The Annals of Statistics
1981, Vol. 9, No. 4, 870–878

ESTIMATING A BOUNDED NORMAL MEAN

BY GEORGE CASELLA[1] AND WILLIAM E. STRAWDERMAN[2]

Cornell University and Rutgers University

The problem of estimating a normal mean has received much attention in recent years. If one assumes, however, that the true mean lies in a bounded interval, the problem changes drastically. In this paper we show that if the interval is small (approximately two standard deviations wide) then the Bayes rule against a two point prior is the unique minimax estimator under squared error loss. For somewhat wider intervals we also derive sufficient conditions for minimaxity of the Bayes rule against a three point prior.

1. Introduction. The problem considered is that of estimating the mean of a normal distribution under the additional assumption that the mean lies in some bounded interval. While the assumption of boundedness of the true mean can be useful in practice, it introduces some surprising difficulties in theory. It has long been known that the sample mean is an inadmissible estimator under quadratic loss. Admissible (proper Bayes) estimators are plentiful, but minimax estimators have not yet been explicitly found.

Ghosh (1964) proved the existence of a unique minimax estimator for a more general problem than considered here. Applying his results to this problem he constructed a sequence of estimators which converge to the minimax estimator. The estimators are rather difficult to evaluate, however, and the limit cannot be explicitly found. Ghosh's results point out that a least favorable prior would put mass on a finite number of points. In this paper we show that if the interval containing the parameter is small enough (approximately two standard deviations wide), then a two point prior is least favorable and the associated Bayes estimator is minimax.

For somewhat wider intervals we derive sufficient conditions for minimaxity of the Bayes estimator against a three point prior. The conditions needed to prove minimaxity seem to be stronger than necessary, however, for we also present numerical evidence that the "three point" Bayes rule is minimax for an interval wider than those covered by our theorem.

In Section 2 we develop the necessary notation. Section 3 contains the results for the two point prior and Section 4 contains the results for the three point prior.

2. Notation. Let x be an observation from a normal population with mean θ and variance 1, i.e., $x \sim n(\theta, 1)$. We assume that $\theta \in [-m, m]$ for some fixed $m > 0$. No generality is lost in assuming the interval to be symmetric about zero.

For any estimator $\delta(x)$ we define the loss incurred by estimating θ with $\delta(x)$ to be

$$L(\theta, \delta(x)) = (\theta - \delta(x))^2.$$

The risk is defined as

$$R(\theta, \delta) = E_\theta(\theta - \delta(x))^2,$$

and the Bayes risk with respect to the prior distribution $\tau(\theta)$ is

$$r(\tau, \delta) = E_\tau E_\theta(\theta - \delta(x))^2.$$

Received July, 1979; revised May, 1980.

[1] Research supported by National Science Foundation grant #MCS79-05771.

[2] Research supported by National Science Foundation grant #MCS75-21343-A02.

AMS 1970 *subject classifications.* Primary 62C99; secondary 62F10.

Key words and phrases. Minimax, normal mean, least favorable prior.

Let τ_m° denote the two point prior putting equal mass on $\pm m$, and let $\delta_m^{\circ}(x)$ denote the Bayes rule against τ_m°. It is straightforward to check that

$$\delta_m^{\circ}(x) = m \tanh(mx) \tag{2.1}$$

Let τ_m^{α} denote the three point prior putting mass α at 0 and $(1/2)(1-\alpha)$ at $\pm m$. The Bayes rule against τ_m^{α} is given by

$$\delta_m^{\alpha}(x) = \frac{(1-\alpha)m \tanh(mx)}{1-\alpha+\alpha \exp\left(\frac{1}{2}m^2\right)\operatorname{sech}(mx)}. \tag{2.2}$$

3. Minimaxity of the Bayes rule against a two point prior. The main result of this section is given in Theorem 3.1, which gives conditions under which $\delta_m^{\circ}(x)$ is a minimax estimator of θ. Before the theorem is proved we must establish two preliminary results.

