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ASYMPTOTIC NORMALITY OF LINEAR COMBINATIONS OF ORDER STATISTICS WITH A SMOOTH SCORE FUNCTION

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Asymptotic normality of linear combinations of order statistics of the form $T_n = n^{-1} \sum J(i/(n+1))X_{in}$ is investigated along with a slightly trimmed version of T_n. Theorem 5 of Stigler (1974) is extended to show asymptotic normality of T_n for a wide class of score functions. In addition, a proof of Theorem 4 of Stigler (1974) is given.

1. Introduction. Let $X_1, \cdots, X_n$ be independent identically distributed having common distribution function F, and let $X_{1n} \leq \cdots \leq X_{nn}$ be their corresponding order statistics. J will be a real valued function defined on (0, 1); J is usually called a score function. We will consider linear combinations of order statistics of the following form:

$$T_n = n^{-1} \textstyle\sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{in}. \tag{1.1}$$

Bennett (1952), Jung (1955) and Chernoff et al. (1967) have shown that if J is properly chosen, T_n can be made into an asymptotically efficient linear estimate of the location or scale of F. Various workers have proven asymptotic normality of T_n under a variety of conditions; see in particular, [3], [5], [11], [14], [16], and [17]. The most general conditions are those obtained by Shorack (1972) and Stigler (1974).

Among the several results of Stigler (1974), his Theorem 5 shows that if

(1.2) $\lim_{x\to\infty} x^{\alpha}(F(-x) + 1 - F(x)) = 0$ *for some* $\alpha > 0$, *and*

(1.3) *J is bounded and continuous* a.e. *with respect to F^{-1} and $J(u) = 0$ whenever $u \in (0, \delta) \cup (1-\delta, 1)$ for some $1/2 > \delta > 0$, then*

$$n^{1/2}(T_n - ET_n) \to_d N(0, \sigma^2(J, F)) \tag{1.4}$$

and

$$\operatorname{Var}(n^{1/2}T_n) \to \sigma^2(J, F), \tag{1.5}$$

where $\sigma^2(J, F) = \int_0^1 \int_0^1 J(u)J(v)(u \wedge v - uv)\, dF^{-1}(u)\, dF^{-1}(v)$. Here $u \wedge v = \min(u, v)$.

Shorack (1972) considers a more general class of statistics called linear combinations of functions of order statistics. The conditions of his Theorem 1 as it applies to T_n can be formulated as follows. Assume

(1.6) *J is continuous* a.e. *with respect to F^{-1}, and if J is continuous at u then $J_n \to J$ converges uniformly to $J(u)$ in some neighborhood of u, where $J_n(u) \equiv J(([nu]+1)/(n+1))$; and for some $M > 0$, $r > 0$, $s > 0$ and $\delta > 0$*

$$|J(u)| < Mu^{1/r-1/2}(1-u)^{1/s-1/2}, \tag{1.7}$$

and

$$|F^{-1}(u)| < Mu^{-1/r+\delta}(1-u)^{-1/s+\delta}; \tag{1.8}$$

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then

(1.9) $$n^{1/2}(T_n - \mu_n) \to_d N(0, \sigma^2(J, F)) \quad \textit{with} \quad \sigma^2(J, F) < \infty,$$

where μ_n *is an appropriately chosen centering constant.* ($\mu_n = \int_{1/n}^{1-1/n} J_n(u) F^{-1}(u)du$ *will always work.*)

We will give an extension of Theorem 5 of Stigler (1974) that will imply asymptotic normality of T_n under somewhat more relaxed conditions than (1.6)-(1.8). In addition, it will be shown that (1.5) is true for a slightly trimmed version of T_n.

2. The main theorem. Let $F_i^{(n)}(x) = P(X_{in} \le x)$ and $F_{ij}^{(n)}(x, y) = P(X_{in} \le x, X_{jn} \le y)$. For notational simplicity, from now on we will drop the superscripts in $F_i^{(n)}$ and $F_{ij}^{(n)}$. We will begin by recording some facts. Observe that

(2.1) $$F_i(x) = P(S_n(x) \ge i), \qquad \text{where} \quad S_n(x) = \textstyle\sum_{i=1}^n I_{(-\infty,x]}(X_i);$$

and

(2.2) $$1 - F_i(x) = P(S_n(x) < i) = P(S_n^*(x) > n - i), \qquad \text{where} \quad S_n^*(x) = \textstyle\sum_{i=1}^n I_{(x,\infty)}(X_i).$$

Let

(2.3) $$g_{ij}(x, y) = F_{ij}(x, y) - F_i(x)F_j(y).$$ By [4]

(2.4) $$g_{ij}(x, y) \ge 0,$$

also

(2.5) $$\operatorname{cov}(X_{in}, X_{jn}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_{ij}(x, y)dx\, dy,$$ see [8].

