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ON MAIN EFFECT PLUS ONE PLANS FOR 2^m FACTORIALS

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In this paper, we obtain main effect plans with $N = m + 1 + (m - h)(h + 1)$ treatments for 2^m factorial experiments, $m = 2^h - 1$, $h(\geq 2)$ an integer, which permit search and estimation of one possible nonzero effect among two factor and higher order interactions.

1. Introduction. The problem of finding main effect plus one plans for 2^m factorials, where $m = 2^h - 1$ and $h(\geqslant 2)$ is an integer, was first considered in [5]. A sufficient condition for a plan to be 'main effect plus one' stated in Theorem 6.2 in [5], was incomplete and hence, incorrect, because one case was not taken into consideration. In this paper, we not only correct the theorem but also present several new results.

2. Search linear models. Consider the linear model

$$E(\mathbf{y}) = A_1\xi_1 + A_2\xi_2, \tag{1}$$

$$V(\mathbf{y}) = \sigma^2 I_N, \tag{2}$$

where $\mathbf{y}$ $(N \times 1)$ is a vector of observations and for $i = 1,2$, $A_i(N \times \nu_i)$ are known matrices, $\xi_i(\nu_i \times 1)$ are vectors of fixed parameters, and σ^2 is a constant which may or may not be known. Furthermore, ξ_1 is completely unknown, but we have partial information about ξ_2. We know that at most k elements of ξ_2 are nonzero and the remaining elements are negligible, where k is a nonnegative integer which may or may not be known. In this paper, we assume k is known to be 1. However, we do not know exactly which element of ξ_2 is nonzero. The problem is to make inferences about the elements of ξ_1 and, moreover, to search the nonzero element of ξ_2 and make inferences about it. Such models are called search linear models with fixed effects and were introduced by Srivastava in [4]. We want A_1 and A_2 to be such that the above problem can be resolved; the underlying design corresponding to A_1 and A_2 is called a search design.

The case when $\sigma^2 = 0$ is called the 'noiseless case', and is of great importance from the design point of view (see [5]). We now state a result of Srivastava in the special case for $k = 1$.

THEOREM 1. *Consider the model* (1, 2) *and let* $\sigma^2 = 0$, $k = 1$. *A necessary and sufficient condition that the search and inference can be completely solved in the noiseless case, is that for every* $(N \times 2)$ *submatrix* A_{20} *of* A_2, *we have*

$$\text{Rank}[A_1 : A_{20}] = \nu_1 + 2. \tag{3}$$

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By 'completely solved', we mean that we will be able to search the nonzero element of ξ_2, without any error, and furthermore obtain estimators of the nonzero element of ξ_2 and the elements of ξ_1 which have variance zero. We say the matrix $[A_1 : A_2]$ has the property P_2^* if the condition (3) is true for every submatrix A_{20} of A_2.

3. Main effect plus one (MEP.1) plans for 2^m factorials. Consider a 2^m factorial experiment. The treatments are denoted by $(x_1, x_2, \cdots, x_m)$, where $x_r = 0$ or $1, r = 1, 2, \cdots, m$. Let the 'true effect' of the treatment $(x_1 \cdots x_m)$ be $\phi(x_1 \cdots x_m)$. We write

$$(4) \quad \phi(x_1 \cdots x_m) = \mu + \Sigma_{i=1}^m \alpha_i F_i + \Sigma_{i_1, i_2=1;\, i_1<i_2}^m \alpha_{i1}\alpha_{i2} F_{i_1 i_2} + \Sigma_{i_1, i_2, i_3=1;\, i_1<i_2<i_3}^m \alpha_{i_1}\alpha_{i_2}\alpha_{i_3} F_{i_1 i_2 i_3} + \cdots + \alpha_1 \cdots \alpha_m F_{12\cdots m},$$

where μ is the general mean, F_i the main effect of the ith factor, $F_{i_1 i_2}$ the two factor interaction between the factors i_1 and i_2, and so on, $\alpha_i = 1$ or -1 according as $x_i = 1$ or $0, i = 1, \cdots, m$. Let $y(x_1, \cdots, x_m)$ be the observed response corresponding to $(x_1, \cdots, x_m)$. Then our model for the noiseless case is