LEMMA 3.1. *Let $x \sim n(\theta, 1)$, $\theta \in [-m, m]$. Let $\delta_m^{\circ}(x)$ be the Bayes estimator* (2.1) *against τ_m°. Then*

$$\max_{\theta\in[-m,,m]} R(\theta, \delta_m^{\circ}) = \max\{R(0, \delta_m^{\circ}), R(m, \delta_m^{\circ})\}.$$

PROOF. For notational convenience let $\delta_m^{\circ}(x) = \delta(x) = m \tanh(mx)$. Note that $\delta'(x) = m^2 - \delta^2(x)$ and $\delta''(x) = -2\delta'(x)\delta(x)$. The derivative of the risk is

$$(d/d\theta)R(\theta, \delta) = 2E_\theta((\theta - \delta(x))(1-\delta'(x))). \tag{3.1}$$

Adding $\pm x$, integrating by parts, and collecting terms yields

$$(d/d\theta)R(\theta, \delta) = E_\theta((x-\delta(x)) - \delta'(x)(x+\delta(x))). \tag{3.2}$$

We will show that the expectation in (3.2), as a function of θ, can have at most three sign changes. This will follow from Theorem 3 and Corollary 2 of Karlin (1957) if we can show that the integrand has at most three sign changes as a function of x.

At $x = 0$ the integrand is zero and, for $x > 0$, it is zero only if

$$1 - \frac{\delta(x)}{x} = \delta'(x)\left(1 + \frac{\delta(x)}{x}\right). \tag{3.3}$$

For $x > 0$ it is straightforward to check that $\delta(x)/x$ and $\delta'(x)$ are both positive and decreasing. Thus $1 - \delta(x)/x$ is increasing and $\delta(x)(1 + \delta(x)/x)$ is decreasing and equation (3.3) can have at most one solution. Since the integrand in (3.1) is an odd function a similar argument shows that equation (3.3) has at most one solution for $x < 0$. Also, since $\delta(x)$ and $\delta'(x)$ are bounded, the integrand in (3.2) is positive for large positive x, hence, as x varies from $-\infty$ to $+\infty$ the order of the sign changes is $-+-+$.

The results of Karlin state that the expectation has at most three sign changes, and the sign changes are in the same order. It is easy to check that

$$E_0((x-\delta(x)) - \delta'(x)(x+\delta(x))) = 0$$

and

$$E_\theta((x-\delta(x)) - \delta'(x)(x+\delta(x))) = -E_{-\theta}((x-\delta(x)) - \delta'(x)(x+\delta(x))).$$

Therefore, if $R(\theta, \delta)$ has an extremum for $\theta > 0$ that extremum must be a minimum, since the sign change in the derivative is from negative to positive. Since $R(\theta, \delta)$ is an even function the same is true for $\theta < 0$. Thus for $\theta \in [-m, m]$, $R(\theta, \delta)$ has its maximum either at $\theta = 0$ or $\theta = m$. □

In order for $\delta_m^\circ(x)$ to be a minimax estimator of θ, its maximum risk must equal its Bayes risk. The next lemma establishes conditions on m that insure that the maximum risk is $R(m, \delta_m^\circ)$.

LEMMA 3.2. *The function*

$$f(m) = R(0, \delta_m^\circ) - R(m, \delta_m^\circ)$$

changes sign only once as m varies from 0 to ∞. *This sign change is from negative to positive and hence there exists a unique value* m_0 *such that* $f(m) \le 0$ *for all* $m \le m_0$.

REMARK. An IBM 360/165 computer was used to evaluate m_0. The value $m_0 = 1.056742$ gives a value of $f(m_0)$ which is zero to six decimal places.

PROOF. To show that $f(m)$ has only one sign change from negative to positive it is sufficient to show that

$$g(m) = E_0 \tanh^2(mx) - E_m(1 - \tanh(mx))^2$$

is increasing, since $f(m) = m^2 g(m)$. Differentiating $g(m)$ with respect to m, we have after an integration by parts

$$(d/dm)g(m) = 2E_0 x \tanh(mx)\operatorname{sech}^2(mx) + 2E_m(x + m)(1 - \tanh(mx))\operatorname{sech}^2(mx). \tag{3.4}$$

The first expectation is positive since the integrand is positive. To see that the second expectation is also positive consider the function

$$h(\theta) = E_\theta(x + m)(1 - \tanh(mx))\operatorname{sech}^2(mx).$$

The integrand has only one sign change, from negative to positive. Thus, proceeding as in Lemma 3.1, if we can show $h(0) > 0$ it follows that $h(m) > 0$ and $g(m)$ is increasing. Now, by symmetry we have

$$h(0) = E_0(m \operatorname{sech}^2(mx) - x \tanh(mx)\operatorname{sech}^2(mx)).$$

Integrating the second term by parts and collecting terms yields

$$h(0) = E_0(m \operatorname{sech}^2(mx)(1 + 2\tanh^2(mx) - \operatorname{sech}^2(mx)) > 0,$$

since the integrand is always positive. Since $h(0) > 0$ the lemma is proved. □

We are now ready to prove the main theorem of this section, asserting the conditions for minimaxity of $\delta_m^\circ(x)$.