Note that

(2.6) $$\textstyle\sum_{i=1}^n \sum_{j=1}^n g_{ij}(x, y) = n(F(x \wedge y) - F(x)F(y)),$$

(2.7) $$g_{ij}(x, y) \le (F_i(x)(1 - F_i(x))^{1/2}(F_j(y)(1 - F_j(y))^{1/2},$$

and

(2.8) $$\left|\textstyle\sum_{i=1}^n \sum_{j=1}^n J\left(\frac{i}{n+1}\right)J\left(\frac{j}{n+1}\right)g_{ij}(x, y)\right|$$

(2.9) $$\le \left(\textstyle\sum_{i=1}^n \sum_{j=1}^n J\left(\frac{i}{n+1}\right)J\left(\frac{j}{n+1}\right)g_{ij}(x, x)\right)^{1/2} \cdot \left(\textstyle\sum_{i=1}^n \sum_{j=1}^n J\left(\frac{i}{n+1}\right)J\left(\frac{j}{n+1}\right)g_{ij}(y, y)\right)^{1/2}.$$

We will first consider the following slightly trimmed linear combination of order statistics:

(2.10) $$S_n = n^{-1}\textstyle\sum_{i=k_1}^{n-k_2+1} J\left(\frac{i}{n+1}\right)X_{in},$$

where $k_1 \ge 1$ and $k_2 \ge 1$ are fixed integers.

In this section, $\mu_n = ES_n$. Define $F^{-1}(u) = \inf\{x : F(x) \ge u\}$ for $0 < u \le 1$.

THEOREM 1. *Let* J *be continuous* a.e. *with respect to* F^{-1}. *If, in addition, for some* $r > 0$, $s > 0$, *and* $M > 0$ J *satisfies* (1.7) *for all* $u \in (0, 1)$;

(2.11) $$\int_0^1 u^{1/r}(1 - u)^{1/s}\, dF^{-1}(u) < \infty;$$

and if

(2.12) k_1 *and* k_2 *are integers chosen so that* $k_1 \geq 2/r$ *and* $k_2 \geq 2/s$, *then*

$$n^{1/2}(S_n - \mu_n) \to_d N(0, \sigma^2(J, F)), \tag{2.13}$$

and

$$\mathrm{Var}(n^{1/2}S_n) \to \sigma^2(J, F) < \infty. \tag{2.14}$$

PROOF. Conditions (1.7) and (2.11) imply that $\sigma^2(J, F) < \infty$. Choose $1/2 > \delta > 0$ and $K > 0$ such that

$$\begin{aligned} |J(u)| &\leq Ku^{1/r-1/2} && \text{for } u \in (0, \delta), \\ \text{and } &\leq K(1-u)^{1/s-1/2} && \text{for } u \in (1-\delta, 1). \end{aligned} \tag{2.15}$$

Now choose ϵ such that $\delta/2 > \epsilon > 0$ and both ϵ and $1 - \epsilon$ are continuity points of F^{-1}. Let n be sufficiently large so that $k_1 < [(n+1)\epsilon]$ and $n - k_2 + 1 > n + 1 - [(n+1)\epsilon]$, where $[x]$ = greatest integer $\leq x$. Now let

$$S_{n\epsilon} = n^{-1} \textstyle\sum_{i=1}^{n} J_\epsilon\left(\frac{i}{n+1}\right) X_{in}, \text{ where } J_\epsilon(u) = J(u) \text{ if } u \in (\epsilon, 1-\epsilon) \text{ and } 0 \text{ otherwise.} \tag{2.16}$$

Theorem 5 of [17] implies that

$$n^{1/2}(S_{n\epsilon} - ES_{n\epsilon}) \to_d N(0, \sigma^2(J_\epsilon, F)), \tag{2.17}$$

and

$$\mathrm{Var}(n^{1/2}S_{n\epsilon}) \to \sigma^2(J_\epsilon, F). \tag{2.18}$$

((2.11) implies that (1.2) is satified for some $\alpha > 0$.)

Hence to prove (2.13) and (2.14) it is sufficient to show

$$\lim_{\epsilon \downarrow 0} \limsup_{n\to\infty} \mathrm{Var}(n^{1/2}(S_n - S_{n\epsilon})) = 0. \tag{2.19}$$

Now, $\mathrm{Var}(n^{1/2}(S_n - S_{n\epsilon})) \leq W_{1n} + W_{2n}$, where

$$W_{1n} = 2\,\mathrm{Var}\left(n^{-1/2} \textstyle\sum_{i=k_1}^{[(n+1)\epsilon]} J\left(\frac{i}{n+1}\right) X_{in}\right), \tag{2.20}$$

and

$$W_{2n} = 2\,\mathrm{Var}\left(n^{-1/2} \textstyle\sum_{i=n+1-[(n+1)\epsilon]}^{n-k_2+1} J\left(\frac{i}{n+1}\right) X_{in}\right). \tag{2.21}$$

We will show that $\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} W_{1n} = 0$. The proof that $\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} W_{2n} = 0$ is merely a change of notation and will not be given.

