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# BALANCED REPEATED MEASUREMENTS DESIGNS

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Two types of repeated measurements designs (RMD), the strongly balanced uniform RMD, and the balanced uniform RMD, are defined and their universal optimality is proved over very broad classes of competing designs. Construction methods for strongly balanced uniform RMD are also given.

**1. Introduction.** In repeated measurements designs (cross over, or change over designs), experimental units are used repeatedly by exposing them to a sequence of different or identical treatments. For a discussion on the use of this kind of design, the readers may consult Hedayat and Afsarinejad (1975), which also provides an extensive bibliography containing more than 130 references on repeated measurements designs.

Using a tool due to Kiefer (1975), Hedayat and Afsarinejad (1978) proved the universal optimality of some *balanced* repeated measurements designs over the class of *uniform* designs. They considered the following model:

Let $t$ treatments be compared via $n$ experimental units in $p$ periods. Altogether $np$ observations are taken. An allocation of the $t$ treatments into the $np$ observations is called a *repeated measurements design* (RMD). If $d$ is an RMD, then let $d(i, j)$ denote the treatment assigned by $d$ in the $i$th period to the $j$th experimental unit. Let $y_{ij}$ be the response obtained under $d(i, j)$. Then all the observations are assumed to be uncorrelated with common variance and

$$E(y_{ij}) = \alpha_i + \beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)}, \qquad (1.1)$$

$$i = 1, \cdots, p, \qquad j = 1, \cdots, n, \qquad \rho_{d(0,j)} = 0 \quad \textit{for all } j,$$

where the unknown constants $\alpha_i$, $\beta_j$, $\tau_{d(i,j)}$ and $\rho_{d(i-1,j)}$ are respectively called *the $i$th period effect, the $j$th experimental unit effect, the direct effect of treatment $d(i, j)$, and the first order residual effect of treatment $d(i-1, j)$.*

For convenience, a repeated measurements design with $t$ treatments, $n$ units and $p$ periods is abbreviated as RMD($t$, $n$, $p$). The collection of all such designs is denoted by $\Omega_{t,n,p}$.

DEFINITION 1.1. A design $d$ is said to be *uniform on the periods* if, in each period, $d$ assigns the same number of units to each treatment.

DEFINITION 1.2. A design is said to be *uniform on the units* if, on each unit, each treatment appears in the same number of periods.

DEFINITION 1.3. A design is called *uniform* if it is uniform on the periods and units.

Hedayat and Afsarinejad (1978) considered the setting in which $t = p$ and $t \mid n$ (say $n = \lambda_1 t$). They also restricted the competing designs to be *uniform*. For a uniform design in $\Omega_{t,\lambda_1 t,t}$, it is clear that a treatment can not be preceded by itself. Thus, a design in $\Omega_{t,\lambda_1 t,t}$ is defined to be

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*balanced* if in the order of application, each treatment is preceded precisely $\lambda_1$ times by each other treatment, i.e., the collection of ordered pairs $(d(i, j), d(i+1, j))$, $1 \le i \le t-1$, $1 \le j \le \lambda_1 t$, contains each ordered pair of *distinct* treatments $\lambda_1$ times. In general (not necessarily $t = p$ and $t \mid n$), an RMD is called *balanced* if in the order of application, each treatment is not preceded by itself and is preceded by the other treatments equally often. Using Kiefer's tool, Hedayat and Afsarinejad proved the universal optimality of a balanced uniform RMD($t$, $\lambda_1 t$, $t$) for the estimation of direct as well as residual effects over the *uniform* designs in $\Omega_{t,\lambda_1 t,t}$.

There is an intrinsic difficulty in removing the restriction of uniformity which Hedayat and Afsarinejad imposed on the competing designs. From an optimality point of view, there is no reason to exclude the designs which allow a treatment to be preceded by itself. When $\lambda_1 > 1$, the $t^2$ ordered pairs of treatments do not appear as nearly equally often as possible in a balanced uniform RMD($t$, $\lambda_1 t$, $t$) and hence its optimality over *all possible designs* is quite doubtful. This also causes difficulty in verifying the maximization of the trace of the information matrices, an important step in applying Kiefer's result. In view of this, it seems unlikely to remove entirely Hedayat and Afsarinejad's restriction. However, we are able to show that a balanced RMD($t$, $\lambda_1 t$, $t$) is universally optimal for the estimation of residual effects over the designs in which each treatment is not preceded by itself. Thus, the restriction of uniformity is relaxed substantially. As to the estimation of direct effects, the restriction of uniformity can also be relaxed to a certain extent. The first purpose of the present paper is to extend and strengthen Hedayat and Afsarinejad's results.

If we want to dispose of the restriction entirely, then a stronger kind of balancing is needed.

DEFINITION 1.4. An RMD($t$, $n$, $p$) $d$ is called *strongly balanced* if the collection of ordered pairs $(d(i, j), d(i+1, j))$, $1 \le i \le p-1$, $1 \le j \le n$, contains each ordered pair of treatments (*distinct or not*) the same number of times, say $\lambda$ times.

Note that in a strongly balanced RMD($t$, $n$, $p$), $\lambda = t^{-2}(p-1)n$.

The second purpose of this paper is to discuss the optimality of some strongly balanced designs. We are able to prove that a strongly balanced uniform design is universally optimal for the estimation of direct as well as residual effects over all possible designs. A more interesting result says that if we repeat the observations in the last period of Hedayat and Afsarinejad's balanced uniform RMD($t$, $\lambda_1 t$, $t$), then the resulting design is universally optimal for direct as well as residual effects over $\Omega_{t,\lambda_1 t,t+1}$. This design is strongly balanced and uniform on the periods but is *not* uniform on the units. The strongly balanced uniform designs were discussed in Berenblut (1964) and the procedure of repeating the observations in the last period was first proposed by Lucas (1957) under the name of *extra-period designs.* Both designs have very nice orthogonality structures which are closely related to the optimality.

The organization of this paper is the following. Section 2 contains some preliminaries. The optimality of some strongly balanced designs is proved in Section 3. Some examples and methods of construction are also presented. The results in this section can be extended in several directions including the case where there are higher-order residual effects and the model in which each unit effect has a higher-dimensional structure. These extensions are treated in Cheng and Wu (1979). Section 4 deals with the extension of Hedayat and Afsarinejad's result on balanced designs. Section 5 contains the proofs of the results in Section 4.

**2. Preliminaries.** We write $C_d$ (resp., $\tilde{C}_d$) as the $C$-matrix or information matrix of the direct (resp., residual) effects when design $d$ is used. An *optimality criterion* is a function $\Phi$: $\mathscr{B}_{t,0} \to (-\infty, \infty]$, where $\mathscr{B}_{t,0}$ is the collection of $t \times t$ nonnegative definite matrices with zero row and column sums. A design is called $\Phi$-optimal if it minimizes $\Phi(C_d)$ or $\Phi(\tilde{C}_d)$ over the competing designs depending on which effects we are interested in. Note that $C_d$, $\tilde{C}_d \in \mathscr{B}_{t,0}$ in our setting.

Kiefer (1975) introduced the notion of universal optimality. A design $d^*$ is called *universally optimal* if it is $\Phi$-optimal for all $\Phi$ satisfying

(2.1)
(i) $\Phi$ is convex,
(ii) $\Phi(bC)$ is nonincreasing in the scalar $b \geq 0$,
(iii) $\Phi$ is invariant under any simultaneous permutation of rows and columns of $C$.

