X})^{-}\mathbf{X}'\mathbf{T}^{-} + \mathbf{W}(\mathbf{I} - \mathbf{TT}^{-}), \tag{5}$$

where $\mathbf{W}$ is an arbitrary matrix of appropriate order. Since (5) is actually a general solution to the equation (3), the lemma follows. □

THEOREM. *Let* $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ *be a general Gauss-Markoff model, and let* $\mathbf{F}$ *be a* $k \times n$ *matrix.*

(i) *A* BLUE *of* $\mathbf{X}\boldsymbol{\beta}$ *is obtainable as a linear function of* $\mathbf{Fy}$ *if and only if*

$$\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{TF}'), \tag{6}$$

or, equivalently,

$$r(\mathbf{X} \vdots \mathbf{VF}') = r(\mathbf{TF}'),$$

where $\mathbf{T}$ *is any matrix as defined in* (4).

(ii) *If the condition of* (i) *is satisfied, then each* BLUE *of* $\mathbf{X}\boldsymbol{\beta}$ *in the transformed model* $\{\mathbf{Fy}, \mathbf{FX}\boldsymbol{\beta}, \mathbf{FVF}'\}$ *is also a* BLUE *of* $\mathbf{X}\boldsymbol{\beta}$ *in the original model* $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$, *and vice versa.*

PROOF. On account of the lemma, a BLUE of $\mathbf{X}\boldsymbol{\beta}$ in the model (1) is expressible as $\mathbf{LFy}$, for some $n \times k$ matrix $\mathbf{L}$, if and only if

$$\mathbf{LFT} = \mathbf{X}(\mathbf{X}'\mathbf{T}^{-}\mathbf{X})^{-}\mathbf{X}'. \tag{7}$$

On the other hand, it is well known that $\mathbf{L}$ satisfying (7) exists if and only if

$$\mathscr{C}\{\mathbf{X}[(\mathbf{X}'\mathbf{T}^{-}\mathbf{X})^{-}]'\mathbf{X}'\} \subset \mathscr{C}(\mathbf{TF}'). \tag{8}$$

But

$$\begin{aligned}\mathscr{C}(\mathbf{X}) &\supset \mathscr{C}\{\mathbf{X}[(\mathbf{X}'\mathbf{T}^{-}\mathbf{X})^{-}]'\mathbf{X}'\}\\ &\supset \mathscr{C}\{\mathbf{X}[(\mathbf{X}'\mathbf{T}^{-}\mathbf{X})^{-}]'\mathbf{X}'(\mathbf{T}^{-})'\mathbf{X}\} = \mathscr{C}(\mathbf{X}),\end{aligned}$$

and so (8) reduces to (6). Further, it is obvious that (6) can be written equivalently as

$$r(\mathbf{X} \vdots \mathbf{TF}') = r(\mathbf{TF}'). \tag{9}$$

But, on account of Theorem 19 in Marsaglia and Styan (1974) and the definition (4) of $\mathbf{T}$, the left-hand side of (9) may be modified as follows:

$$\begin{aligned}r(\mathbf{X} \vdots \mathbf{TF}') &= r(\mathbf{X}) + r[(\mathbf{I} - \mathbf{XX}^{-})\mathbf{TF}']\\ &= r(\mathbf{X}) + r[(\mathbf{I} - \mathbf{XX}^{-})\mathbf{VF}']\\ &= r(\mathbf{X} \vdots \mathbf{VF}').\end{aligned}$$

This completes the proof of part (i) of the theorem.

To prove part (ii) observe first that (6) implies

(10) $$\mathscr{C}[\mathbf{X}'(\mathbf{T}^-)'\mathbf{X}] \subset \mathscr{C}[\mathbf{X}'(\mathbf{T}^-)'\mathbf{T}\mathbf{F}'].$$

According to the definition of **T**, $\mathbf{X}'(\mathbf{T}^-)'\mathbf{T} = \mathbf{X}'$ and $\mathscr{C}[\mathbf{X}'(\mathbf{T}^-)'\mathbf{X}] = \mathscr{C}(\mathbf{X}')$, and therefore (10) reduces to

$$\mathscr{C}(\mathbf{X}') \subset \mathscr{C}(\mathbf{X}'\mathbf{F}').$$

This shows that the functions $\mathbf{X}\boldsymbol{\beta}$, which are obviously estimable in the original model (1), are also estimable in the transformed model (2). In view of the lemma, a statistic **LFy** is a BLUE of $\mathbf{X}\boldsymbol{\beta}$ in the model (2) if and only if

(11) $$\mathbf{LFTF}' = \mathbf{X}[\mathbf{X}'\mathbf{F}'(\mathbf{FTF}')^-\mathbf{FX}]^-\mathbf{X}'\mathbf{F}'.$$

Using now the fact that, on account of (6),

$$\mathbf{X} = \mathbf{TF}'\mathbf{M}$$

for some $k \times p$ matrix **M**, the equation (11) may be written in the form

(12) $$\mathbf{LFTF}' = \mathbf{X}(\mathbf{X}'\mathbf{T}^-\mathbf{X})^-\mathbf{M}'\mathbf{FTF}'.$$

Since **T** is nonnegative definite, (12) is equivalent to (7), thus showing that the sets of BLUEs of $\mathbf{X}\boldsymbol{\beta}$ in models (1) and (2) coincide. □

The conditions of the theorem simplify in the case where

(13) $$\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{V}).$$

The relation (13) is known (Zyskind and Martin, 1969) as a necessary and sufficient condition for a statistic $\mathbf{X}(\mathbf{X}'\mathbf{V}^-\mathbf{X})^-\mathbf{X}'\mathbf{V}^-\mathbf{y}$ to represent a BLUE of $\mathbf{X}\boldsymbol{\beta}$ irrespective of the choice of a g-inverse of **V**. The simplification is an immediate consequence of the fact that if (13) holds then one possible choice of **U** in (4) is $\mathbf{U} = \mathbf{O}$.

COROLLARY. *Let* $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$ *be a Gauss-Markoff model wherein* $\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{V})$, *and let* **F** *be a* $k \times n$ *matrix. Then a* BLUE *of* $\mathbf{X}\boldsymbol{\beta}$ *is obtainable as a linear function of* **Fy** *if and only if*

(14) $$\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{VF}'),$$

or, equivalently,

(15) $$r(\mathbf{X} \,\vdots\, \mathbf{VF}') = r(\mathbf{VF}').$$

If this is the case, then each BLUE *of* $\mathbf{X}\boldsymbol{\beta}$ *in the transformed model* $\{\mathbf{Fy}, \mathbf{FX}\boldsymbol{\beta}, \mathbf{FVF}'\}$ *is also a* BLUE *of* $\mathbf{X}\boldsymbol{\beta}$ *in the original model* $\{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \mathbf{V}\}$, *and vice versa.*

A further simplification of the theorem is possible in a still more special case of model (1), when $\mathbf{V} = \mathbf{I}$. The conditions (14) and (15) occurring in the corollary reduce then to

$$\mathscr{C}(\mathbf{X}) \subset \mathscr{C}(\mathbf{F}')$$

and

$$r(\mathbf{X} \,\vdots\, \mathbf{F}') = r(\mathbf{F}'),$$

respectively.

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