Observe that by (2.5), (2.8), and (2.9), $W_{1n} \leq 2(\int_{-\infty}^{\infty} M_n(x)^{1/2}\, dx)^2$, where

$$M_n(x) = n^{-1} \textstyle\sum_{i=k_1}^{[(n+1)\epsilon]} \sum_{j=k_1}^{[(n+1)\epsilon]} \left| J\left(\frac{i}{n+1}\right) J\left(\frac{j}{n+1}\right) \right| g_{ij}(x, x). \tag{2.22}$$

Thus it will be enough to show that

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \int_{-\infty}^{\infty} (M_n(x))^{1/2}\, dx = 0. \tag{2.23}$$

For this purpose we need to consider what happens on the following sets:

$$A_{1n} = \{x : n^{-1} \leq F(x) \leq 2\epsilon\}, \qquad A_{2n} = \{x : F(x) < n^{-1}\},$$

$$A_{3n} = \{x: 2\epsilon \le F(x) \le 1 - n^{-1}\}, \qquad A_{4n} = \{x: 1 - n^{-1} < F(x) \le 1\},$$

$$\Delta_{1n}(x) = \left\{i: k_1 \le i \le [(n+1)\epsilon], \frac{i}{n+1} < F(x)\right\},$$

$$\Delta_{2n}(x) = \left\{i: k_1 \le i \le [(n+1)\epsilon], \frac{i}{2(n+1)} \le F(x) \le \frac{i}{n+1}\right\},$$

and

$$\Delta_{3n}(x) = \left\{i: k_1 \le i \le [(n+1)\epsilon], n^{-1} \le F(x) < \frac{i}{2(n+1)}\right\}.$$

Let $m_{ij}(x) = n^{-1} \,|\, J(i/(n+1)J(j/(n+1) \,|\, g_{ij}(x, x)$. By Minkowski's inequality and (2.7),

$$\int_{-\infty}^{\infty} (M_n(x))^{1/2}\, dx \le \sum_{l=1}^{3} \int_{A_{1n}} (\textstyle\sum_{i,j \in \Delta_{ln}(x)} m_{ij}(x))^{1/2}\, dx$$

$$+ \sum_{l=2}^{4} \int_{A_{ln}} \sum_{i=k_1}^{[(n+1)\epsilon]} n^{-1/2} \left| J\left(\frac{i}{n+1}\right) \right| (F_i(x)(1 - F_i(x))^{1/2}\, dx \equiv \sum_{k=1}^{6} I_{kn} \text{ (say)}.$$

To complete the proof it is sufficient to show that $\lim_{\epsilon\downarrow 0} \lim_{n\to\infty} I_{in} = 0$ for each $i = 1, \cdots, 6$. A proof will only be given here for the case when $i = 1$. The interested reader is referred to [9] for proofs for the other cases. $C_1, \cdots, C_3$ will be constants such that the stated inequalities are true independent of n and ϵ.

Case 1. $0 < r < 2$.

PROOF. Observe that for $x \in A_{1n}$ and $i \in \Delta_{1n}(x)$, $\left| J\left(\frac{i}{n+1}\right) \right| \le K(F(x))^{1/r - 1/2}$; hence by (2.6), $I_{1n} \le C_1 V_{\epsilon n}$, where $V_{\epsilon n} = \int_{A_{1n}} (F(x))^{1/r}\, dx$. It is easy to see by (2.11) that $\lim_{\epsilon\downarrow 0} \lim_{n\to\infty} V_{\epsilon n} = 0$.

Case 2. $r \ge 2$.

PROOF. *Let* $B_{1n} = \{i: k_1 \le i \le [(n+1)\epsilon], F(x)/2 \le i/(n+1) \le F(x)\}$, and $B_{2n} = \{i: k_1 \le i \le [(n+1)\epsilon], i/(n+1) \le F(x)/2\}$. By Minkowski's inequality, $I_{1n} \le I_{11n} + I_{21n}$; where $I_{l1n} = \int_{A_{1n}} (\sum_{i,j \in B_{ln}(x)} m_{ij}(x))^{1/2}\, dx$ for $l = 1$ or 2. Proceeding as in Case 1, $I_{11n} \le C_1 V_{\epsilon n}$. Observe that

$$I_{21n} \le \int_{A_{1n}} \sum_{i \in B_{2n}(x)} n^{-1/2} \,|\, J(i/(n+1)) \,|\, (F_i(x)(1 - F_i(x)))^{1/2}\, dx. \tag{2.24}$$

By Chebyshev's inequality, we see that for $i \in B_{2n}(x)$ and $x \in A_{1n}$,

$$1 - F_i(x) \le P(S_n(x) - nF(x) \le i - nF(x)) \le C_2(nF(x))^{-1}(1 - F(x)).$$

Now (2.15) and an integral approximation shows that the expression in (2.24) is $\le C_3 V_{\epsilon n}$.