Note that if $d^*$ is universally optimal then it is $D$-, $A$-, and $E$-optimal. Kiefer (1975) obtained a very simple sufficient condition for universal optimality. A simple algebraic argument shows that $d^*$ is universally optimal as long as $d^*$ maximizes tr $C_d$ (or tr $\tilde{C}_d$) and $C_{d^*}$ (or $\tilde{C}_{d^*}$) is *completely symmetric* in the sense that all the diagonal elements of $C_{d^*}$ (or $\tilde{C}_{d^*}$) are equal and all the off-diagonals are the same.

For each design $d \in \Omega_{t,n,p}$, define

$n_{diu}$ = number of appearances of treatment $i$ on unit $u$,
$\tilde{n}_{diu}$ = number of appearances of treatment $i$ on unit $u$ in the first $p - 1$ periods.
$l_{dik}$ = number of appearances of treatment $i$ in period $k$,
$m_{dij}$ = number of appearances of treatment $i$ preceded by treatment $j$ on the same unit,

where $i, j = 1, \cdots, t$, $u = 1, \cdots, n$, $k = 1, \cdots, p$. It follows that

$$r_{di} \equiv \textstyle\sum_{u=1}^{n} n_{diu} = \sum_{k=1}^{p} l_{dik} = \text{number of appearances of treatment } i,$$

$$\tilde{r}_{di} \equiv \textstyle\sum_{u=1}^{n} \tilde{n}_{diu} = \sum_{k=1}^{p-1} l_{dik} = \sum_{j=1}^{t} m_{dji} = \text{number of appearances of treatment } i \text{ in the first } p-1 \text{ periods.} \tag{2.2}$$

The following relations are fundamental:

$$\textstyle\sum_{i=1}^{t} n_{diu} = p, \ \sum_{i=1}^{t} \tilde{n}_{diu} = p - 1, \ \sum_{i=1}^{t} l_{dik} = n,$$

$$s_{di} \equiv \text{number of appearances of treatment } i \text{ in the last } p-1 \textit{ periods} \tag{2.3}$$

$$= \textstyle\sum_{k=2}^{p} l_{dik} = \sum_{j=1}^{t} m_{dij}, \ \sum_{i=1}^{t} r_{di} = np, \ \sum_{i=1}^{t} \tilde{r}_{di} = n(p-1).$$

From (2.2) and (2.3), uniformity on units and equal $m_{dij}$ imply the uniformity in the first and last periods.

In vector notation, for any $d \in \Omega_{t,n,p}$,

$$Ey_d = X_d\theta, \tag{2.4}$$

where $y_d$ is the $np \times 1$ vector of observations, $X_d$ is the design matrix, and $\theta = (\tau_1, \tau_2, \cdots, \tau_t, \rho_1, \cdots, \rho_t, \alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_n)'$. Then

$$X_d'X_d = \begin{bmatrix} D_d & M_d & N_{dp} & N_{du} \\ M_d' & \tilde{D}_d & \tilde{N}_{dp} & \tilde{N}_{du} \\ N_{dp}' & \tilde{N}_{dp}' & nI_p & J_{p,n} \\ N_{du}' & \tilde{N}_{du}' & J_{n,p} & pI_n \end{bmatrix}, \tag{2.5}$$

where $I_p$ is the identity matrix of order $p$, $J_{p,n}$ is the $p \times n$ matrix of ones, $D_d = \text{diag}(r_{d1}, \cdots, r_{dt})$, $\tilde{D}_d = \text{diag}(\tilde{r}_{d1}, \cdots, \tilde{r}_{dt})$, $M_d = [m_{dij}]_{1\leq i,j\leq t}$, $N_{dp} = [l_{dik}]_{1\leq i\leq t, 1\leq k\leq p}$, $\tilde{N}_{dp} = [\tilde{l}_{dik}]_{1\leq i\leq t,1\leq k\leq p}$ with $\tilde{l}_{di1} = 0$, $\tilde{l}_{dik} = l_{di,k-1}$ for $k \geq 2$, $N_{du} = [n_{diu}]_{1\leq i\leq t,1\leq u\leq n}$, and $\tilde{N}_{du} = [\tilde{n}_{diu}]_{1\leq i\leq t,1\leq u\leq n}$.

From (2.5), the information matrix for estimating direct and residual effects jointly is

$$\begin{bmatrix} D_d & M_d \\ M_d' & \tilde{D}_d \end{bmatrix} - \begin{bmatrix} N_{dp} & N_{du} \\ \tilde{N}_{dp} & \tilde{N}_{du} \end{bmatrix} \begin{bmatrix} nI_p & J_{p,n} \\ J_{n,p} & pI_n \end{bmatrix}^{-} \begin{bmatrix} N_{dp}' & \tilde{N}_{dp}' \\ N_{du}' & \tilde{N}_{du}' \end{bmatrix} = \begin{bmatrix} C_{d11} & C_{d12} \\ C_{d21} & C_{d22} \end{bmatrix}, \tag{2.6}$$

where

$$C_{d11} = D_d - n^{-1}N_{dp}N_{dp}' - p^{-1}N_{du}N_{du}' + n^{-1}p^{-1}N_{du}J_{n,n}N_{du}',$$

$$C_{d12} = C_{d21}' = M_d - n^{-1}N_{dp}\tilde{N}_{dp}' - p^{-1}N_{du}\tilde{N}_{du}' + n^{-1}p^{-1}N_{du}J_{n,n}\tilde{N}_{du}',$$

$$C_{d22} = \tilde{D}_d - n^{-1}\tilde{N}_{dp}\tilde{N}_{dp}' - p^{-1}\tilde{N}_{du}\tilde{N}_{du}' + n^{-1}p^{-1}\tilde{N}_{du}J_{n,n}\tilde{N}_{du}',$$

and $A^-$ denotes a generalized inverse of $A$.

Then

(2.7) $$C_d = C_{d11} - C_{d12}C_{d22}^-C_{d21}$$

and

(2.8) $$\tilde{C}_d = C_{d22} - C_{d21}C_{d11}^-C_{d12}.$$

The following lemmas are useful for later developments.

LEMMA 2.1. *The row sums and column sums of $C_{d12}$ and $M_d - n^{-1}N_{dp}\tilde{N}'_{dp}$ are all equal to zero, and the column sums of $M_d - p^{-1}N_{du}\tilde{N}'_{du}$ are all equal to zero.*

PROOF. From (2.2) and (2.3), it can be easily checked that the $j$th column sums of $M_d$, $n^{-1}N_{dp}\tilde{N}'_{dp}$, $p^{-1}N_{du}\tilde{N}'_{du}$, and $n^{-1}p^{-1}N_{du}J_{n,n}\tilde{N}'_{du}$ are all equal to $\tilde{r}_{dj}$. Also, the $i$th row sums of $M_d$ and $n^{-1}N_{dp}\tilde{N}'_{dp}$ are all equal to $s_{di}$ as defined in (2.3) and the $i$th row sums of $p^{-1}N_{du}\tilde{N}'_{du}$ and $n^{-1}p^{-1}N_{du}J_{n,n}\tilde{N}'_{du}$ are both equal to $p^{-1}(p-1)r_{di}$. The desired results follow. □

From Lemma 2.1, it follows easily that the row and column sums of $C_d$ and $\tilde{C}_d$ are zero for any $d \in \Omega_{t,n,p}$.