THEOREM 3.1. *If* $x \sim n(\theta, 1)$ *and* $\theta \in [-m, m]$, $0 \le m \le m_0$ *then* $\delta_m^\circ(x) = m \tanh(mx)$ *is minimax against squared error loss and* τ_m° *is a least favorable distribution.*

PROOF. From Lemma 3.1 and Lemma 3.2

$$\max_{\theta \in [-m,m]} R(\theta, \delta_m^\circ) = R(m, \delta_m^\circ) = r(\tau_m^\circ, \delta_m^\circ). \tag{3.5}$$

Theorem 1, page 90, of Ferguson (1967) can now be used to finish the proof. For completeness we note that (3.5) implies

$$\inf_\delta \sup_\tau r(\tau, \delta) \le r(\tau_m^\circ, \delta_m^\circ) \le \sup_\tau \inf_\delta r(\tau, \delta) \tag{3.6}$$

and thus τ_m° is least favorable. The inequalities in (3.6) are, in fact, equalities so $r(\tau_m^\circ, \delta_m^\circ)$ is the minimax value of the risk and $\delta_m^\circ(x)$ is the minimax estimator. □

If $x \sim n(\theta, \sigma^2)$, σ^2 known, then the restriction on m becomes $0 \le m \le m_0\,\sigma$ and $\delta_m^\circ(x) = m \tanh(mx/\sigma)$ is a minimax estimator of θ. Thus, if an experimenter can assert the value of θ to within one standard deviation. $\delta_m^\circ(x)$ can provide a substantial decrease in the risk over that of the usual estimator $\delta(x) = x$. Table 1 gives values of the minimax risk for $m = .1, 1.05(.1)$ and Figure 1 shows the graphs of the risk function $R(\theta, \delta_m^\circ)$ for $m = .25, 1(.25)$. We also want to note that at $\theta = 1$ all risk functions were below the value of 1, the risk of x.

An obvious competitor for $\delta_m^\circ(x)$ is the maximum likelihood estimator (MLE), $u_m(x)$, where

$$u_m(x) = \begin{cases} m & \text{if } x \geq m \\ x & \text{if } |x| < m \\ -m & \text{if } x \leq -m. \end{cases}$$

For $|\theta| < m$ it is clear that $u_m(x)$ dominates x. The following theorem shows when $\delta_m^\circ(x)$ dominates $u_m(x)$.

THEOREM 3.2. *If $m < 1$ then $R(\theta, \delta_m^\circ) < R(\theta, u_m)$ for all $\theta \in [-m, m]$.*

PROOF. The proof uses a sign change argument similar to that of Lemma 3.1 and will only be sketched. Let $\delta = \delta_m^\circ$ and $u = u_m$. Using an integration by parts the difference in risks $\Delta(\delta, u, \theta) = R(\theta, \delta) - R(\theta, u)$ can be expressed as

$$\Delta(\delta, u, \theta) = E_\theta[(\delta(x) - u(x))(\delta(x) + u(x) - 2x) - 2(I_{(-m,m)}(x) - \delta'(x))].$$

If $m < 1$, the function inside the expectation has only one sign change, from negative to positive, as x varies from 0 to ∞. Since δ is Bayes with respect to the two point prior with mass at $\pm m$, it follows that $\Delta(\delta, u, m)$ is negative and hence $\Delta(\delta, \mathrm{u}, \theta)$ is negative for all $|\theta| < m$. □

Figure 2 displays the risk functions of δ_m° and u_m for m equal to .5 and .1. For $|\theta| < m \leq 1$ there is a definite advantage in using δ_m° rather than u_m. In fact, for values of m close to 1, numerical evidence suggests that the maximum risk of δ_m° is less than the minimum risk of u_m (for $|\theta| < m$). For large θ, u_m dominates δ_m°, but only by a small amount since, as θ increases, $\delta_m^\circ \simeq m = u_m$.

4. The Bayes rule against a three point prior. In this section we demonstrate that m_0 is the largest value of m for which $\delta_m^\circ(x)$ is minimax, and investigate the minimaxity of the Bayes rule against a three point prior. We start with the following lemma, which displays a prior that will later be seen to be more unfavorable than τ_m°.

