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A DENSITY-QUANTILE FUNCTION APPROACH TO OPTIMAL SPACING SELECTION[1]

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In this paper design techniques for continuous parameter time series regression analysis are employed to develop a general approach to optimal spacing selection for the linear estimation of location and scale parameters by sample quantiles from uncensored or censored samples. The spacings derived from this approach are asymptotically optimal in the sense that they result in near optimal asymptotic relative efficiencies for large values of k, the number of spacing elements. A comparison with the optimum efficiencies for several distribution types indicates that the asymptotically optimum spacings perform well for $k \geq 7$. The regression framework is also utilized to develop sufficient conditions for optimal spacing unicity and to obtain asymptotically optimal spacings for quantile estimation.

1. Introduction. In the location and scale parameter model it is assumed that the distribution function (df) for the elements of a random sample, $X_1, \cdots, X_n$, has the form

$$F(x) = F_0\left(\frac{x-\mu}{\sigma}\right) \tag{1.1}$$

where F_0 is a known distributional form and μ and σ are respectively location and scale parameters. Usually, μ and/or σ require estimation from the data. This paper explores the properties of a particular variety of location and scale parameter estimators, the asymptotically best linear unbiased estimators based on sample quantiles.

It will be assumed that F is absolutely continuous with probability density function (pdf) f. The *quantile function* is defined to be $Q(u) = F^{-1}(u)$, $0 \leq u \leq 1$. As a consequence of (1.1), f and Q have the forms

$$f(x) = \frac{1}{\sigma} f_0\left(\frac{x-\mu}{\sigma}\right) \tag{1.2}$$

and

$$Q(u) = \mu + \sigma Q_0(u) \tag{1.3}$$

where f_0 and Q_0 are the pdf and quantile function corresponding to F_0. The *density-quantile function* for F_0, denoted d_0, is the composition of f_0 and Q_0, i.e., $d_0(u) = f_0(Q_0(u))$, $0 \leq u \leq 1$.

Using the sample order statistics, $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$, define the *sample quantile function*, $\tilde{Q}$, by

$$\tilde{Q}(u) = X_{(j)}, \qquad \frac{j-1}{n} < u \leq \frac{j}{n}, \qquad j = 1, \cdots, n. \tag{1.4}$$

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A *spacing* for the sample quantiles is a k-tuple, $T = \{u_1, \cdots, u_k\}$, whose elements satisfy $0 < u_1 < u_2 < \cdots < u_k < 1$. Let D^k denote the set of all spacings.

Given $T \in D^k$ the corresponding sample quantiles, under certain restrictions, have been shown by Mosteller [15] to have a normal limiting distribution with means $Q(u_i)$, $i = 1, \cdots, k$, and covariances

$$\text{(1.5)} \qquad \operatorname{Cov}\{\tilde{Q}(u_i), \tilde{Q}(u_j)\} = \frac{\sigma^2}{n} \frac{u_i - u_i u_j}{d_0(u_i) d_0(u_j)}, \qquad 0 < u_i \le u_j < 1, \qquad i, j = 1, \cdots, k.$$

Thus generalized least squares may be utilized to obtain asymptotically best linear unbiased estimators (ABLUE's), $\mu^*(T)$ and $\sigma^*(T)$, of μ and σ. Formulas for these estimators and their asymptotic relative efficiencies (ARE's) have been derived by Ogawa [16].