LEMMA 2.2. *The following matrices are all nonnegative definite: $C_{d11}$, $D_d - n^{-1}N_{dp}N'_{dp}$, $D_d - p^{-1}N_{du}N'_{du}$, $N_{dp}N'_{dp} - p^{-1}N_{dp}J_{p,p}N'_{dp}$, $N_{du}N'_{du} - n^{-1}N_{du}J_{n,n}N'_{du}$, $C_{d22}$, $\tilde{D}_d - n^{-1}\tilde{N}_{dp}\tilde{N}'_{dp}$, $\tilde{D}_d - p^{-1}\tilde{N}_{du}\tilde{N}'_{du}$, $\tilde{N}_{dp}\tilde{N}'_{dp} - p^{-1}\tilde{N}_{dp}J_{p,p}\tilde{N}'_{dp}$, $\tilde{N}_{du}\tilde{N}'_{du} - n^{-1}\tilde{N}_{du}J_{n,n}\tilde{N}'_{du}$.*

The proof of Lemma 2.2 is straightforward. For example, $N_{du}N'_{du} - n^{-1}N_{du}J_{n,n}N'_{du} = N_{du}(I_n - n^{-1}J_{n,n})N'_{du}$, where the middle matrix is nonnegative definite; and $D_d - n^{-1}N_{dp}N'_{dp}$ is the $C$-matrix for the direct effects when the residual and unit effects are not present in the model.

LEMMA 2.3. *For any positive integers $s$ and $t$, the minimum of $\sum_{i=1}^{s} n_i^2$ subject to $\sum_{i=1}^{s} n_i = t$, where the $n_i$'s are nonnegative integers, is obtained when $t - s[t/s]$ of the $n_i$'s are equal to $[t/s] + 1$ and the others are equal to $[t/s]$, where $[t/s]$ is the largest integer $\leq t/s$.*

LEMMA 2.4. *Let $d^*$ be a uniform* RMD$(t, n, p)$. *Then $d^*$ maximizes* tr $C_{d11}$ *and* tr $C_{d22}$ *over $d \in \Omega_{t,n,p}$.*

PROOF. We only prove that $d^*$ maximizes tr $C_{d22}$. The proof for the maximization of tr $C_{d11}$ is similar and in fact is even simpler.

For any $d \in \Omega_{t,n,p}$,

$$\begin{aligned}\operatorname{tr} C_{d22} &= n(p-1) - n^{-1}\sum_{i=1}^{t}\sum_{j=1}^{p-1} l_{dij}^2 - p^{-1}\sum_{i=1}^{t}\sum_{u=1}^{n}\tilde{n}_{diu}^2 + n^{-1}p^{-1}\sum_{i=1}^{t}\tilde{r}_{di}^2\\ &= n(p-1) - p^{-1}\sum_{i=1}^{t}\sum_{u=1}^{n}\tilde{n}_{diu}^2 - n^{-1}\Big[\sum_{i=1}^{t}\sum_{j=1}^{p-1}(l_{dij} - (p-1)^{-1}\tilde{r}_{di})^2\\ &\quad + \{(p-1)^{-1} - p^{-1}\}\sum_{i=1}^{t}\tilde{r}_{di}^2\Big] \quad \text{since } \tilde{r}_{di} = \sum_{j=1}^{p-1} l_{dij}.\end{aligned}$$

For a uniform design $d^*$, $\sum_{i=1}^{t}\sum_{j=1}^{p-1}(l_{d^*ij} - (p-1)^{-1}\tilde{r}_{d^*i})^2 = 0$, so it suffices to show that $d^*$ minimizes $\sum_{i=1}^{t}\sum_{u=1}^{n}\tilde{n}_{diu}^2$ and $\sum_{i=1}^{t}\tilde{r}_{di}^2$. This follows immediately from Lemma 2.3 since $\sum_{i=1}^{t}\sum_{u=1}^{n}\tilde{n}_{diu} = n(p-1) = \sum_{i=1}^{t}\tilde{r}_{di}$, $\tilde{r}_{d^*i}$ are all equal, and $|\tilde{n}_{d^*iu} - \tilde{n}_{d^*i'u'}| \leq 1$, $\forall\,(i,u) \neq (i', u')$. □

Similarly, the following lemma can be established.

LEMMA 2.5. *Let $d^*$ be an* RMD$(t, n, p)$ *which is uniform on the periods and is uniform on the units in the first $p-1$ periods,* i.e., *if the last period of $d^*$ is deleted, then the resulting design is uniform on the units. Then $d^*$ maximizes* tr $C_{d11}$ *and* tr $C_{d22}$ over $d \in \Omega_{t,n,p}$.

PROOF. The proof is similar to Lemma 2.4 except that in the present case $\tilde{n}_{d^*iu}$ are all equal, and $| n_{d^*iu} - n_{d^*i'u'} | \le 1$, $\forall\ (i, u) \ne (i', u')$. □

### 3. Optimality of strongly balanced repeated measurements designs.

THEOREM 3.1. *Let $d^*$ be a strongly balanced uniform design in $\Omega_{t,n,p}$. Then $d^*$ is universally optimal for the estimation of direct as well as residual effects over $\Omega_{t,n,p}$.*

PROOF. By Proposition 1 of Kiefer (1975), it suffices to show that both $C_{d^*}$ and $\tilde{C}_{d^*}$ are completely symmetric and $d^*$ maximizes tr $C_d$ and tr $\tilde{C}_d$ over $d \in \Omega_{t,n,p}$.

The existence of a uniform design in $\Omega_{t,n,p}$ implies that $t \mid n$ and $t \mid p$. Let $n = \lambda_1 t$ and $p = \lambda_2 t$. Then it can easily be checked that $C_{d^*12} = M_{d^*} - n^{-1} N_{d^*p} \tilde{N}'_{d^*p} - p^{-1} N_{d^*u} \tilde{N}'_{d^*u} + n^{-1} p^{-1} N_{d^*u} J_{n,n} \tilde{N}'_{d^*u} = t^{-1}\lambda_1(p-1)J_{t,t} - n^{-1}\lambda_1^2(p-1)J_{t,t} - p^{-1}\lambda_2(r-\lambda_1)J_{t,t} + n^{-1}p^{-1}r(r-\lambda_1)J_{t,t} \equiv 0$, where $r \equiv t^{-1}np = n\lambda_2 = p\lambda_1$. Therefore $C_{d^*} = C_{d^*11} = t^{-1}np(I_t - t^{-1}J_{t,t})$ and $\tilde{C}_{d^*} = C_{d^*22} = t^{-1}n(p-1)(I_t - t^{-1}J_{t,t})$ which are completely symmetric.

For any $d \in \Omega_{t,n,p}$, $C_d = C_{d11} - C_{d12}C_{d22}^{-}C_{d21} \le C_{d11}$ and $\tilde{C}_d = C_{d22} - C_{d21}C_{d11}^{-}C_{d12} \le C_{d22}$, where $A \le B$ means that $B - A$ is nonnegative definite. Hence

$$\begin{aligned} \text{tr } C_d \le \text{tr } C_{d11} &\le \text{tr } C_{d^*11} \qquad \text{(Lemma 2.4)} \\ &= \text{tr } C_{d^*} \end{aligned}$$

Similarly, tr $\tilde{C}_d \le$ tr $\tilde{C}_{d^*}$. □

Thus, a strongly balanced uniform RMD is optimal in a very strong sense. We now show a simple construction for this kind of design. If there exists a strongly balanced uniform RMD($t$, $n$, $p$) then $t^2 \mid n$ and $p = \lambda_2 t$ with $\lambda_2 \ge 2$. In case $\lambda_2$ is even, we have the following:

THEOREM 3.2. *If $t^2 \mid n$ and $p/t$ is an even integer, then there exists a strongly balanced uniform* RMD($t$, $n$, $p$).