LEMMA 4.1. *If $m > m_0$ there exists a unique three point prior $\tau_m^{\alpha^*}$ putting mass α^* on zero and (1/2) $(1 - \alpha^*)$ on $\pm m$ such that*

$$R(0, \delta_m^{\alpha^*}) = R(m, \delta_m^{\alpha^*}), \tag{4.1}$$

where $\delta_m^{\alpha^*}(x)$ is the Bayes rule (2.2) against $\tau_m^{\alpha^*}$ under squared error loss.

PROOF. Let τ_m^α be the three point prior which puts mass α on zero and (1/2) $(1 - \alpha)$ on $\pm m$. The Bayes rule is given by (2.2). Using the same technique as in the proof of Lemma

TABLE 1.
Selected values of the minimax risk, $m \leq 1.05$

m	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.0	1.05
Risk	.010	.038	.083	.138	.199	.262	.321	.374	.417	.450	.461

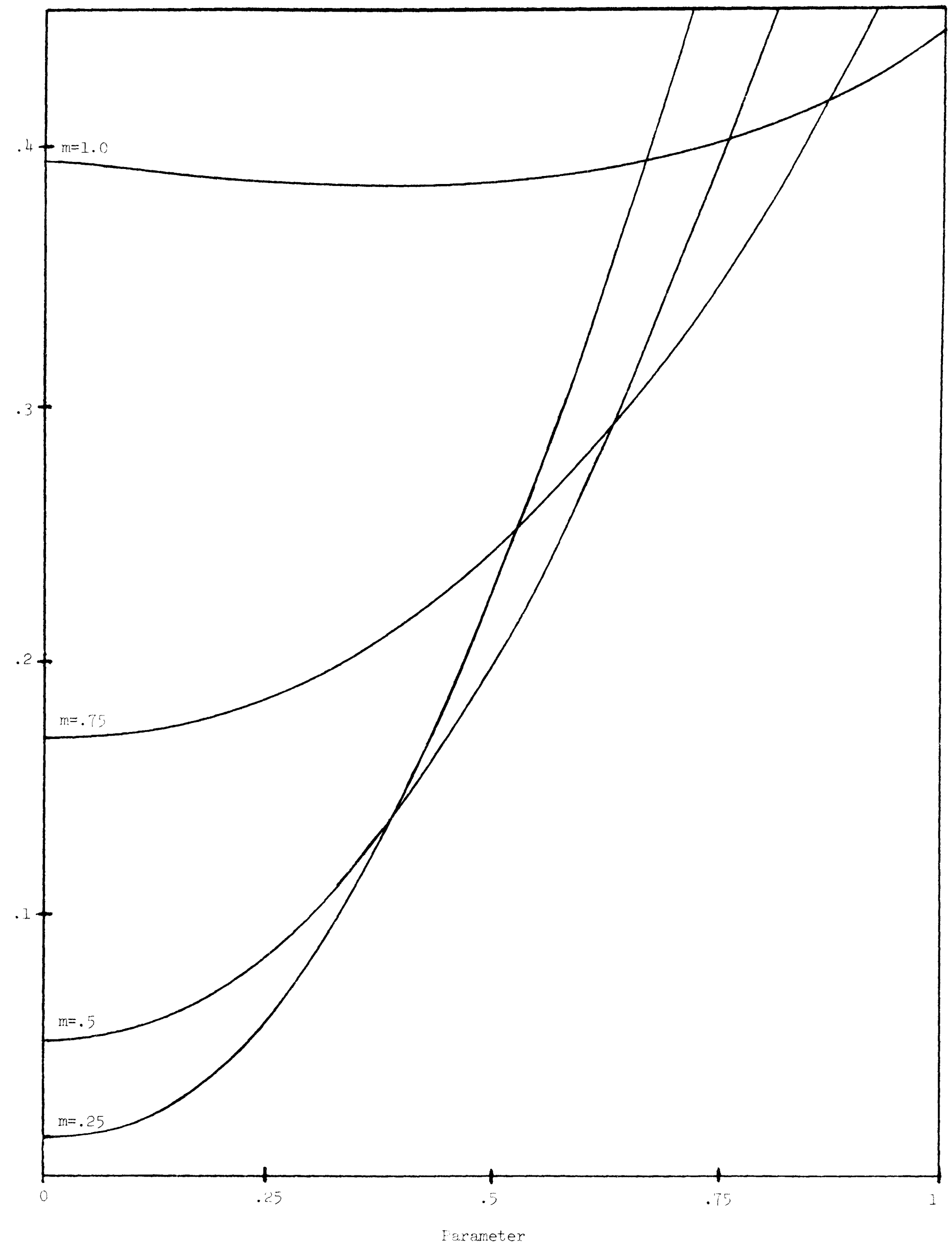

FIG. 1. *Risk functions of* $\delta_m^{\circ}(x)$.