$$(5) \qquad y(x_1 \cdots x_m) = \phi(x_1 \cdots x_m).$$

Let T be a design consisting of the N treatments $(x_{r1}, x_{r2}, \cdots, x_{rm}), r = 1, \cdots, N$. We shall write T as a $(N \times m)$ matrix with the rth row $(x_{r1}, x_{r2}, \cdots, x_{rm})$. Let $\mathbf{y}(N \times 1)$ be the observation vector corresponding to T. We now define two vectors of parameters $\xi_1(\nu_1 \times 1)$ and $\xi_2(\nu_2 \times 1)$, where

$$(6) \qquad \nu_1 = m + 1, \qquad \nu_2 = 2^m - \nu_1,$$

$$(7) \qquad \xi_1' = (\mu; F_1, \cdots, F_m),$$

$$\xi_2' = (F_{12}, \cdots, F_{m-1, m}, \cdots, F_{12\cdots m}).$$

Consider the model (1, 2) with $\nu_1, \nu_2, \xi_1, \xi_2$ as in (6), (7), and A_1, A_2 as determined by T and (4). The Hadamard product of two vectors $\mathbf{a} = (a_1, a_2, \cdots, a_N)'$ and $\mathbf{b} = (b_1, b_2 \cdots b_N)'$ is defined as $\mathbf{a} * \mathbf{b} = (a_1 b_2, \cdots, a_N b_N)'$. It is important to note that the column of A_2 corresponding to the element $F_{u_1 u_2 \cdots u_l} (l \geqslant 2)$ of ξ_2 is the Hadamard product of the columns of A_1 corresponding to the elements $F_{u_1}, F_{u_2}, \cdots, F_{u_l}$ of ξ_1.

The main effect plans (or the designs of resolution III) allow estimation of ξ_1 under the assumption $\xi_2 = \mathbf{0}$ in our model.

DEFINITION 1. A search design T for a 2^m factorial experiment is said to be a 'main effect plus one plan', when in the search model $(1, 2)\nu_1, \nu_2, \xi_1, \xi_2$ are as in (6) and (7), $k = 1$; and A_1 and A_2 are determined by T and (4).

For brevity, we write 'main effect plus one plan' as MEP.1 plan.

4. Conditions on existence of MEP.1 plans for 2^m factorials. We now consider the problem of obtaining 'good' MEP.1 plans. Suppose $T = T_1 + T_2$, where for $i = 1, 2, T_i(N_i \times m)$ are designs with N_i observations, and $T(N \times n)$ with $N = N_1$

$+ N_2$. For $i = 1, 2$, let $\mathbf{y}_i(N_i \times 1)$ be the observation vector corresponding to T_i. We write

(8) $$E\mathbf{y}_1 = A_{11}\xi_1 + A_{12}\xi_2,$$

(9) $$E\mathbf{y}_2 = A_{21}\xi_1 + A_{22}\xi_2,$$

where for $i = 1, 2$, A_{i1} and A_{i2} are $(N_i \times \nu_1)$ and $(N_i \times \nu_2)$ matrices. Let T_1 be such that Rank $(A_{11}) = \nu_1$. We have $\mathbf{y}' = (\mathbf{y}'_1, \mathbf{y}'_2)$ and $A'_i = [A'_{1i}, A'_{2i}]$ for $i = 1, 2$. We first take a design T_1 with N_1 treatments which is a 'good' main effect plan. We then consider the problem of selecting a design $T_2(N_2 \times m)$ with N_2 treatments so that the design $T = T_1 + T_2$ with $N = N_1 + N_2$ treatments is a MEP.1 plan.

Suppose $m = 2^h - 1$, where $h(\geqslant 2)$ is a positive integer. We now consider the classical optimal main effect plan $T_1(N_1 \times m)$, where $N_1 = m + 1$, for which A_{11} has the following 'group structure'. Consider the colu̇mns of A_{11} corresponding to $F_1, \cdots, F_h$ and the $(m + 1 \times h)$ submatrix of A_{11}. In this submatrix, the 2^h vectors $(x_1, \cdots, x_h)$, $x_i = +1$ or -1, appear as rows. The other columns of A_{11} are filled by taking the Hadamard product of any two or more of these h columns. Clearly, $A'_{11}A_{11} = N_1 I_{N_1}$, where I_{N_1} is the identity matrix of order N_1; and, furthermore, the district columns of A_{12} are the same as the columns of A_{11}. We want $T_2(N_2 \times m)$ to be such that the matrix