PROOF. Suppose $t^2 \mid n$ and $p = \lambda_2 t$ where $\lambda_2$ is an even integer. Denote the $t$ treatments by the nonnegative residues modulo $t$. Assign the treatments to the first two periods so that each treatment receives the same number of units in each period, and each ordered pair of treatments appears the same number of times. This is possible since $t^2 \mid n$. Let the rows correspond to the units and the columns correspond to the periods. Then an $n \times 2$ array is constructed.

To each symbol in this $n \times 2$ array add $i$ (mod $t$) ($i = 1, 2, \cdots, t-1$), and put the two columns obtained in this way in the $(2i+1)$st and $(2i+2)$nd periods. Then we get an $n \times 2t$ plan which is obviously a strongly balanced uniform RMD($t$, $n$, $2t$). A strongly balanced uniform RMD ($t$, $n$, $\lambda_2 t$) with $\lambda_2$ even is obtained by piecing $\lambda_2/2$ copies of this design together. □

For example, the following is a strongly balanced uniform RMD(3, 9, 6);

(3.1)

| | units | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| periods | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 |
| | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 |
| | 1 | 1 | 1 | 2 | 2 | 2 | 0 | 0 | 0 |
| | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 |
| | 2 | 2 | 2 | 0 | 0 | 0 | 1 | 1 | 1 |
| | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 |

A similar method to construct a strongly balanced uniform RMD($t$, $t^2$, $2t$) was reported in Berenblut (1964).

Another kind of optimal strongly balanced RMD is provided by the following:

THEOREM 3.3. *Let $n = \lambda_1 t$, $p = \lambda_2 t + 1$, $\lambda_1, \lambda_2 \geq 1$, and let $d^*$ be a strongly balanced* RMD($t$, $n$, $p$) *which is uniform on the periods and is uniform on the units in the first $p - 1(= \lambda_2 t)$ periods. Then $d^*$ is universally optimal for the estimation of direct as well as residual effects over $\Omega_{t,n,p}$.*

PROOF. It can be easily checked that $C_{d^*12} = M_{d^*} - n^{-1}N_{d^*p}\tilde{N}'_{d^*p} - p^{-1}N_{d^*u}\tilde{N}'_{d^*u} + n^{-1}p^{-1}N_{d^*u}J_{n,n}\tilde{N}'_{d^*u} = \lambda_1\lambda_2 J_{t,t} - n^{-1}(p-1)\lambda_1^2 J_{t,t} - p^{-1}\lambda_2(\lambda_1 + n\lambda_2)J_{t,t} + n^{-1}p^{-1}r(r-\lambda_1)J_{t,t} \equiv 0$, where $r = t^{-1}np = p\lambda_1 = \lambda_2 n + \lambda_1$. Hence $C_{d^*} = C_{d^*11}$ and $\tilde{C}_{d^*} = C_{d^*22}$ which are completely symmetric. Theorem 3.3 follows from Lemma 2.5 and the same argument as in the proof of Theorem 3.1. □

A simple example of the design $d^*$ in Theorem 3.3 is the one obtained by repeating the observations in the last period of a balanced uniform RMD($t$, $\lambda_1 t$, $t$). The resulting design is an RMD($t$, $\lambda_1 t$, $t + 1$) which is clearly uniform on the periods and is uniform on the units in the first $p - 1(= t)$ periods. In the original design, each treatment is preceded by any *other* treatment $\lambda_1$ times. By repeating the observations in the last period we make each treatment also precede itself $\lambda_1$ times since $n = \lambda_1 t$. So the augmented design is strongly balanced. In summary, we have

COROLLARY 3.3.1. *Let $d^*$ be obtained by repeating the observations in the last period of a balanced uniform* RMD($t$, $\lambda_1 t$, $t$). *Then $d^*$ is universally optimal for the direct as well as residual effects over $\Omega_{t,\lambda_1 t,t+1}$.*

Therefore, from each of the Hedayat and Afsarinejad's balanced uniform designs, we can construct an optimum design satisfying the conditions in Theorem 3.3. The conditions for the existence of the designs in Theorem 3.3 are less restrictive than those for strongly balanced uniform designs. It is not necessary that $t^2 \mid n$ and $p \geq 2t$. So it is possible to construct optimal designs with fewer units and periods.

Designs satisfying the conditions in Theorem 3.3 can also be constructed from strongly balanced uniform ones. Let $d^*$ be a strongly balanced uniform RMD($t$, $n$, $p$). Then since $t^2 \mid n$, an extra period can be added so that the resulting design is still strongly balanced. This design certainly satisfies the conditions in Theorem 3.3 and hence is universally optimal over $\Omega_{t,n,p+1}$.

If we restrict the competing designs to a smaller class, then some stronger optimality results can be proved. Let $\Omega^*_{t,n,p} = \{d \in \Omega_{t,n,p}: r_{d_1} = r_{d_2} = \cdots = r_{d_t}\}$, i.e., $\Omega^*_{t,n,p}$ is the collection of all equally replicated designs in $\Omega_{t,n,p}$. Then we have

THEOREM 3.4. *Let $d^*$ be a strongly balanced uniform* RMD($t$, $n$, $p$). *Then $d^*$ minimizes the variance of the best linear unbiased estimator of any contrast among the direct effects $\{\tau_i\}_1^t$ over $\Omega^*_{t,n,p}$.*

PROOF. By a theorem of Ehrenfeld (1955), it suffices to show that $C_{d^*} \geq C_d$ for any $d \in \Omega^*_{t,n,p}$. Since both $C_d$ and $C_{d^*}$ have zero row sums, it is enough to show that $x'C_{d^*}x \geq x'C_d x$ for any $t \times 1$ vector $x$ such that $x'1_t = 0$ where $1_t$ is the $t \times 1$ vector of ones. By the computation in the proof of Theorem 3.1, $C_{d^*} = C_{d^*11} = rI_t - aJ_{t,t}$ for some constant $a$, where $r = t^{-1}np$. On the other hand, for any $d \in \Omega^*_{t,n,p}$, $C_d = C_{d11} - C_{d12}C^-_{d22}C_{d21} \leq C_{d11} = rI_t - n^{-1}N_{dp}N'_{dp} - (p^{-1}N_{du}N'_{du} - n^{-1}p^{-1}N_{du}J_{n,n}N'_{du}) \leq rI_t$, since by Lemma 2.2, both $n^{-1}N_{dp}N'_{dp}$ and $p^{-1}N_{du}N'_{du} - n^{-1}p^{-1}N_{du}J_{n,n}N'_{du}$ are nonnegative definite. Thus

$$x'C_d x \leq x'(rI_t)x = x'(rI_t - aJ_{t,t})x \qquad (\text{since } x'J_{t,t} = 0)$$
$$= x'C_{d^*}x. \qquad \square$$

Similarly, we have

THEOREM 3.5. *Let $d^*$ be an* RMD($t$, $n$, $p$) *satisfying the conditions in Theorem* 3.3. *Then $d^*$ minimizes the variance of the best linear unbiased estimator of any contrast among the residual effects $\{\rho_i\}_1^t$ over $\Omega^{**}_{t,n,p}$,* where $\Omega^{**}_{t,n,p} = \{d \in \Omega_{t,n,p}: \tilde{r}_{d1} = \cdots = \tilde{r}_{dt}\}$.

**4. Optimality of balanced repeated measurements designs.** In this section we will discuss the optimality of balanced uniform RMD over the class of designs $d$ in $\Omega_{t,n,p}$ for which $m_{dii} = 0$ for $1 \le i \le t$ (i.e., no treatment is allowed to be preceded by itself in $d$). The collection of such designs is denoted by $\Lambda_{t,n,p}$. In order to prove optimality we sometimes impose uniformity on units or periods on the class of competing designs. A related paper by Hedayat and Afsarinejad (1978) gave the same optimality result under the more restrictive assumption that the competing designs are uniform on both units and periods.