3.2 it is easy to verify that

$$\frac{d}{d\alpha}(R(0, \delta_m^{\alpha}) - R(m, \delta_m^{\alpha})) < 0,$$

so that the difference in risks is decreasing. The lemma will be proved if we can show that equation (4.1) has a solution for $0 \le \alpha \le 1$.

At $\alpha = 0$ it follows from Lemma 3.2 that $R(0, \delta_m^{\circ}) > R(m, \delta_m^{\circ})$ and at $\alpha = 1$ we have $R(0, \delta_m^1) < R(m, \delta_m^1)$ (since $\delta_m^1 \equiv 0$). Thus $(0, \delta_m^{\alpha})$ changes from positive to negative as α

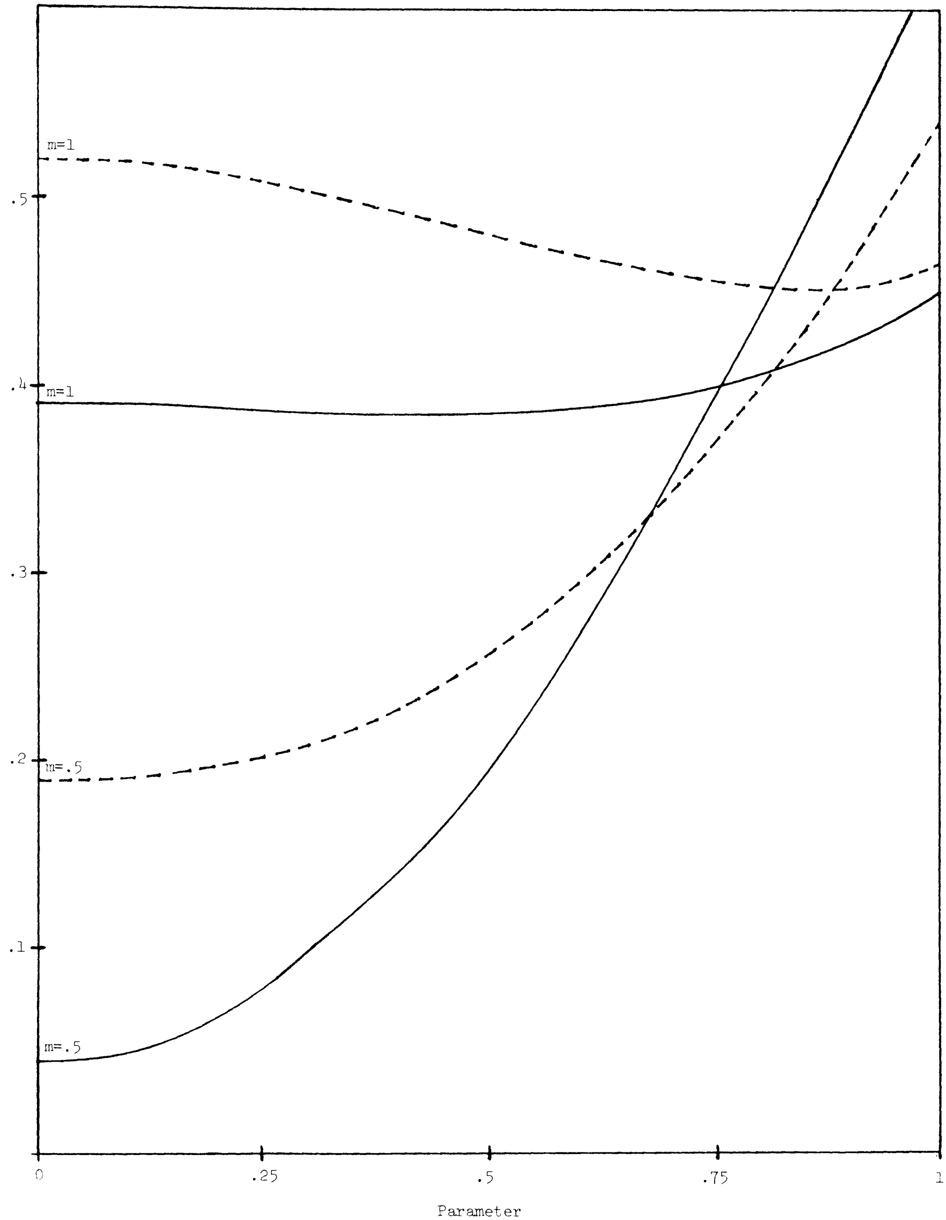

FIG. 2. *Comparison of risk functions of the* MLE (*dashed lines*) *and* δ_m° (x) (*solid lines*).