Suppose $T = \{u_1, \cdots, u_k\} \in D^k$ and let $u_0 = 0$, $u_{k+1} = 1$. Assuming that $d_0(0) = d_0(1) = d_0(0)Q_0(0) = d_0(1)Q_0(1) = 0$, define $A(T)$ as the 2×2 matrix with elements

$$\text{(1.6)} \qquad a_{11}(T) = \sum_{i=1}^{k+1} \frac{[d_0(u_i) - d_0(u_{i-1})]^2}{u_i - u_{i-1}}$$

$$\text{(1.7)} \qquad a_{12}(T) = a_{21}(T) = \sum_{i=1}^{k+1} \frac{[d_0(u_i) - d_0(u_{i-1})][d_0(u_i)Q_0(u_i) - d_0(u_{i-1})Q_0(u_{i-1})]}{u_i - u_{i-1}}$$

and

$$\text{(1.8)} \qquad a_{22}(T) = \sum_{i=1}^{k+1} \frac{[d_0(u_i)Q_0(u_i) - d_0(u_{i-1})Q_0(u_{i-1})]^2}{u_i - u_{i-1}}.$$

The usual intrinsic accuracy matrix will be denoted by A and has elements

$$\text{(1.9)} \qquad a_{11} = E\left[\left(\frac{f'(X)}{f(X)}\right)^2\right]$$

$$\text{(1.10)} \qquad a_{12} = a_{21} = E\left[X\left(\frac{f'(X)}{f(X)}\right)^2\right]$$

and

$$\text{(1.11)} \qquad a_{22} = E\left[\left(X\frac{f'(X)}{f(X)}\right)^2\right] - 1.$$

Using this notation, the ARE's, given by Ogawa [16], for the various estimation situations are as follows:

(i) When σ is known the ARE of $\mu^*(T)$ is

$$\text{(1.12)} \qquad \operatorname{ARE}\{\mu^*(T)\} = \frac{a_{11}(T)}{a_{11}}.$$

(ii) When μ is known the ARE of $\sigma^*(T)$ is

$$\text{(1.13)} \qquad \operatorname{ARE}\{\sigma^*(T)\} = \frac{a_{22}(T)}{a_{22}}.$$

(iii) When both μ and σ are unknown the ARE of $(\mu^*(T), \sigma^*(T))$ is

$$\text{(1.14)} \qquad \operatorname{ARE}\{\mu^*(T), \sigma^*(T)\} = \frac{|A(T)|}{|A|},$$

where $|\cdot|$ denotes the matrix determinant.

To obtain estimators with maximum efficiency, T is chosen to maximize one of (1.12)–(1.14) or equivalently $a_{11}(T)$, $a_{22}(T)$ or $|A(T)|$. A spacing which results in a maximum value for one of these quantities will be termed an *optimal spacing*. The problem of finding optimal spacings will be termed the *optimal spacing problem.*

The classical approach to the optimal spacing problem hinges upon obtaining solutions to a simultaneous equation system which results from the differentiation of the ARE expression being considered with respect to $Q_0(u_i)$, for $i = 1, \cdots, k$. The solutions provide a set of optimal spacing candidates and frequently must be found through numerical methods. This type of approach (with some modifications) has been utilized by Bloch [1], Chan [2], Eisenberger and Posner [3], Gupta and Gnanadesikan [7], Hassanein [8, 9, 10, 11, 12, 13], Kulldorf and Vännman [14], and Ogawa [16] to obtain optimal or near-optimal spacings for various choices of F_0.

The principal objective of this paper is to develop a computationally simple approach to spacing selection that results in spacings nearly as efficient as those of optimal character. In Section 2 an asymptotic (as $k \to \infty$) solution to the optimal spacing problem, applicable to both censored and uncensored samples, is seen to provide the desired procedure. Section 4 contains further applications of the techniques developed in Section 2 to the solution of other "optimal spacing" problems.

2. Asymptotically optimal spacings. Estimating location and scale parameters given a (possibly) censored set of order statistics, $X_{(np)}, \cdots, X_{(nq)}$, can be formulated as using the sample quantile function over the interval $[p, q] \subset [0, 1]$. Using this fact, Parzen [19] has shown that for large samples the linear estimation of μ and σ by sample quantiles can be considered as a regression problem for continuous parameter time series through use of the model

$$d_0(u)\tilde{Q}(u) = \mu d_0(u) + \sigma d_0(u)Q_0(u) + \sigma_B B(u), \qquad u \in [p, q], \tag{2.1}$$

where $B(\cdot)$ is a Brownian bridge process on $[p, q]$ with covariance kernel

$$K(u_1, u_2) = u_1 - u_1u_2, \qquad p \le u_1 \le u_2 \le q, \tag{2.2}$$

and $\sigma_B = \sigma/\sqrt{n}$. Estimators of μ and σ, denoted $\hat{\mu}_{p,q}$ and $\hat{\sigma}_{p,q}$, based on the entire sample are then derived through use of the reproducing kernel Hilbert space (RKHS) techniques developed by Parzen [17, 18].