More precisely, recall that the definition of a uniform RMD$(t, n, p)$ implies $n = \lambda_1 t$ and $p = \lambda_2 t$ for some $\lambda_1$, $\lambda_2$. Therefore, for the rest of the section, we will only consider RMD$(t, \lambda_1 t, \lambda_2 t)$. For a balanced uniform RMD$(t, \lambda_1 t, \lambda_2 t)$ $d^*$,

$$(4.1)\qquad n_{d^*iu} = \lambda_2, \qquad l_{d^*ik} = \lambda_1, \qquad m_{d^*ii} = 0, \qquad m_{d^*ij} = \frac{\lambda_1(p-1)}{t-1} = \lambda,$$

for all $i$, $u$, $k$ and $i \neq j$, which, in turn, give

$$(4.2)\qquad \begin{aligned} &C_{d^*11} = \lambda_1 p(I_t - t^{-1}J_{t,t}), \qquad C_{d^*12} = C'_{d^*21} = -\lambda(I_t - t^{-1}J_{t,t}),\\ &C_{d^*22} = \lambda_1(p - 1 - p^{-1})(I_t - t^{-1}J_{t,t}). \end{aligned}$$

Therefore, $C_{d^*}$ and $\tilde{C}_{d^*}$ are both completely symmetric. To prove universal optimality, it remains to show that $d^*$ maximizes $\operatorname{tr}(C_{d11} - C_{d12}C_{d22}^{-}C_{d21})$ and $\operatorname{tr}(C_{d22} - C_{d21}C_{d11}^{-}C_{d12})$ over the class of competing designs. However, unlike the strongly balanced RMD, the $C_{d^*12}$ matrix in (4.2) is not zero. It is thus necessary to compute the very intractable $C_{d11}^{-}$ and $C_{d22}^{-}$ for an arbitrary $d$, which is the major technical difficulty for the optimality proof. In the case of $C_{d11}^{-}$, the following technique circumvents this problem. Since $N_{dp}N'_{dp}$ and $N_{du}N'_{du} - n^{-1}N_{du}J_{n,n}N'_{du}$ are nonnegative definite from Lemma 2.2, $C_{d11} \le D_d$ for any $d \in \Omega_{t,n,p}$. From Theorem 5(i) of Wu (1980), there exists a $g$-inverse $C_{d11}^{-}$ of $C_{d11}$ such that $C_{d11}^{-} \ge D_d^{-1}$ for any $d$. Furthermore, $D_{d^*}^{-1}$ is a $g$-inverse of $C_{d^*11}$ for any balanced uniform RMD $d^*$. Therefore, to prove that $d^*$ maximizes $\operatorname{tr}(C_{d22} - C_{d21}C_{d11}^{-}C_{d12})$ over $\Lambda_{t,\lambda_1 t,\lambda_2 t}$, it suffices to prove that $d^*$ maximizes the more tractable $\operatorname{tr}(C_{d22} - C_{d21}D_d^{-1}C_{d12})$ over $\Lambda_{t,\lambda_1 t,\lambda_2 t}$. With this basic idea in mind, we will defer the technical proofs to the next section and only summarize the optimality results here.

THEOREM 4.1. *Let $d^*$ be a balanced uniform design in $\Lambda_{t,\lambda_1 t,\lambda_2 t}$, $t \ge 3$. Then $d^*$ is universally optimal for the estimation of residual effects over the class of designs $d$ in $\Lambda_{t,\lambda_1 t,\lambda_2 t}$ with $\bar{r}_{di} = \lambda_1(p-1)$ for all $i$,* i.e., *$d$ is equally-replicated in the first $p-1$ periods.*

The reason for imposing the equal $\bar{r}_{di}$ requirement on the class of competing designs is purely technical. For $\lambda_2 = 1$, such a restriction can be removed.

THEOREM 4.2. *Let $d^*$ be a balanced uniform design in $\Lambda_{t,\lambda_1 t,t}$, $t \ge 3$. Then $d^*$ is universally optimal for the estimation of residual effects over $\Lambda_{t,\lambda_1 t,t}$.*

The proof of optimality for the estimation of direct effects is more difficult. Unlike the residual effect case, $C_{d^*22}^{-} - \tilde{D}_{d^*}^{-1} = \lambda^{-1}(p - 1 - p^{-1})I_t - \lambda^{-1}(p-1)^{-1}I_t$ is not zero and the minimization of $\operatorname{tr}(C_{d12}C_{d22}^{-}C_{d21})$ over $\Lambda_{t,\lambda_1 t,\lambda_2 t}$ can not be replaced by the minimization of the more tractable $\operatorname{tr}(C_{d12}\tilde{D}_d^{-1}C_{d21})$ over $\Lambda_{t,\lambda_1 t,\lambda_2 t}$. To make the computation possible, we impose a uniformity condition on each unit and the last period.

THEOREM 4.3. *Let $d^*$ be a balanced uniform design in $\Lambda_{t,\lambda_1 t,\lambda_2 t}$. Then $d^*$ is universally optimal for the estimation of direct effects over the class of designs in $\Lambda_{t,\lambda_1 t,\lambda_2 t}$ which are uniform on each unit and the last period.*

When the period effect is not present in model (1.1), the optimality results are even stronger.

The $C_{dij}$ matrices in (2.6) become

$$C_{d11} = D_d - p^{-1}N_{du}N'_{du}, \qquad C_{d12} = C'_{d21} = M_d - p^{-1}N_{du}\tilde{N}'_{du}, \tag{4.3}$$
$$C_{d22} = \tilde{D}_d - p^{-1}\tilde{N}_{du}\tilde{N}'_{du},$$

Let $d^*$ be a RMD($t$, $\lambda_1 t$, $\lambda_2 t$) which is

(i) uniform on each unit

(ii) uniform in the last period

(4.4) (iii) $m_{d^*ij} = \dfrac{\lambda_1(p-1)}{t-1} = \lambda$ for $i \neq j$ and $m_{d^*ii} = 0$ (i.e., in the order of application, each treatment is preceded by each other treatment the same number of times.)

From (2.2) and (2.3), one can easily verify that (i) and (iii) imply (ii) and the uniformity in the first period.

THEOREM 4.4. *Assume there is no period effect in model* (1.1). *The design $d^*$ defined in* (4.4) *is universally optimal for the estimation of residual effects over $\Lambda_{t,\lambda_1 t,\lambda_2 t}$. The same design is also universally optimal for the estimation of direct effects over the class of designs in $\Lambda_{t,\lambda_1 t,\lambda_2 t}$ which are uniform on each unit and the last period.*

**5. Proofs.** The optimality proofs require some preliminary lemmas on constrained minimization. Unless otherwise stated, all the variables below are assumed to be real.

LEMMA 5.1. *The minimum of $\sum_{i=1}^t x_i^2$, subject to $\sum_{i=1}^t x_i = d$, is achieved by taking $x_i = \dfrac{d}{t}$ for all $i$ and is equal to $\dfrac{d^2}{t}$.*

LEMMA 5.2. *The minimum of $\sum_{i=1}^t \dfrac{x_i^2}{r_i}$, subject to $\sum_{i=1}^t x_i = d$, $r_i \geq 0$ and $\sum_{i=1}^t r_i = e > 0$, is achieved by taking $x_i = \dfrac{d}{e} r_i$ for all $i$ and is equal to $\dfrac{d^2}{e}$.*

LEMMA 5.3. *Suppose two sequences $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ are similarly ordered in the sense that $(x_i - x_j)(y_i - y_j) \geq 0$ for all $i, j$, then*

$$\sum_{i=1}^n x_i y_i \geq \frac{1}{n}\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right).$$

The first two lemmas can be easily proved by the method of Lagrange multiplier. Lemma 5.3 can be found in Hardy, Littlewood and Polya (1967), page 43.