The following corollary extends Theorem 1 to the statistic T_n.

COROLLARY 1. *Assume the conditions of Theorem* 1. *In addition, assume that* $E\,|X^-|^r < \infty$ *and* $E\,|X^+|^s < \infty$, *then* $n^{1/2}(T_n - \mu_n) \to_d N(0, \sigma^2(J, F))$.

PROOF. It is sufficient to show that if $r < 2$ and $k_1 \ge 2/r$, $n^{-1/2} \sum_{i=1}^{k_1 - 1} J(i/(n+1))X_{in} \to 0$ in probability, and if $s < 2$ and $k_2 \ge 2/s$ that $n^{-1/2} \sum_{i=n-k_2+2}^{n} J(i/(n+1))X_{in} \to 0$ in probability. The proof of these facts is straightforward and left to the reader.

REMARK 1. Several workers, [3], [5], and [14], have considered a more general class of statistics called linear combinations of functions of order satistics. For an extension of Theorem 1 to this class of statistics refer to [9]. Also conditions for the asymptotic normality of linear combinations of order statistics in the non i.i.d. case have been investigated by [12], [15] and [17]. For an extension of Theorem 7 of Stigler (1974) along the lines of Theorem 1 also see [9].

REMARK 2. In [9] asymptotically efficient estimates of the location and scale of various distributions are constructed based on S_n. Theorem 1 shows that the variances of these estimates converge to the asymptotically optimum variance even though the underlying distribution may not have a finite variance. Also refer to [6] for another application of Theorem 1.

3. On Centering. It is often of interest for estimation purposes to know when the centering constant μ_n can be replaced by $\mu(J, F) = \int_0^1 J(u)F^{-1}(u)\,du$ in Theorem 1 and Corollary 1. As in [14] a sufficient condition is that $|J'(u)| < Mu^{-3/2+1/r+\delta}(1-u)^{-3/2+1/s+\delta}$ for some $\delta > 0$ and $M > 0$. Also refer to [6]. Here we will supply a proof for Theorem 4 of Stigler (1974). See [18] and [19].

THEOREM 2. (Theorem 4 of Stigler (1974)). *Assume that J is bounded and satisfies a Hölder condition of order $\alpha > ½$ except perhaps at a finite number of continuity points of F^{-1} and $\int_0^1 (u(1-u))^{1/2}\,dF^{-1}(u) < \infty$, then $n^{1/2}(ET_n - \mu(J, F)) \to 0$ as $n \to \infty$.*

PROOF. First consider the statistic $L_n = \sum_{i=1}^n J_n(i/(n+1))F^{-1}(U_{in})$, where $U_{1n} \le \cdots \le U_{nn}$ are the other statistics of n independent uniform (0, 1) random variables $U_1 \cdots, U_n$ and $J_n(i/(n+1)) = \int_{(i-1)/n}^{i/n} J(u)\,du$ for $i = 1, \cdots, n$. We will first show that $EL_n - \mu(J, F) = o(n^{-1/2})$. By integration by parts $L_n - \mu(J, F) = \int_0^1 (\psi(G_n(u)) - \psi(u))\,dF^{-1}(u)$, where $\psi(u). = \int_u^1 J(v)\,dv$ and G_n is the empirical distribution based on $U_1, \cdots, U_n$. Pick any $\epsilon > 0$ and $\delta > 0$ such that $\int_{1-\delta}^1 (u(1-u))^{1/2}\,dF^{-1}(u) < \epsilon$ and $\int_0^\delta (u(1-u))^{1/2}\,dF^{-1}(u) < \epsilon$. Observe that $|n^{1/2}(EL_n - \mu(J, F))|$ is less than or equal to

$$\sqrt{n}E\int_{1-\delta}^{1} |\Delta_n(u)|\,dF^{-1}(u) + \sqrt{n}E\int_{\delta}^{1-\delta} |\Delta_n(u)|\,dF^{-1}(u) + \sqrt{n}E\int_0^{\delta} |\Delta_n(u)|\,dF^{-1}(u)$$

$$\equiv I_{1n} + I_{2n} + I_{3n},$$

where
$$\Delta_n(u) = \psi(G_n(u)) - \psi(u) + J(u)(G_n(u) - u).$$

It is easily seen by Fubini's theorem and the fact that $E|G_n(u) - u| \le n^{-1/2}(u(1-u))^{1/2}$ that $I_{1n} + I_{3n} \le 4\epsilon \sup_{0\le u\le 1}|J(u)|$. Proceeding in a manner similar to the techniques of Boos (1979), it can be shown that $I_{2n} \to 0$. See [10] for more details.