In this section an approach to order statistic selection for location and scale parameter estimation will be developed through the use of model (2.1) and regression design techniques. Much of the work which follows is motivated by the RKHS approach to regression design for continuous parameter time series used by Sacks and Ylvisaker [20, 21]. To adapt their techniques for use with model (2.1) some preliminaries are needed.

The RKHS generated by K, $H(K)$, is known to consist of L^2 differentiable functions (cf. Parzen [19]). The inner product of two functions, f and g, in $H(K)$ is

$$\langle f, g\rangle_{p,q} = \int_p^q f'(s)g'(s)\,ds + \frac{f(p)g(p)}{p} + \frac{f(q)g(q)}{1-q}. \tag{2.3}$$

The norm of $f \in H(K)$ is denoted $\| f \|_{p,q}$. If f is twice differentiable, integration by parts and use of the reproducing property in (2.3) shows that

$$\begin{aligned} f(u) = &-\int_p^q f''(s)K(u, s)\,ds \\ &+ K(u, p)\left[\frac{1}{p}f(p) - f'(p)\right] + K(u, q)\left[\frac{1}{1-q}f(q) + f'(q)\right]. \end{aligned} \tag{2.4}$$

Given observations obtained from model (2.1) by sampling at a set of noncoincident points, $T = \{u_1, \cdots, u_k\}$, in $[p, q]$, μ and σ may be estimated by generalized least squares. These estimators will be denoted $\mu^*_{p,q}(T)$ and $\sigma^*_{p,q}(T)$.

Let P_T denote the $H(K)$ projection operator for the subspace generated by $\{K(\cdot, u):u \in T\}$. It can be shown that when d_0 and the product of d_0 and Q_0, $d_0 \cdot Q_0$, are in $H(K)$ the variance-covariance matrix for $(\mu^*_{p,q}(T), \sigma^*_{p,q}(T))$ is $\sigma_B^2 A_{p,q}(T)^{-1}$ where

$$(2.5) \qquad A_{p,q}(T) = \begin{bmatrix} \| P_T d_0 \|^2_{p,q} & \langle P_T d_0, P_T d_0 \cdot Q_0 \rangle_{p,q} \\ \langle P_T d_0, P_T d_0 \cdot Q_0 \rangle_{p,q} & \| P_T d_0 \cdot Q_0 \|^2_{p,q} \end{bmatrix}.$$

The variance-covariance matrix for $(\hat{\mu}_{p,q}, \hat{\sigma}_{p,q})^t$ is $\sigma_B^2 A^{-1}_{p,q}$ where

$$(2.6) \qquad A_{p,q} = \begin{bmatrix} \| d_0 \|^2_{p,q} & \langle d_0, d_0 \cdot Q_0 \rangle_{p,q} \\ \langle d_0, d_0 \cdot Q_0 \rangle_{p,q} & \| d_0 \cdot Q_0 \|^2_{p,q} \end{bmatrix}.$$

Using (2.3) and the reproducing property it follows that functions in $H(K)$ vanish at 0 and 1. This fact along with consideration of the explicit form of $A_{p,q}(T)$ lead to the conclusion that observations taken at 0 and 1 provide no information. Consequently, these values will not be considered as design points. Since design points can be taken arbitrarily close to 0 and 1 this convention does not preclude the use of extreme order statistics in estimation. The preceeding discussion suggests the following definition.