varies from 0 to 1. Since the difference is continuous and decreasing, there is a unique α which satisfies equation (4.1). □

We can now use Lemma 4.1 to show that m_0 is the largest value for which the two point prior τ_m° is least favorable.

THEOREM 4.1. *If $m > m_0$ then τ_m° is not least favorable.*

PROOF. Let α^* be the solution to $R(0, \delta_m^{\alpha^*}) = R(m, \delta_m^{\alpha^*})$. Since δ_m° is Bayes against τ_m°

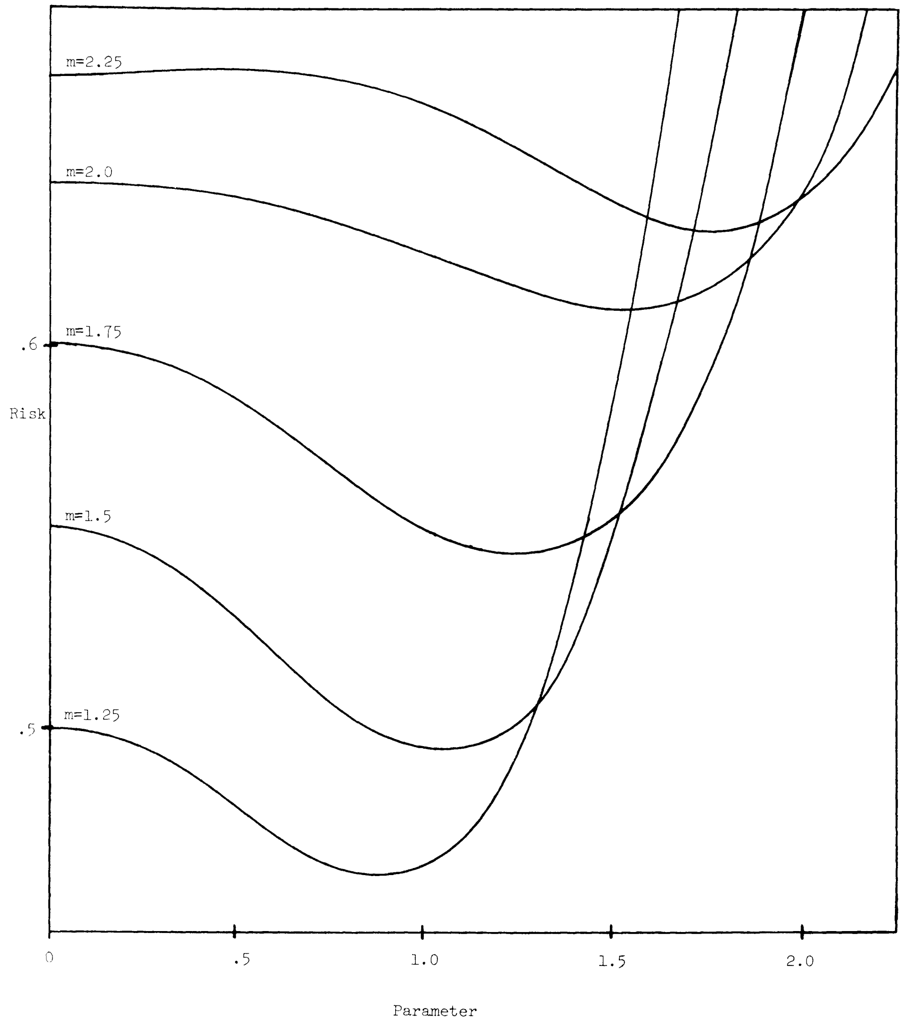

FIG. 3. *Risk functions of* $\delta_m^{\alpha^*}(x)$ *for* $m = 1.25, 1.5, 1.75, 2.0$, *and the risk function of the Bayes estimator against a four point prior for* $m = 2.25$.