Now to complete the proof of the theorem. Without loss of generality we will assume that J is left continuous and has jumps $b_1, \cdots, b_k$ at $0 < a_1 < \cdots < a_k < 1$ respectively. Set $J_d(u) = \sum_{i=1}^k b_i I(a_i < u)$ and $J_c(u) = J(u) - J_d(u)$. Note that J_c is continuous and satisfies a Hölder condition of order $\alpha > ½$. Now $n^{1/2}|ET_n - EL_n|$ is less than or equal to

$$n^{1/2}\sum_{i=1}^n \int_{(i-1)/n}^{i/n} \left|J_c(u) - J_c\left(\frac{i}{n+1}\right)\right| du\,E|F^{-1}(U_{in})| \tag{3.1}$$

$$+ 2n^{-1/2}\sum_{i=1}^k |b_i|\,E|X_{[na_i]+1,n}|. \tag{3.2}$$

It is not too difficult to see that (3.1) $\le Mn^{1/2-\alpha}E|X_1|$ for some finite $M > 0$, and application of the theorem of Sarkadi (1974) shows that (3.2) $\to 0$.

REFERENCES

[1] Bennett, C. A. (1952). Asymptotic properties of ideal linear estimators. Ph.D. Thesis, Univ. Michigan.

[2] Boos, D. D. (1979). A differential for L-statistics. *Ann. Statist.* **7** 955–959.

[3] Chernoff, H., Gastwirth, J. L. and Johns, M. V. (1967). Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation. *Ann. Math. Statist.* **38** 52–72.

[4] Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of random variables, with application. *Ann. Math. Statist.* **38** 1466–1474.

[5] Govindarajulu, Z. (1968). Asymptotic normality of linear combinations of functions of order statistics, II. *Proc. Nat. Acad. Sci.* **59** 713–719.

[6] Govindarajulu, Z. and Mason, D. M. (1980). A strong representation for linear combinations of order statistics with application to fixed-width confidence intervals for location and scale parameters. Univ. Kentucky Tech. Report No. 154.

[7] Jung, J. (1955). On linear estimates defined by a continuous weight function. *Ark. Mat.* **3,** 199–209.

[8] Lehmann, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.

[9] Mason, D. M. (1979a). Asymptotic normality of linear combinations of order statistics with a smooth score function. Univ. Kentucky Dept. Statist. Tech. Report No. 149.

[10] Mason, D. M. (1979b). On centering linear combinations of order statistics. University of Kentucky Department of Statistics Technical Report No. 148.

[11] Moore, D. S. (1968). An elementary proof of asymptotic normality of linear functions of order statistics. *Ann. Math. Statist.* **39** 263–265.

[12] Ruymgaart, F. H. and Van Zuijlen, M. C. A. (1977). Asymptotic normality of linear combinations of order statistics in the non i.i.d. case. Proc. Koninklijke Nederlandse Akademic van Wettenschappen, Ser. A, Math. Sci. **80** 432–447.

[13] Sarkadi, K. (1974). On the convergence of the expectation of the sample quantile. Colloq. Math. Soc. János Bolyai. Limit Theorems of Probability Theory 341–346. Keszthely.

[14] Shorack, G. R. (1972). Functions of order statistics. *Ann. Math. Statist.* **43** 412–427.

[15] Shorack, G. R. (1973). Convergence of reduced empirical and quantile processes with applications to functions of order statistics in the non i.i.d. case. *Ann Statist.* **1** 146–152.

[16] Stigler, S. M. (1969). Linear functions of order statistics. *Ann. Math. Statist.* **40** 770–788.

[17] Stigler, S. M. (1974). Linear functions of order statistics with smooth weight functions. *Ann. Statist.* **2** 676–693.

[18] Stigler, S. M. (1979). Correction to linear functions of order statistics with smooth weight functions. *Ann. Statist.* **1** 466.

[19] Wesley, R. A. (1977). Contributions to the theory of robust regression. Stanford Dept. Statist. Tech. Report No. 2.

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THE CAUCHY MEAN VALUE PROPERTY AND LINEAR FUNCTIONS OF ORDER STATISTICS[1]

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An estimator is said to have the Cauchy Mean Value property if the estimate obtained from a pooled sample is always between the estimates obtained from the two original samples. It is shown that the only linear functions of order statistics with this property are the arithmetic mean, percentiles and weighted midranges.

1. Introduction, notation, examples and statement of theorem. An estimator is said to have the Cauchy Mean Value (CMV) property if the estimate obtained from a pooled sample is always between the estimates obtained from the original two samples. In many statistical situations the CMV property seems natural. CMV estimators (including percentiles, midranges and the arithmetic, geometric and harmonic means) have been used in isotonic regression. (See Robertson and Wright (1974).) This note is a proof that the CMV property is restrictive in that the class of linear functions of order statistics contains essentially three CMV estimators.