DEFINITION 1. When $0 < p < q < 1$ a k-point design for model (2.1) is a k-tuple, $\{u_1, \cdots, u_k\}$, whose elements satisfy

$$(2.7) \qquad p \leq u_1 < u_2 < \cdots < u_k \leq q.$$

In the event $p = 0$ $(q = 1)$ *the left-hand (right-hand) inequality in* (2.7) *is taken to be a strict inequality.* The set of all k-point designs on $[p, q]$ is denoted by $D^k_{p,q}$.

In the uncensored case, $p = 0$, $q = 1$, it is readily verified that $\mu^*_{0,1}(T) = \mu^*(T)$, $\sigma^*_{0,1}(T) = \sigma^*(T)$, $A_{0,1}(T) = A(T)$, $A_{0,1} = A$, and $D^k_{0,1} = D^k$. Thus, the optimal spacing problem is identical to the problem of optimal design selection, in the minimum variance (maximum information) sense, for model (2.1).

The term spacing and optimal spacing will now be used as being synonymous with the designs and optimal designs in $D^k_{p,q}$. Thus, for instance, a spacing, $T^* \in D^k_{p,q}$, would be optimal for the estimation of μ when σ is known if

$$\| P_{T^*} d_0 \|^2_{p,q} = \sup_{T \in D^k_{p,q}} \| P_T d_0 \|^2_{p,q}.$$

To develop a general solution to the spacing selection problem an asymptotic (as $k \to \infty$) version of spacing optimality will be used.

DEFINITION 2. A spacing sequence $\{T_k\}^\infty_{k=1}$, $T_k \in D^k_{p,q}$, is asymptotically optimal for the estimation of μ when σ is known if

$$(2.8) \qquad \lim_{k\to\infty}(\| d_0 \|^2_{p,q} - \sup_{T \in D^k_{p,q}} \| P_T d_0 \|^2_{p,q})(\| d_0 \|^2_{p,q} - \| P_{T_k} d_0 \|^2_{p,q})^{-1} = 1.$$

A scale parameter version of Definition 2 is obtained by replacing d_0 with $d_0 \cdot Q_0$ in (2.8). For simultaneous parameter estimation the following definition will be used.

DEFINITION 3. A spacing sequence $\{T_k\}^\infty_{k=1}$, $T_k \in D^k_{p,q}$, is asymptotically optimal for the simultaneous estimation of μ and σ if

$$(2.9) \qquad \lim_{k\to\infty}(| A_{p,q} | - \sup_{T \in D^k_{p,q}} | A_{p,q}(T) |)(| A_{p,q} | - | A_{p,q}(T_k) |)^{-1} = 1.$$

The use of the determinant in (2.9) is motivated by the classical approach to optimal spacing selection. From a regression design point of view other criteria might be considered. This topic is amenable to analysis along the lines of Theorem 1 but will not be pursued here.

Spacing sequences may be constructed through the use of density functions. Let h be a continuous density on $[p, q]$ with corresponding df H. The kth element in the spacing sequence *generated* by h is composed of the design points

$$(2.10)\qquad u_i = \begin{cases} H^{-1}\left(\frac{i-1}{k-1}\right), & i = 1, \cdots, k, \quad \text{when } 0 < p < q < 1, \\ H^{-1}\left(\frac{i}{k}\right), & i = 0, \cdots, k-1, \quad \text{when } 0 < p < q = 1, \\ H^{-1}\left(\frac{i}{k}\right), & i = 1, \cdots, k, \quad \text{when } 0 = p < q < 1, \\ H^{-1}\left(\frac{i}{k+1}\right), & i = 1, \cdots, k, \quad \text{when } p = 0, \quad q = 1. \end{cases}$$

To solve (asymptotically) the problem of optimal spacing selection it suffices to construct an optimal density, i.e., a density which generates an asymptotically optimal design sequence. Optimal densities for the various estimation situations are provided by Theorem 1.