(10) $$[A_1 : A_2] = \begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix}$$

has the property P_2^*. It can be checked that the matrix in (10) has P_2^* if and only if (iff) the matrix

(11) $$\begin{bmatrix} N_1^{-1}A'_{11} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & I_{N_2} \end{bmatrix} \begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix}$$

has P_2^*. The matrix in (11) can be written as

(12) $$\begin{bmatrix} I_{N_1} & \vdots & N_1^{-1}A'_{11}A_{12} \\ \cdots & \cdots & \cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix} = [B_1 : B_2], \quad \text{say.}$$

We therefore want T_2 to be such that for any two distinct columns $\mathbf{b}_1$, $\mathbf{b}_2$ of B_2

(13) $$\text{Rank}[B_1, \mathbf{b}_1, \mathbf{b}_2] = m + 3.$$

Suppose $\mathbf{e}_i(N_1 \times 1)$ is a vector whose ith coordinate is 1 and whose other coordinates are zero. Define

(14) $$B_1 = \begin{bmatrix} \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{N_1} \\ \cdots \\ \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_{N_1} \end{bmatrix}, \mathbf{b}_j = \begin{bmatrix} \mathbf{d}_j \\ \hline \boldsymbol{\delta}_j \end{bmatrix}, j = 1, 2,$$

where $\boldsymbol{\alpha}_i(N_2 \times 1)$, $i = 1, \cdots, N_1$, $\mathbf{d}_j(N_1 \times 1)$ and $\boldsymbol{\delta}_j(N_2 \times 1)$. We consider two cases (a) $\mathbf{d}_1 \neq \mathbf{d}_2$ and (b) $\mathbf{d}_1 = \mathbf{d}_2$, separately. (a) $\mathbf{d}_1 \neq \mathbf{d}_2$. Suppose $\mathbf{d}_1 = \mathbf{e}_{i_1}$ and $\mathbf{d}_2 = \mathbf{e}_{i_2}$.

THEOREM 2. *The following are equivalent:*

(i) $\text{Rank}[B_1, b_1, b_2] = m + 3$;

(ii) $\text{Rank}\begin{bmatrix} \mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \mathbf{d}_1, \mathbf{d}_2 \\ \cdots\cdots \\ \boldsymbol{\alpha}_{i_1}, \boldsymbol{\alpha}_{i_2}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \end{bmatrix} = 4$;

(iii) $\text{Rank}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{\alpha}_{i1} & \boldsymbol{\alpha}_{i2} & \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 \end{bmatrix} = 4$;

(iv) $\mathbf{0} \neq (\delta_1 - \alpha_{i_1}) \neq \pm (\delta_2 - \alpha_{i_2}) \neq \mathbf{0}$.

PROOF. The equivalence of (i)–(iii) is easy to see from the structure of B_1 and $\mathbf{e}_k (k = 1, \cdots, N_1)$. We prove only the equivalence of (iii) and (iv). Denote the 4 columns of the matrix in (iii) by $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$. Then consider

$$\theta_1 \mathbf{C}_1 + \theta_2 \mathbf{C}_2 + \theta_3 \mathbf{C}_3 + \theta_4 \mathbf{C}_4 = \mathbf{0}, \tag{15}$$

where θ_1, θ_2, θ_3 and θ_4 are constants (real numbers). Clearly, $\theta_1 + \theta_3 = 0$ and $\theta_2 + \theta_4 = 0$. Thus, we have

$$\theta_1(\boldsymbol{\delta}_1 - \boldsymbol{\alpha}_{i_1}) + \theta_2(\boldsymbol{\delta}_2 - \boldsymbol{\alpha}_{i_2}) = \mathbf{0}. \tag{16}$$

The rest follows from the fact that the distinct elements of $\boldsymbol{\delta}_k - \boldsymbol{\alpha}_{i_k}$, $k = 1, 2$, are 0, 2 and -2. This completes the proof of the theorem.

(b) $\mathbf{d}_1 = \mathbf{d}_2$. Suppose $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{e}_{i_1}$.