Unsubscripted capital letters will denote sets of numbers. If A contains n numbers, they will be denoted $A_1, A_2, \cdots, A_n$, where $A_1 \leq \cdots \leq A_n$. A will be called a sample. If A (size m) and B (size n) are samples, $(A \cup B)_j$ will be a member of the pooled sample. A member of the r-fold replication of A will be $(A^r)_i$. If an estimator L_n is a linear function of order statistics $L_n(A) = \sum_{i=1}^n c_{ni}A_i$ for every n.

One immediate consequence of the CMV property is that if $L_m(A) = L_n(B)$, then L_n cannot be CMV unless $L_{n+m}(A \cup B) = L_m(A)$. Therefore $L_{rm}(A^r) = L_m(A)$ for all r and all A of size m. Taking $A = \{1\}$, CMV linear functions of order statistics must satisfy $\sum_{i=1}^n c_{ni} = c_{11}$, and it suffices to characterize those CMV linear functions of order statistics with $c_{11} = 1$.

Example 1. The arithmetic mean is given by $L_n(A) = \sum_{i=1}^n n^{-1}A_i$. $(c_{ni} = n^{-1})$.

Example 2. A weighted percentile can be defined by $L_n(A) = \theta A_{np} + (1 - \theta)A_{np+1}$ if np is integer and $L_n(A) = A_{[np]}$ otherwise, for θ and p in $[0, 1]$ and $[np]$ the least integer in $\{1, \cdots, n\}$ greater than or equal to np. If p is irrational, $c_{n,[np]} = 1$ and $c_{ni} = 0$ for all other i.

Example 3. A weighted midrange is defined by $L_n(A) = \theta A_1 + (1 - \theta)A_n$, where $0 \leq \theta \leq 1$, and $c_{n1} = \theta = 1 - c_{nn}$ and $c_{ni} = 0$ for all other i.

These examples are the only examples.

Theorem 1.1. *If $\{L_n, n \geq 1\}$ is a linear function of order statistics with the Cauchy Mean Value property, then L_n is a constant multiple of the arithmetic mean, of a weighted percentile, or of a weighted midrange.*

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2. Association of CMV estimators and monotone functions.

LEMMA 2.1. *If L_n is a CMV linear combination of order statistics, then there exists a real-valued function g on $[0, 1]$ such that $g(k/n) = \sum_{i=1}^{k} c_{ni}$ for $1 \le k \le n$ and every integer n.*

Such a function will be referred to as a CMV function. CMV functions for the examples are easy to obtain: for Example 1, $g(k/n) = k/n$; for Example 2, $g(s) = \theta I_{\{s \ge p\}} + (1-\theta) I_{\{s>p\}}$; and for Example 3, $g(s) = \theta I_{\{s>0\}} + (1-\theta) I_{\{s=1\}}$. Section 4 proves Theorem 1.1 by showing that these are the only CMV functions.

PROOF OF LEMMA 2.1. yt $A = \{0\}^k \cup \{1\}^{n-k}$. If L_n is CMV, $L_{rn}(A^r) = L_n(A)$, for all r, k and n. Take s rational and n an integer such that ns is also an integer. The equation above implies that $g(s) = 1 - L_n(\{0\}^{ns} \cup \{1\}^{n-ns})$ is well defined for all rational s in that $g(s)$ does not depend on n. The lemma follows by extending g to irrationals in any (measurable) manner.

THEOREM 2.1. *All CMV functions can be taken to be nondecreasing on $[0, 1]$.*

PROOF. Since every monotone function on the rationals in $[0, 1]$ can be extended to a monotone (measurable) function on $[0, 1]$, it suffices to show CMV functions are monotone on the rationals. We assume $c_{mj} \ge 0$, $1 \le j \le m$, $m < N$ and sketch the induction on N. ($N = 2$ follows from $c_{11} = 1$.)

Let A consist of $N-1$ distinct positive numbers. Set $M = N(N-1)$. Since L_n is CMV, $L_M(A^N) = L_{N-1}(A)$. Define

$$\gamma_2 = \sum_{i=(N-1)(k-1)+1}^{N(k-1)} c_{Mi} \qquad \text{and} \qquad \gamma_3 = \sum_{i=N(k-1)+1}^{(N-1)k} c_{Mi}.$$

Lemma 2.1 implies $\gamma_2 + \gamma_3 = c_{N,k}$. If $k = 1$, $\gamma_2 = 0$ trivially. If $k \ge 2$, modify sample A to obtain A' by increasing A_{k-1} to $(A_{k-1} + A_k)/2$. Next set $A^* = A^{N-k+1} \cup (A')^{k-1}$. The CMV property and the inductive hypothesis imply $L_M(A^*) \ge L_M(A^N)$. Direct calculations show $L_M(A^*) = L_M(A^N) + (A_k - A_{k-1})\gamma_2/2$. Since A has no ties, γ_2 is nonnegative. The proof for γ_3 is similar. Therefore $c_{N-k} \ge 0$. The theorem follows from Lemma 2.1.