THEOREM 1. *Under the assumption that the regression function or functions under consideration admit a representation of the form* (2.4), *the following conclusions hold*:

(i) *The density*

$$(2.11)\qquad h^*(u) = [d_0(u)'']^{2/3}/\lambda_\mu,$$

where $\lambda_\mu = \int_p^q [d_0(s)'']^{2/3}\, ds$, *generates a spacing sequence*, $\{T_k^*\}$, *which is asymptotically optimal for the estimation of* μ *when* σ *is known and satisfies*

$$(2.12)\qquad lim_{k\to\infty} k^2(\| d_0 \|_{p,q}^2 - \| P_{T_k^*} d_0 \|_{p,q}^2) = \lim_{k\to\infty} k^2(\| d_0 \|_{p,q}^2 - \sup_{T\in D_{p,q}^k} \| P_T d_0 \|_{p,q}^2) = \lambda_\mu^3.$$

(ii) *The density*

$$(2.13)\qquad h^*(u) = [\{d_0(u)Q_0(u)\}'']^{2/3}/\lambda_\sigma,$$

where $\lambda_\sigma = \int_p^q [\{d_0(s)Q_0(s)\}'']^{2/3}\, ds$, *generates a spacing sequence*, $\{T_k^*\}$, *which is asymptotically optimal for the estimation of* σ *when* μ *is known and satisfies*

$$(24)\qquad \begin{aligned} &\lim_{k\to\infty} k^2(\| d_0 \cdot Q_0 \|_{p,q}^2 - \| P_{T_k^*} d_0 \cdot Q_0 \|_{p,q}^2) \\ &\qquad = \lim_{k\to\infty} k^2(\| d_0 \cdot Q_0 \|_{p,q}^2 - \sup_{T\in D_{p,q}^k} \| P_T d_0 \cdot Q_0 \|_{p,q}^2) = \lambda_\sigma^3. \end{aligned}$$

(iii) *Let* $\psi(u) = ([d_0(u)]'', [d_0(u)Q_0(u)]'')^t$. *The density*

$$(2.15)\qquad h^*(u) = [\psi(u)^t A_{p,q}^{-1} \psi(u)]^{1/3}/\lambda_{\mu,\sigma},$$

where $\lambda_{\mu,\sigma} = \int_p^q [\psi(s)^t A_{p,q}^{-1} \psi(s)]^{1/3}\, ds$, *generates a spacing sequence*, $\{T_k^*\}$, *that is asymptotically optimal for the simultaneous estimation of* μ *and* σ *and satisfies*

$$(2.16)\qquad lim_{k\to\infty} k^2(|A_{p,q}| - |A_{p,q}(T_k^*)|) = \lim_{k\to\infty} k^2(|A_{p,q}| - \sup_{T\in D_{p,q}^k} |A_{p,q}(T)|) = \lambda_{\mu,\sigma}^3.$$

PROOF. For the case $0 < p < q < 1$ conclusions (i)–(iii) are an immediate consequence of the form of RKHS and results obtained by Sacks and Ylvisaker (see Theorem 3.1 and Remarks 3.3 and 3.5 of [20] as well as Theorem 4.2 and Remark 2.1 of [21]). For the other cases the spacing sequences are obtained by taking the $(k+1)$th or $(k+2)$th element of the spacing sequences generated by (2.11), (2.13), and (2.15) when $0 < p < q < 1$ and deleting design points at 0 and/or 1. Results (i)–(iii) now follow from this fact and the discussion preceeding Definition 1.

3. Comparison and discussion. Theorem 1 provides an asymptotic solution to the optimal spacing problem. An indication of how well the asymptotically optimal spacings perform for finite k can be obtained, for $p = 0$, $q = 1$, through comparison with the optimal or near-optimal spacings found by other authors. Although the asymptotically optimal spacings usually have lower ARE's than those of optimal character, their ARE's will usually be quite similar for $k \geq 7$. This is illustrated in Table 1 where the ratio of the ARE of the asymptotically optimal spacing to that of the optimal (or nearly optimal) spacing, denoted ARE(asymp. opt.)/ARE(opt.), is given for $k = 7$ and several choices of F_0. In fact, for the logistic, when σ is known, and the Pareto, when μ is known and ν, the shape parameter, is one, the asymptotically optimal spacings agree with the optimal spacings given by Gupta and Gnanadesikan [7] and Kulldorf and Vännman [14] respectively.