LEMMA 5.4. *Suppose $(x_i)_{i=1}^t$, $\lambda_2$ and $t$ are all nonnegative integers. Then the minimum of $\sum_{i=1}^{t-1} x_i^2 + x_t(x_t - 1)$, subject to $x_t \geq 1$ and $\sum_{i=1}^t x_i = \lambda_2 t$, is achieved by taking $x_i = \lambda_2$ and is equal to $\lambda_2^2 t - \lambda_2$.*

PROOF. We want to show that

$$\sum_{i=1}^t x_i^2 - x_t \geq \lambda_2^2 t - \lambda_2 \tag{5.1}$$

for $\sum_{i=1}^t x_i = \lambda_2 t$, $x_t \geq 1$ and $x_i$ are nonnegative integers. (5.1) is equivalent to

$$\sum_{i=1}^t x_i^2 - \lambda_2^2 t \geq x_t - \lambda_2. \tag{5.2}$$

From Lemma 5.1, the left-hand side of (5.2) is always nonnegative. Therefore, (5.2) holds if $x_t \le \lambda_2$. On the other hand, if $x_t = \lambda_2 + m$, $m$ is a positive integer, then

$$\sum_{i=1}^{t} x_i^2 = \sum_{i=1}^{t-1} x_i^2 + x_t^2 \ge (t-1)\left(\frac{\lambda_2 t - \lambda_2 - m}{t-1}\right)^2 + (\lambda_2 + m)^2 = \lambda_2^2 t + m^2 \frac{t}{t-1}.$$

This implies that $\sum_{i=1}^{t} x_i^2 - \lambda_2^2 t \ge m^2 \frac{t}{t-1} > m = x_t - \lambda_2$, proving (5.2). □

Lemma 5.5. *The matrix $A = [a_{ij}]$ with $a_{ii} = 1$ for $1 \le i \le n$, $a_{ij} = b$ for $|i-j| = 1$ and $a_{ij} = 0$ otherwise has eigenvectors* $v_i = \left(\sin\frac{i\pi}{n+1}, \sin\frac{2i\pi}{n+1}, \cdots, \sin\frac{ni\pi}{n+1}\right)$ *with corresponding eigenvalues* $1 + 2b\cos\frac{i\pi}{n+1}$ *for $1 \le i \le n$. In particular, $A$ is nonsingular when* $|b| < \left(2\cos\frac{\pi}{n+1}\right)^{-1}$

This can be verified by routine matrix multiplication and trigonometric identities. It is stated in Noble (1969), page 307.

Lemma 5.6. *For $0 \le c \le [2t(p-1)\lambda_1^2]^{-1}$, the minimum of* $F(\mathbf{x}) = \sum_{i=1}^{t}\sum_{k=1}^{p-1} x_{ik}^2 + c(\sum_{i=1}^{t}\sum_{k=2}^{p} x_{ik}x_{i,k-1})^2$, *subject to $\sum_{i=1}^{t} x_{ik} = \lambda_1 t$ for $1 \le k \le p$, is achieved by taking $x_{ik} = \lambda_1$ for all $i, k$.*

Proof. It is obvious that the minimum must be attained at some bounded $x_{ik}$ values. Therefore, the minimum must also be a local minimum. Rewrite $F(\mathbf{x})$ as

$$\sum_{i=1}^{t-1}\sum_{k=1}^{p-1} x_{ik}^2 + \sum_{k=1}^{p-1}(\lambda_1 t - \sum_{i=1}^{t-1} x_{ik})^2 + c\{\sum_{i=1}^{t-1}\sum_{k=2}^{p} x_{ik}x_{i,k-1} + \sum_{k=2}^{p}(\lambda_1 t - \sum_{i=1}^{t-1} x_{ik})(\lambda_1 t - \sum_{i=1}^{t-1} x_{i,k-1})\}^2.$$

A necessary condition for local minimum is

$$\frac{\partial}{\partial x_{ik}} F(\mathbf{x}) = 0, \quad \text{for} \quad 1 \le i \le t-1,\ 1 \le k \le p,$$

which gives

$$\begin{aligned} &x_{i1} - x_{t1} + cA(\mathbf{x})(x_{i2} - x_{t2}) = 0, \\ (5.3)\quad &x_{ik} - x_{tk} + cA(\mathbf{x})(x_{i,k+1} - x_{t,k+1} + x_{i,k-1} - x_{t,k-1}) = 0 \quad \text{for} \quad 2 \le k \le p-1 \\ &cA(\mathbf{x})(x_{i,p-1} - x_{t,p-1}) = 0, \end{aligned}$$

where $A(\mathbf{x}) = \sum_{i=1}^{t}\sum_{k=2}^{p} x_{ik}x_{i,k-1}$.

(i) If $A(\mathbf{x}) = 0$ at the local minimum, then (5.3) implies $x_{ik} = x_{tk}$ for $1 \le i \le t-1$, $1 \le k \le p-1$, which is equivalent to $x_{ik} = \lambda_1$ for $1 \le i \le t$, $1 \le k \le p-1$. But this, in turn, implies that $A(\mathbf{x}) > 0$. Therefore, at the local minimum $A(x) \ne 0$.

(ii) Since $\sum_{i=1}^{t}\sum_{k=1}^{p-1} x_{ik}^2$ is minimized by $x_{ik} = \lambda_1$ for $1 \le i \le t$, $1 \le k \le p-1$ and the corresponding $A(\mathbf{x})$ value is $t(p-1)\lambda_1^2$, any $\mathbf{x} = (x_{ik})$ with $|A(\mathbf{x})| > t(p-1)\lambda_1^2$ can not give the global minimum of $F(\mathbf{x})$.

(iii) For any $\mathbf{x}$ with $0 < |A(\mathbf{x})| \le t(p-1)\lambda_1^2$, we want to show that the only solution of (5.3) is $x_{ik} - x_{tk} = 0$ for all $1 \le i \le t-1$, $1 \le k \le p$. Since (5.3) is equivalent to $B\mathbf{y}_i = 0$ for $1 \le i \le t-1$, where $\mathbf{y}_i = (x_{i1} - x_{t1}, \cdots, x_{ip} - x_{tp})$ and $B = [b_{ik}]$ with $b_{ii} = 1$ for $i < p$, $b_{pp} = 0$, $b_{ik} = cA(\mathbf{x})$ for all $|i-k| = 1$ and $b_{ik} = 0$ otherwise, it remains to prove the nonsingularity of $B$. From $cA(\mathbf{x}) > 0$ and the special form of $B$, it can be seen that $B$ is nonsingular if and only if its upper left $(p-2) \times (p-2)$ submatrix, called $B_{11}$, is nonsingular. (This can be done by showing that the $p$ column or row vectors of $B$ are linearly independent iff the $p-2$ column or row vectors of $B_{11}$ are linearly independent.) In particular, $B$ is always nonsingular for $p \le 3$, $cA(\mathbf{x}) > 0$. The matrix $B_{11}$ is a special form of the matrix $A$ in Lemma 5.5 with $b = cA(\mathbf{x})$

and $n = p - 2$. From Lemma 5.5, $B_{11}$ is nonsingular since $cA(\mathbf{x}) \le (2t(p-1)\lambda_1^2)^{-1}t(p-1)\lambda_1^2 \le \frac{1}{2}$.