3. Two equations satisfied by CMV functions.

THEOREM 3.1. *If g is a CMV function, and $0 \le a < a + \delta < b < b + \delta \le 1$, all rational, are such that $g(a) < g(a+\delta)$ and $g(b) < g(b+\delta)$, then (1) and (2) hold.*

$$(1) \qquad [g(a+\delta) - g(a)]\left[g(b+\delta) - g\left(\frac{a+b+2\delta}{2}\right)\right] = [g(b+\delta) - g(a+\delta)]\left[g\left(a + \frac{\delta}{2}\right) - g(a)\right].$$

$$(2) \qquad [g(b) - g(a)]\left[g\left(\frac{a+\delta+b+\delta}{2}\right) - g\left(\frac{a+\delta+b}{2}\right)\right] = [g(b+\delta) - g(b)]\left[g\left(\frac{a+\delta+b)}{2}\right) - g\left(a + \frac{\delta}{2}\right)\right].$$

PROOF. Select N such that Na, Nb and $N\delta$ are all integers, with $N\delta > 1$. Choose W, X, Y and Z positive satisfying $X > Z(g(a+\delta) - g(a))/(g(b+\delta) - g(a+\delta)) > W(g(b) - g(a))/(g(b+\delta) - g(b)) > -W > -Z > -Y$. Construct samples A and B by taking $A_i = B_i = -Y$ for $i \le Na$ and $A_i = B_i = X$, $i \ge N(b+\delta) + 1$. Complete A by taking $A_i = -Z$, Na

$+ 1 \leq i \leq N(a + \delta)$ and $A_i = Z(g(a + \delta) - g(a))/(g(b + \delta) - g(a + \delta))$, $N(a + \delta) + 1 \leq i \leq N(b + \delta)$. Finish B with $B_i = -W$, $Na + 1 \leq i \leq Nb$ and $B_i = W(g(b) - g(a))/(g(b + \delta) - g(b))$, $Nb + 1 \leq i \leq N(b + \delta)$. $L_{2N}(A \cup B)$ can be calculated from the definitions of A and B, yielding an expression linear in Z and W as constrained above. Since A and B satisfy $L_n(A) = L_n(B)$, if L_N is CMV, $L_{2N}(A \cup B)$ must also equal $L_N(A)$ for all Z and W. Equations (1) and (2) are obtained by setting the coefficients of Z and W to zero.

4. Proof of Theorem 1.1. Theorem 1.1 follows from two propositions. Proposition 4.1 states the CMV functions increase either at one point (Example 2), at 0 and at 1 (Example 3) or at all points in [0, 1]. Proposition 4.2 states that the only strictly increasing CMV function is the identity function (Example 1).

We extend the terminology of Boas (1972, page 121) to call x a point of constancy of a nondecreasing function g if g does not increase at x and to refer to an interval on which g does not increase as an interval of constancy.

PROPOSITION 4.1. *If g is a* CMV *function and g is constant at some point, then g increases at* 0 *and at* 1 *or at only one point.*

PROOF. Since nondecreasing functions cannot have isolated points of constancy, if g is a CMV function constant at a point, g is constant on an interval. Assume this interval is to the right of a point at which g increases (or use $g'(s) = 1 - g(1 - s)$). From Lemma 4.1, the point at which g increases is isolated. If g increases at any other points, it must increase at a pair of points surrounding an interval of constancy. By Lemma 4.2, g increases only at 0 and at 1. □

LEMMA 4.1. *If g generates L_n, and there exist t, u, v such that $0 \leq t < u < v \leq 1$ and g is strictly increasing on $[t, u]$ and g is constant on $(u, v]$, then L_n is not* CMV.