Table 1 also provides examples of explicit solutions for H^{*-1}. Evaluation of H^{*-1} will usually require interpolation in a tabulation of H^* obtained through numeric integration. However, given a tabulation of H^* or H^{*-1}, spacings are easily computed for any value of k. This is in contrast to the classical approach where complicated spacing calculations are often required for each value of k.

Some of the H^{*-1} functions in Table 1 have been derived by Särndal [22] and Chernoff [3] through the use of variational methods. The techniques utilized here dispense with certain difficulties in Särndal's approach associated with zeros for h^*.

4. Other related results. Consider first the problem of optimal spacing selection for estimation of the τth population quantile, $Q(\tau)$, $\tau \in [p, q]$. In view of (1.3), this is a special instance of optimal spacing selection for the estimation of $l_1\mu + l_2\sigma$, where $l = (l_1, l_2)^t$ is a fixed known vector. Since, for $T \in D^k_{p,q}$

$$V\{l_1\mu^*_{p,q}(T) + l_2\sigma^*_{p,q}(T)\} = \sigma^2_B \operatorname{tr}\{A_{p,q}(T)^{-1}ll^t\}$$

an equivalent problem is the selection of a T which minimizes $\operatorname{tr}\{A_{p,q}(T)^{-1}ll^t\}$. Using results given by Sacks and Ylvisaker [21] it follows that

$$h^*(u) = \{\psi(u)^t A^{-1}_{p,q} ll^t A^{-1}_{p,q}\psi(u)\}^{1/3}/\lambda, \tag{4.1}$$

where $\lambda = \int_p^q \{\psi(s)^t A^{-1}_{p,q} ll^t A^{-1}_{p,q}\psi(s)\}^{1/3}\, ds$, generates an asymptotically optimal spacing sequence for the estimation of $l_1\mu + l_2\sigma$. Taking $l = \{1, Q_0(\tau)\}^t$ provides an optimal density for the estimation of $Q(\tau)$.

Another problem in optimal spacing selection stems from the possible nonuniqueness of optimal solutions. Conditions which imply optimal spacing unicity can be obtained

TABLE 1
The H^{*-1} *functions and spacing efficiency comparisons for selected distributions and parametric assumptions.*

Distribution	H^{*-1} (u)	Parametric assumptions	ARE(Asymp. opt.)/ ARE(opt.)*
Normal ($F_0 = \Phi$)	$\Phi(\sqrt{3}\,\Phi^{-1}(u))$	σ known	.998
Exponential	$1 - (1 - u)^3$	μ known	.988
Pareto	$1 - (1 - u)^{3\nu/2 + \nu}$	μ known, ν=.5μ,	.999
		known, ν=2	.999
Extreme value	u^3	σ known	.998
Cauchy	numeric tabulation	μ known	.991
Logistic	numeric tabulation	μ and σ unknown	1.01

* The optimal or near optimal spacings and ARE's used in the calculations for rows 1–6 of the table may be found in references [16 or 22], [22], [14], [8], [2], and [10] respectively.

through an application of Theorem 2.1 of Eubank, Smith, and Smith [6]. Thus if g represents either d_0 *or* $d_0 \cdot Q_0$, then $g \in C^2[a, b]$ with $g'' > 0$ on $[p, q]$ and $\log g''$ concave on (p, q) suffices to insure that for each integer k there exists unique optimal spacings. Examples of distributions satisfying these hypotheses are the logistic (when σ is known) and the Pareto (when μ is known and $\nu < 1$).

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