(iv) From (iii), a necessary condition for any $\mathbf{x}$ with $0 < A(\mathbf{x}) \le t(p-1)\lambda_1^2$ is that $x_{ik} = z_k$ for $1 \le i \le t$, $1 \le k \le p$. From the constraint condition $\sum_{i=1}^{t} x_{ik} = \lambda_1 t$, we have $x_{ik} = z_k = \lambda_1$. Together with (ii), we have shown that the global minimum of $F(\mathbf{x})$ is achieved by taking $x_{ik} = \lambda_1$ for all $i$, $k$. □

LEMMA 5.7. *For $0 \le c \le [tp(p-1)\lambda_1^2]^{-1}$, the minimum of $F(x) = \sum_{i=1}^{t}\sum_{k=1}^{p-1} x_{ik}^2 - p^{-1}\sum_{i=1}^{t}(\sum_{k=1}^{p-1} x_{ik})^2 + c(\sum_{i=1}^{t}\sum_{k=2}^{p} x_{ik}x_{i,k-1})^2$, subject to $\sum_{i=1}^{t} x_{ik} = \lambda_1 t$ for $1 \le k \le p$, is achieved by taking $x_{ik} = \lambda_1$ for all $i$, $k$.*

PROOF. Since the argument is very similar to that of Lemma 5.6, the proof will be sketched briefly. By rewriting $x_{tk} = \lambda_1 t - \sum_{i=1}^{t-1} x_{ik}$ in $F(\mathbf{x})$, a necessary condition for the local minimum of $F(\mathbf{x})$ is,

$$(5.4)\qquad \begin{aligned} &x_{i1} - x_{t1} - p^{-1}\textstyle\sum_{k=1}^{p-1}(x_{ik} - x_{tk}) + cA(\mathbf{x})(x_{i2} - x_{t2}) = 0,\\ &x_{ik} - x_{tk} - p^{-1}\textstyle\sum_{k=1}^{p-1}(x_{ik} - x_{tk})\\ &\quad + cA(\mathbf{x})(x_{i,k+1} - x_{t,k+1} + x_{i,k-1} - x_{t,k-1}) = 0, \qquad \textit{for}\quad 2 \le k \le p-1,\\ &cA(\mathbf{x})(x_{i,p-1} - x_{t,p-1}) = 0, \end{aligned}$$

where $A(\mathbf{x}) = \sum_{i=1}^{t}\sum_{k=2}^{p} x_{ik}x_{i,k-1}$.

A very similar argument to (i), (ii) of the proof of Lemma 5.6 shows that $|A(\mathbf{x})|$ can not be equal to zero or greater than $t(p-1)\lambda_1^2$. For any $\mathbf{x}$ with $0 < |A(\mathbf{x})| \le t(p-1)\lambda_1^2$, it remains to show that the only solution of (5.4) is $x_{ik} - x_{tk} = 0$ for all $1 \le i \le t-1$, $1 \le k \le p$. The linear system (5.4) is defined as $\tilde{B}\mathbf{y}_i = 0$ for $1 \le i \le t-1$, $\mathbf{y}_i = (x_{i1} - x_{t1}, \cdots, x_{ip} - x_{tp})$, $\tilde{B} = B - p^{-1}J_p$, where $B$ is given in the proof of Lemma 5.6, $J_p$ is the $p \times p$ matrix with all entries 1. Since $cA(\mathbf{x}) > 0$, $\tilde{B}$ is nonsingular iff its upper left $(p-2) \times (p-2)$ submatrix $\tilde{B}_{11}$ is nonsingular. Note that $\tilde{B}_{11} = B_{11} - p^{-1}J_{p-2}$, $B_{11}$ is defined in the proof of Lemma 5.6 and $J_{p-2}$ is the $(p-2) \times (p-2)$ matrix with all entries 1. From Lemma 5.5, the smallest eigenvalue of $B_{11}$ is $1 - 2c|A(\mathbf{x})|\cos\frac{\pi}{p-1}$ and the largest eigenvalue of $p^{-1}J_{p-2}$ is $\frac{p-2}{p}$. Therefore, $\tilde{B}_{11}$ is nonsingular since $p^{-1} - |cA(\mathbf{x})|\cos\frac{\pi}{p-1} > 0$, which is satisfied since $|A(\mathbf{x})| \le t(p-1)\lambda_1^2$ and $c \le (pt(p-1)\lambda_1^2)^{-1}$. □

PROOF OF THEOREM 4.1. $\tilde{C}_{d^*}$ is clearly completely symmetric. It remains to show that tr $C_{d22}$ − tr $C_{d21}C_{d11}^{-}C_{d12}$ is maximized by $d^*$ over the specified class of designs. Since $C_{d11}^{-} = D_d^{-1}$ for $d = d^*$ and in general there exists a generalized inverse $C_{d11}^{-} \ge D_d^{-1}$ (see the paragraph before Theorem 4.1), we proceed to show that tr $C_{d22}$ − tr $C_{d21}D_d^{-1}C_{d12}$ is maximized by $d^*$.

First, note that

$$\operatorname{tr} C_{d22} = \sum_{i=1}^{t} \tilde{r}_{di} - n^{-1}\sum_{i=1}^{t}\sum_{k=1}^{p-1} l_{dik}^2 - p^{-1}\sum_{i=1}^{t}\sum_{u=1}^{n} \tilde{n}_{diu}^2 + n^{-1}p^{-1}\sum_{i=1}^{t} \tilde{r}_{di}^2,$$

and

$$\operatorname{tr} C_{d21}D_d^{-1}C_{d12} = \sum_{i=1}^{t} r_{di}^{-1}\sum_{j=1}^{t}\Big(m_{dij} - n^{-1}\sum_{k=2}^{p} l_{dik}l_{dj,k-1} - p^{-1}\sum_{u=1}^{n} n_{diu}\tilde{n}_{dju} + n^{-1}p^{-1}r_{di}\tilde{r}_{dj}\Big)^2.$$

Since $\tilde{r}_{di}$ is constant over the class of competing designs and $\sum_{i=1}^{t}\sum_{u=1}^{n} \tilde{n}_{diu}^2$ is minimized by $d^*$ according to Lemma 2.3, it suffices to show that $d^*$ minimizes

$$(5.5)\qquad n^{-1}\sum_{i=1}^{t}\sum_{k=1}^{p-1} l_{dik}^2 + \sum_{i=1}^{t} r_{di}^{-1}\sum_{j=1}^{t}(m_{dij} - q_{dij})^2,$$

where $q_{dij} = n^{-1}\sum_{k=2}^{p} l_{dik}l_{dj,k-1} + p^{-1}\sum_{u=1}^{n} n_{diu}\tilde{n}_{dju} - n^{-1}p^{-1}r_{di}\tilde{r}_{dj}$, subject to the constraints $\sum_{i=1}^{t} l_{dik} = n$, $m_{dii} = 0$, $\sum_{i=1}^{t} r_{di} = np$ and $\sum_{j=1}^{t}(m_{dij} - q_{dij}) = \sum_{i=1}^{t}(m_{dij} - q_{dij}) = 0$ (corresponding to the fact that the row and column sums of $C_{d12}$ are zero). This minimization problem will be solved in several steps.

(i) $\sum_{j=1}^{t} (m_{dij} - q_{dij})^2 = \sum_{j,j\neq i} (m_{dij} - q_{dij})^2 + q_{dii}^2$ and the constraint is $\sum_{j,j\neq i} (m_{dij} - q_{dij}) = q_{dii}$, since $m_{dii} = 0$. From Lemma 5.1, this is minimized by taking $m_{dij} - q_{dij} = \frac{1}{t-1} q_{dii}$ for $j \neq i$ and the minimum is $\frac{t}{t-1} q_{dii}^2$. It can be checked that $m_{dij} - q_{dij} = \frac{1}{t-1} q_{dii}$ holds for $d = d^*$.