THEOREM 3. *The following are equivalent:*

(i) $\text{Rank}[B_1, \mathbf{b}_1, \mathbf{b}_2] = m + 3$;

(ii) $\text{Rank}\begin{bmatrix} \mathbf{e}_{i_1}, \mathbf{d}_1, \mathbf{d}_2 \\ \cdots\cdots \\ \boldsymbol{\alpha}_{i_1}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \end{bmatrix} = 3$;

(iii) $\text{Rank}\begin{bmatrix} 1 & 1 & 1 \\ \cdots & \cdots & \cdots \\ \boldsymbol{\alpha}_{i_1} & \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 \end{bmatrix} = 3$;

(iv) $\mathbf{0} \neq (\boldsymbol{\delta}_1 - \boldsymbol{\alpha}_{i_1}) \neq (\boldsymbol{\delta}_2 - \boldsymbol{\alpha}_{i_1}) \neq \mathbf{0}$.

PROOF. Similar to the proof of Theorem 2.

The results stated below are very useful in finding MEP.1 plans.

LEMMA 1. *If* $\boldsymbol{\delta}_1 * \boldsymbol{\alpha}_{i_1} \neq \boldsymbol{\delta}_2 * \boldsymbol{\alpha}_{i_2}$, i.e., *the vectors* $\boldsymbol{\delta}_1 * \boldsymbol{\alpha}_{i_1}$ *and* $\boldsymbol{\delta}_2 * \boldsymbol{\alpha}_{i_2}$ *are distinct, and, moreover,* $\boldsymbol{\delta}_k * \boldsymbol{\alpha}_{i_k} \neq \mathbf{1}$ *for* $k = 1, 2$, *where* $\mathbf{1}$ *is a vector with all elements unity, then*

$$\mathbf{0} \neq (\boldsymbol{\delta}_1 - \boldsymbol{\alpha}_{i_1}) \neq \pm (\boldsymbol{\delta}_2 - \boldsymbol{\alpha}_{i_2}) \neq \mathbf{0}. \tag{17}$$

PROOF. We write, for $k = 1, 2$,

$$(\boldsymbol{\delta}_k - \boldsymbol{\alpha}_{i_k}) = \boldsymbol{\alpha}_{i_k} * (\boldsymbol{\delta}_k * \boldsymbol{\alpha}_{i_k} - \mathbf{1}). \tag{18}$$

The rest is clear.

LEMMA 2. *Suppose* $\boldsymbol{\delta}_1 * \boldsymbol{\alpha}_{i_1}$ *and* $\boldsymbol{\delta}_2 * \boldsymbol{\alpha}_{i_2}$, i.e., *two vectors* $\boldsymbol{\delta}_1 * \boldsymbol{\alpha}_{i_1}$ *and* $\boldsymbol{\delta}_2 * \boldsymbol{\alpha}_{i_2}$, *are identical. If there are two treatments in* T_2 *for which*

(a) *the elements in* $\boldsymbol{\delta}_1 * \boldsymbol{\alpha}_{i_1}$ *and* $\boldsymbol{\delta}_2 * \boldsymbol{\alpha}_{i_2}$ *are equal to* -1 *for both cases, and*

(b) *the elements in* $\boldsymbol{\alpha}_{i_1}$ *and* $\boldsymbol{\alpha}_{i_2}$ *are equal in one case and different in the other case, then,*

$$\mathbf{0} \neq (\boldsymbol{\delta}_1 - \boldsymbol{\alpha}_{i_1}) \neq \pm (\boldsymbol{\delta}_2 - \boldsymbol{\alpha}_{i_2}) \neq \mathbf{0}. \tag{19}$$

The converse is also true.

PROOF. The result can be seen by considering (18).

LEMMA 3. *A necessary and sufficient condition that* $\mathbf{0} \neq (\boldsymbol{\delta}_1 - \boldsymbol{\alpha}_{i_1}) \neq (\boldsymbol{\delta}_2 - \boldsymbol{\alpha}_{i_1}) \neq \mathbf{0}$ *is that*

(a) *for* $j = 1, 2$, $\boldsymbol{\delta}_j * \boldsymbol{\alpha}_{i_1} \neq \mathbf{1}$ (*or, equivalently*, $\boldsymbol{\delta}_j \neq \boldsymbol{\alpha}_{i_1}$),
and

(b) $\boldsymbol{\delta}_1 * \boldsymbol{\alpha}_{i_1} \neq \boldsymbol{\delta}_2 * \boldsymbol{\alpha}_{i_1}$ (*or, equivalently*, $\boldsymbol{\delta}_1 \neq \boldsymbol{\delta}_2$).