PROOF. The proof of this lemma consists of the construction of a counter-example. Choose n sufficiently large that there exists k such that $t \leq (k - 2)/n < (2k - 1)/2n < u \leq k/n < (k + 1)/n \leq v$. These inequalities and the assumptions of the lemma imply that $c_{n,k-1}$, $c_{n,k}$, $c_{2n,2k-3}$, $c_{2n,2k-2}$, $c_{2n,2k-1}$, and $c_{2n,2k}$ are positive and that $c_{n,k+1}$ and $c_{2n,2k+1}$ are zero. Let A be any sample of size n, with no ties. Modify sample A to create two new samples. For sample B, replace A_{k-1} by $A_{k-1} - x/c_{n,k-1}$ and A_k by $A_k + x/c_{n,k}$. For sample C, replace A_{k+1} by $A_{k+1} - x$, where $(A_{k+1} - A_k)c_{n,k}/(1 + c_{n,k}) < x < c_{n,k}(A_{k+1} - A_k)$ and $c_{n,k-1}(A_{k-1} - A_{k-2}) > x$. A_{k-2} can be chosen to ensure the last inequality. These constraints define the ordering of the sample $B \cup C$. It follows that $L_{2n}(B \cup C) = L_{2n}(A^2) - x[c_{2n,2k-3}/c_{n,k-1} + c_{2n,2k}] + (A_{k+1} - A_k)c_{2n,2k}$. As x ranges over an open interval, $L_{2n}(B \cup C)$ varies. Since samples B and C were chosen so that $L_n(B) = L_n(C) = L_n(A)$, L_n cannot be CMV.

LEMMA 4.2. *If g is* CMV *and there exist a, b, and δ such that $0 \leq a < a + \delta < b < b + \delta \leq 1$, $\delta < (b - a)/2$ and*

$$g(a) < g(a + \delta) = g(b) < g(b + \delta), \tag{3}$$

then g is constant on (0, 1) *and discontinuous at* 0 *and at* 1.

PROOF. First we show that (3) holds for $a = 0$. The constraint on δ implies that $(a + b + 2\delta)/2$ is in the interval of constancy. Since g is CMV, equation (1) of Theorem 3.1 applies. Substituting $g(a + \delta) = g(b)$ in (1) and reducing gives $g(a + \delta/2) = g(b)$.

If a is positive, set $a' = \max(a - \delta/2, 0)$. By the monotonicity of g, $g(a') < g(a' + \delta) = g(b)$. Because $g(a + \delta/2) = g(b)$ holds for $a = a'$, $g(b) = g(a' + \delta/2) = g(\max(a, \delta/2))$. Since $g(b) > g(a)$, it follows that $a < \delta/2$, $a' = 0$, and $g(0) < g(\delta) = g(b) < g(b + \delta)$.

Next we show that the interval of constancy extends leftwards to the origin. Define a sequence b_n recursively by $b_o = b$ and $b_n = b_{n-1}$ if $g(b_{n-1}) < g(b_{n-1} + \delta/2^n)$ and $b_n = b_{n-1} + \delta/2^n$ otherwise. Inductively, $g(\delta/2^{n+1}) = g(b_n) = g(b) < g(b_n + \delta/2^n)$, for every n, and hence $g(0+) = g(b)$ or g is constant on $(0, b]$. Since $g(0) < g(\delta/2^n) = g(0+)$, g is discontinuous at zero. Similarly, g is constant on $[b, 1)$ and discontinuous at one. □

PROPOSITION 4.2. *If g is a strictly increasing* CMV *function, then* $g(x) = x$.

PROOF. If g is a strictly increasing CMV function, Lemma 2.1 implies that all $c_{ni} > 0$. Set $n = 2^N > 8$. Substituting $a = j/n$, $b = (j+2)/n$ and $\delta = 1/n$ in (1) and expressing the resulting equation in terms of c_{ni} shows that $c_{2n,2j+1} = (c_{n,j+1} + c_{n,j+3})/(c_{n,j+2} + c_{n,j+3})$, $1 \le 2j + 1 \le 2n - 5$. Using (2) in place of (1) implies $[c_{n,j+1} + c_{n,j+2}]c_{2n,2j+4} = c_{n,j+3}[c_{2n,2j+3} + c_{2n,2j+2}]$, $0 \le j \le n - 3$. If $j \le n - 4$, eliminating the $c_{2n,i}$ terms implies $c_{n,j+4} - c_{n,j+2} = -c_{n,j+1}(c_{n,j+4} - c_{n,j+2})/(c_{n,j+2} + c_{n,j+3})$. Since $c_{n,j+1}/(c_{n,j+2} + c_{n,j+3})$ is finite and positive, $c_{n,j+2} = c_{n,j+4}$, or $c_{n,1} = c_{n,3} = \cdots = c_{n,n-1}$ and $c_{n,2} = c_{n,4} = \cdots = c_{n,n}$. From Lemma 2.1, $c_{n/2,i} = c_{n,1} + c_{n,2} = c_{n/2,1}$ for every i, or $c_{n/2,i} = 2/n$ and $g(k/2^N) = k/2^N$, $0 \le k \le 2^N$. Since g is monotone, $g(x) = x$, $0 \le x \le 1$. □

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REFERENCES

BOAS, R. P., JR. (1972). *A Primer of Real Functions*. Quinn & Boden.

ROBERTSON, T. AND WRIGHT, F. T. (1974). A norm-reducing property for isotonized Cauchy mean value functions, *Ann. Statist.* **2** 1302–1307.

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