we have $R(m, \delta_m^{\circ}) < R(m, \delta_m^{\alpha^*})$ and since $R(m, \delta_m^{\alpha^*}) = r(\tau_m^{\alpha^*}, \delta_m^{\alpha^*})$, we have $r(\tau_m^{\circ}, \delta_m^{\circ}) < r(\tau_m^{\alpha^*}, \delta_m^{\alpha^*})$ so τ_m° is not least favorable. □

As the interval containing θ widens the least favorable prior will put mass on an increasing number of points. Thus, it seem reasonable to inquire at this point if the three point prior $\tau_m^{\alpha^*}$ is least favorable for some values of m. The associated Bayes estimator $\delta_m^{\alpha^*}(x)$ has the property that its risk is constant on the points of the prior. Hence, if it can be shown that the maximum risk is attained at these points it will follow that $\delta_m^{\alpha^*}(x)$ is minimax.

It might be conjectured that for $m_0 < m \leq 2$ (say) that $\delta_m^{\alpha^*}(x)$ is minimax. Indeed, from numerical evidence (see Figure 3) this seems to be the case. Unfortunately, as might be expected from the complex form of $\delta_m^{\alpha^*}(x)$, this conjecture is quite difficult to prove. However, Theorem 4.2 asserts the minimaxity of $\delta_m^{\alpha^*}(x)$ for m in a subset of the interval $[m_0, 2]$, approximately for $1.4 \leq m \leq 1.6$. We again need a preliminary lemma.

TABLE 2.
Selected values of the Bayes risk of $\delta_m^{\alpha^*}$, $1.1 \le m \le 2.0$.

m	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Risk	.472	.492	.513	.535	.556	.577	.596	.615	.631	.645
α^*	.050	.143	.214	.269	.318	.345	.371	.392	.408	.420