The set $G = \{\mu;\ F_1, \cdots, F_m;\ F_{12}, \cdots, F_{m-1,m};\ F_{123}, \cdots, F_{m-2,m-1,m};\ F_{1234}, \cdots;\ F_{12\cdots m}\}$ forms a commutative group of order 2^m in which μ is the identity element, $F_1, \cdots, F_m$ can be regarded as generators, each F_i is idempotent (i.e., $F_i^2 = \mu$, $i = 1, \cdots, m$), and multiplication is defined as in ordinary algebra. For convenience in notation we denote μ by F_0. We now consider the factorial effects, which are elements of ξ_2, so that the corresponding columns in the matrix A_{12} are $(1 \cdots 1)'$, i.e., the same as the column in A_{11} corresponding to μ. It follows from the nature of T_1 that they are three factor and higher order interactions and their number is $2^{m-h} - 1$. We shall name these factorial effects as $(\pi_1, \cdots, \pi_{m-h};\ \pi_{12}, \cdots, \pi_{m-h-1,m-h};\ \pi_{123}, \cdots;\ \pi_{12\cdots m-h})$ where the π's are factorial effects in ξ_2. (Note that $\pi_{i_1 i_2 \cdots i_l} = \pi_{i_1}\pi_{i_2} \cdots \pi_{i_l}$). The π effects are known as defining contrasts. The set $H = \{\mu;\ \pi_1, \cdots, \pi_{m-h};\ \pi_{12}, \cdots, \pi_{12\cdots m-h}\}$ is a commutative group of order 2^{m-h} in which μ is the identity element, $\pi_1, \cdots, \pi_{m-h}$ are generators, and $\pi_i^2 = \mu$, $i = 1, \cdots, m-h$. Thus H is a subgroup of G. It can be easily seen that, for $i = 1, \cdots, m$, $F_iH = \{F_ih | h \in H\}$ are cosets of H in G. The columns $\boldsymbol{\delta}_k * \boldsymbol{\alpha}_{i_k}$, $\boldsymbol{\delta}_k * \boldsymbol{\alpha}_{i_l}$ in Lemmas 1–3 are, in fact, the columns corresponding to the π's.

THEOREM 4. *A necessary and sufficient condition that the design* $T = T_1 + T_2$ *with* $N = N_1 + N_2$ *treatments is a* MEP.1 *plan is that*

(i) *the* $2^{m-h} - 1$ *columns in* A_{22} *corresponding to* π's *are all distinct and, moreover, none has all elements* $+1$.

(ii) *for any* $\pi_{w_1 w_2 \cdots w_l}$ *and for any two factorial effects* F_u *and* $F_v (0 \leqslant u, v \leqslant m;\ u \neq v)$, *there are at least two treatments* $\mathbf{X}_i' = (x_{i_1}, x_{i_2}, \cdots, x_{i_m})$, $i = 1, 2$, *such that*

(a) $x_{iw_1} x_{iw_2} \cdots x_{iw_l} = -1$ *for* $i = 1, 2$,

(b) $x_{1u} = x_{1v}$ *and* $x_{2u} \neq x_{2v}$.

PROOF. The pairs whose members are elements in ξ_2 can be any of the 4 types:

$$(\pi_{u_1 \cdots u_k}, \pi_{v_1 \cdots v_l}), \quad (F_u \pi_{u_1 \cdots u_k}, F_u \pi_{v_1 \cdots v_l}),$$

$$(F_u \pi_{u_1 \cdots u_k}, F_v \pi_{v_1 \cdots v_l}), \quad (F_u \pi_{u_1 \cdots u_k}, F_v \pi_{u_1 \cdots u_k}),$$

where $u \neq v$, $k \geqslant 1$, $l \geqslant 1$. The proof is now immediate from Theorems 2–3 and Lemmas 1–3.

Note that Theorem 4 is a correction of Srivastava's earlier result where the case corresponding to the pair $(F_u \pi_{u_1 \cdots u_k}, F_v \pi_{u_1 \cdots u_k})$ has been ignored.

COROLLARY 1. *Under the conditions of Theorem* 4, *we have*

$$N_2 \geqslant (m - h)(h + 1). \tag{20}$$

PROOF. We need at least $m - h$ treatments to satisfy the condition (i) in Theorem 4. For any $\pi_{w_1 \cdots w_l}$, we need at least $h + 1$ treatments to satisfy (ii); the column in A_{22} corresponding to $\pi_{w_1 \cdots w_l}$ for these $h + 1$ treatments is a $(h + 1 \times 1)$ vector with all elements -1. It is now easy to see that N_2 must at least be $(m - h)(h + 1)$.