(ii) From Lemma 5.2, the minimum of $\sum_{i=1}^{t} r_{di}^{-1} q_{dii}^2$, subject to $\sum_{i=1}^{t} r_{di} = np$, is achieved by taking $r_{di} = \lambda_2 n$ and $q_{dii} = t^{-1} \sum_{i=1}^{t} q_{dii}$ and is equal to $(t\lambda_2 n)^{-1} (\sum_{i=1}^{t} q_{dii})^2$. Again, for $d^*$, it can be checked that $r_{d^*i} = \lambda_2 n$ and $q_{d^*ii} = t^{-1} \sum_{i=1}^{t} q_{d^*ii}$ do hold.

(iii) In $\sum_{i=1}^{t} q_{dii} = n^{-1} \sum_{i=1}^{t} \sum_{k=2}^{p} l_{dik} l_{di,k-1} + p^{-1} \sum_{i=1}^{t} \sum_{u=1}^{n} n_{diu} \tilde{n}_{diu} - n^{-1} p^{-1} \sum_{i=1}^{t} r_{di} \tilde{r}_{di} = n^{-1} \sum_{i=1}^{t} \sum_{k=2}^{p} l_{dik} l_{di,k-1} + p^{-1} \sum_{i=1}^{t} \sum_{u=1}^{n} (n_{diu} - n^{-1} r_{di}) \cdot (\tilde{n}_{diu} - n^{-1} \tilde{r}_{di})$, $\sum_{u=1}^{n} (n_{diu} - n^{-1} r_{di})(\tilde{n}_{diu} - n^{-1} \tilde{r}_{di}) \geq 0$ and equals zero for $d^*$ (which gives $n_{d^*iu} = \lambda_2 = n^{-1} r_{d^*i}$). This follows from Lemma 5.3 if one can show that $(n_{diu})_{u=1}^{n}$ and $(\tilde{n}_{diu})_{u=1}^{n}$ are similarly ordered for all $i$. For any $u$ and $v$, if $n_{diu} > n_{div}$, then $n_{diu} \geq n_{div} + 1$ which implies $\tilde{n}_{diu} \geq \tilde{n}_{div}$ and $(n_{diu} - n_{div})(\tilde{n}_{diu} - \tilde{n}_{div}) \geq 0$ holds. Other cases are proved similarly. Therefore $(\sum_{i=1}^{t} q_{dii})^2 \geq n^{-2} (\sum_{i=1}^{t} \sum_{k=2}^{p} l_{dik} l_{di,k-1})^2$ and equality holds for $d^*$.

(iv) From (i) (ii) (iii), it remains to show that

$$n^{-1} \sum_{i=1}^{t} \sum_{k=1}^{p-1} l_{dik}^2 + \frac{t}{t-1} \frac{1}{nt\lambda_2} \frac{1}{n^2} (\sum_{i=1}^{t} \sum_{k=2}^{p} l_{dik} l_{di,k-1})^2,$$

subject to $\sum_{i=1}^{t} l_{dik} = n$ for all $i$, is minimized by $d^*$ (which gives $l_{d^*ik} = \lambda_1$ for all $i$, $k$). This follows easily from Lemma 5.6, since $[(t-1)\lambda_2 n^2]^{-1} \leq [2t(p-1)\lambda_1^2]^{-1}$ holds for $t \geq 3$. □

PROOF OF THEOREM 4.2. The proof will be given by simply modifying that of Theorem 4.1. All the notations are the same. Only the steps involving different arguments are given below.

The complete symmetry of $\tilde{C}_{d^*}$ is obvious. We want to show that $d^*$ also maximizes $\mathrm{tr}(C_{d22} - C_{d21} D_d^{-1} C_{d12})$ over $\Lambda_{t,\lambda_1,t,t}$. Since $\tilde{r}_{di}$ is not assumed constant, the minimization problem (5.5) is replaced, in this case, by

$$n^{-1} \sum_{i=1}^{t} \sum_{k=1}^{p-1} l_{dik}^2 - n^{-1} p^{-1} \sum_{i=1}^{t} (\sum_{k=1}^{p-1} l_{dik})^2 + \sum_{i=1}^{t} r_{di}^{-1} \sum_{j=1}^{t} (m_{dij} - q_{dij})^2, \tag{5.6}$$

subject to the same set of constraints. Here, $\tilde{r}_{di} = \sum_{k=1}^{p-1} l_{dik}$ is used. From (i) (ii) (iii) of the proof of Theorem 4.1, the third term of (5.6) is greater than or equal to $[(t-1)\lambda_2 n^3]^{-1} (\sum_{i=1}^{t} \sum_{k=2}^{p} l_{dik} l_{di,k-1})^2$ and equality holds for $d^*$. The proof will be completed by showing that

$$\sum_{i=1}^{t} \sum_{k=1}^{p-1} l_{dik}^2 - p^{-1} \sum_{i=1}^{t} (\sum_{k=1}^{p-1} l_{dik})^2 + [(t-1)\lambda_2 n^2]^{-1} (\sum_{i=1}^{t} \sum_{k=2}^{p} l_{dik} l_{di,k-1})^2$$

is minimized by $d^*$ which gives $l_{d^*ik} = \lambda_1$ for all $i$, $k$. However, this follows from Lemma 5.7, since $[(t-1)\lambda_2 n^2]^{-1} \leq [tp(p-1)\lambda_1^2]^{-1}$ for $p = t (\Leftrightarrow \lambda_2 = 1)$. □

PROOF OF THEOREM 4.3. The complete symmetry of $C_{d^*}$ follows easily from computation (5.7). Since the class of competing designs are uniform on each unit and in the last period, one can easily verify that

$$\begin{aligned} C_{d11} &= \lambda_1 p(I - t^{-1} J_{t,t}), \qquad C_{d12} = C_{21}' = M_d - t^{-1}\lambda_1(p-1) J_{t,t}, \\ C_{d22} &= \lambda_1 (p - 1 - p^{-1})[I_t - t^{-1} J_{t,t}]. \end{aligned} \tag{5.7}$$

Since tr $C_{d11}$ is a constant and $C_{d22}^{-} = \lambda_1^{-1}(p - 1 - p^{-1})^{-1} I_t$, $\mathrm{tr}(C_{d11} - C_{d12} C_{d22}^{-} C_{d21})$ is maximized iff tr $C_{d12} C_{d21}$ is minimized. However, tr $C_{d12} C_{d21} = \sum_{i,j=1}^{t} (m_{dij} - t^{-1}\lambda_1(p-1))^2$ is minimized, over the class of designs with $m_{dii} = 0$, by $d^*$ (which gives $m_{d^*ij} = t^{-1}\lambda_1(p-1)$ for all $i \neq j$). □

PROOF OF THEOREM 4.4. The idea of proof is essentially the same as that of Theorems 4.1 and 4.3. The only difference is for the residual effect case, where part (iii) of the proof of Theorem 4.1 is replaced by the minimization of $\sum_{u=1}^{n} \sum_{i=1}^{t} n_{diu}\tilde{n}_{diu}$, subject to $\sum_{i=1}^{t} n_{diu} = p$, which is minimized by $n_{d^*iu} = \lambda_2$ according to Lemma 5.4. Note that $n_{diu} = \tilde{n}_{diu}$ for all $i$ not equal to some $i_o$ and $n_{di_ou} = \tilde{n}_{di_ou} + 1$. □

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