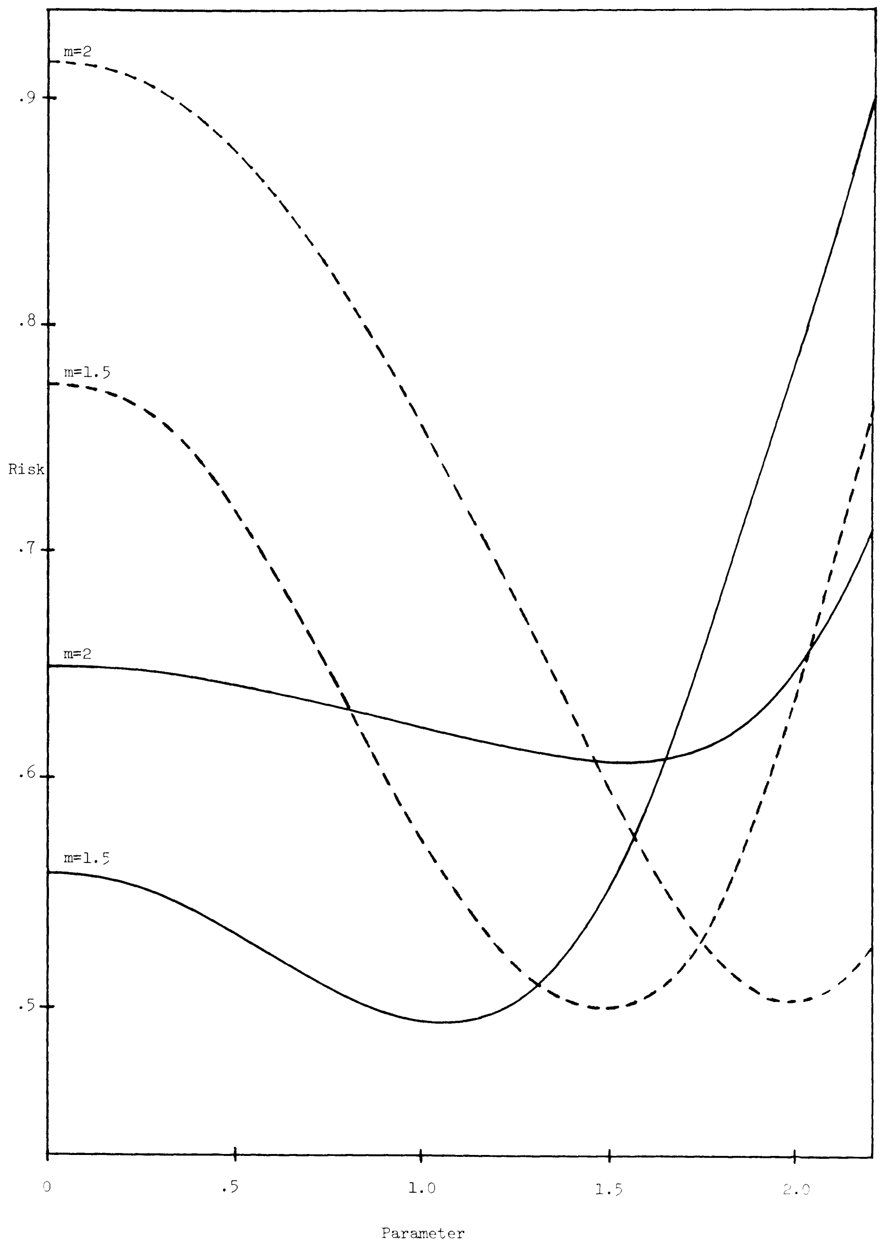

FIG. 4. *Comparison of the risk functions of the* MLE (*dashed lines*) *and* $\delta_m^{\alpha^*}(x)$ (*solid lines*).

LEMMA 4.2. *Let $\delta_m^\alpha(x)$ be the Bayes estimator against τ_m^α. If α and m satisfy*

$$(m^2 - 1)(m^2 - 1 + \exp(m^2/2))^{-1} \le \alpha \le 2(2 + \exp(m^2/2))^{-1} \tag{4.2}$$

then $\max_{-m<\theta<m} R(\theta, \delta_m^\alpha) = \max\{R(0, \delta_m^\alpha), R(m, \delta_m^\alpha)\}$.

PROOF. The proof is very similar to that of Lemma 3.1 and is omitted. We remark that expression (4.2) insures that, for $x > 0$, $(d/dx)(\delta_m^\alpha(x)) < 1$ and $(d^2/dx^2)(\delta_m^\alpha(x)) < 0$. These conditions are needed to apply the sign change arguments of Lemma 3.1. □

THEOREM 4.2. *Let $\tau_m^{\alpha^*}$ be the three point prior described in Lemma 4.1, and let $\delta_m^{\alpha^*}(x)$ be the associated Bayes rule under squared error loss. If α^* satisfies (4.2) then $\delta_m^{\alpha^*}(x)$ is minimax and $\tau_m^{\alpha^*}$ is least favorable.*

PROOF. Similar to that of Theorem 3.1. □

For $m = 1.1, 2.0(.1)$ values of α^* have been calculated and are presented in Table 2. The only values which satisfy condition (4.2) are those for which $1.4 \le m \le 1.6$. This is disappointing but, due to the difficult form of $\delta_m^{\alpha^*}(x)$ we don't expect much improvement over these bounds. Figure 3 shows the risk function of $\delta_m^{\alpha^*}(x)$ for $m = 1.25, 2.0$ (.25). In each case the function indicates that the estimator is minimax, hence our conjecture that $\delta_m^{\alpha^*}(x)$ is minimax for $m_0 \le m \le 2$. Numerical evidence suggests that $\delta_m^{\alpha^*}(x)$ will not be minimax for values of m much greater than 2. Indeed, we include in Figure 3 the risk function, for $m = 2.25$, of the Bayes estimator against the *four* point prior with mass .255 on ±.5633 and mass .245 on ±2.25. Calculations suggest that this estimator is minimax with a maximum risk of .675 attained at the points of the prior.

Figure 4 compares the risk of $\delta_m^{\alpha^*}(x)$ with the MLE. Since the risk functions cross inside the interval $[0, m]$, there is no hope for a result analagous to Theorem 3.2. However, the graphs indicate that $\delta_m^{\alpha^*}(x)$ is preferable to the MLE since it can provide a large risk improvement for small values of θ, and not be significantly worse for large values of θ.

REFERENCES

FERGUSON, THOMAS S. (1967). *Mathematical Statistics: A Decision Theoretic Approach*. Academic, New York.

GHOSH, M. N. (1964). Uniform approximation of minimax point estimates. *Ann. Math. Statist.* **35** 1031–1047.

KARLIN, SAMUEL, (1957). Pólya type distributions, II. *Ann. Math. Statist.* **28** 281–309.

BIOMETRICS UNIT
337 WARREN HALL
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

DEPARTMENT OF STATISTICS
HILL CENTER
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903