5. Equations over GF(2). Consider the equation

$$A\mathbf{x} = \mathbf{C} \tag{21}$$

over $GF(2)$, where A is a known $(m - h \times m)$ matrix with $\text{Rank}(A) = m - h$, $\mathbf{C}$ is a known vector of order $(m - h \times 1)$. The distinct elements of the matrix A and the vector $\mathbf{C}$ are 0 and 1. The number of solutions of (21) is equal to $N_0 = 2^h$. Let T be a $(N_0 \times m)$ matrix whose rows are solutions of (21). We then have the following result.

THEOREM 5. *Consider the equation* (21). *Then*

(i) Rank $T = h$, *if* $\mathbf{C} = \mathbf{0}$,

and

(ii) Rank $T = h + 1$ *if* $\mathbf{C} \neq 0$.

Consider the case $\mathbf{C} \neq 0$ and take a set of $h + 1$ independent solutions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{h+1}$. Then the other $2^h - h - 1$ solutions can be expressed as the linear combinations of $\mathbf{x}_i (i = 1, \cdots, h + 1)$. Consider

$$\mathbf{y} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + , \cdots, + \alpha_{h+1} \mathbf{x}_{h+1}, \tag{22}$$

where α_i's are either 0 or 1 and the weight of the vector $(\alpha_1, \alpha_2, \cdots, \alpha_{h+1})$ is an odd number. (The weight of a vector is the number of nonzero elements in it.) In fact, $\mathbf{y}$'s are all solutions of (21). Notice that

$$2^h = \binom{h+1}{1} + \binom{h+1}{3} + \binom{h+1}{5} + \cdots. \tag{23}$$

Suppose T_0 is a $(h + 1 \times m)$ matrix whose rows are $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{h+1}$. If $\mathbf{C} = \mathbf{0}$, then we take a set of independent solutions $\mathbf{x}_1^*, \mathbf{x}_2^*, \cdots, \mathbf{x}_h^*$ and the null vector $\mathbf{x}_{h+1}^* = (0, \cdots, 0)'$. We now form a $(h + 1 \times m)$ matrix T_0 whose rows are $\mathbf{x}_1^*, \mathbf{x}_2^*, \cdots, \mathbf{x}_{h+1}^*$. Thus for any $\mathbf{C}$, we form a $(h + 1 \times m)$ matrix T_0. We write

$$T_0 = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m], \tag{24}$$

where $\mathbf{b}_i$ is a $(h + 1)$ vector.

Suppose the weight of any vector (except the null vector) in the row space of A is greater than or equal to 3. Then it is well known that $T'(m \times N)$ is an orthogonal array (OA) of strength 2 (see [6]).

THEOREM 6. *Consider the equation* (21) *and the matrix* T_0 *in* (24). *Then*

(i) $\mathbf{b}_i \neq \mathbf{0}, \mathbf{1}$, *for* $i = 1, 2, \cdots, m$,

(ii) $\mathbf{b}_i + \mathbf{b}_j \neq \mathbf{0}, \mathbf{1}$ *for* $i, j = 1, 2, \cdots, m$ *and* $i \neq j$, *where* $\mathbf{0}$ *and* $\mathbf{1}$ *denote the vectors all of whose elements are* 0's *and* 1's *respectively*.

PROOF. Note that T_0 is a submatrix of T, where T is an OA of strength 2 and hence, of strength 1. Furthermore, any row in T, which is not in T_0, is a linear combination of the rows in T_0. It can now be checked that if (i) and (ii) are not true, then T_0 cannot be an OA of strength 2. This completes the proof of the theorem.

6. Construction of MEP.1 plans. We now discuss a method of construction of MEP.1 plans with $N = m + 1 + (m - h)(h + 1)$ treatments for a 2^m factorial where $m = 2^h - 1$, and h ($\geqslant 2$) is a positive integer. It is well known that there is a correspondence between the factorial effects for a 2^m factorial and the equations over $GF(2)$ (see [1]). For example, the factorial effect $F_{i_1 i_2 \cdots i_l}$ corresponds to the equation $x_{i_1} + x_{i_2} + \cdots + x_{i_l} = C$, where C is either 0 or 1. Consider the $m - h$ factorial effects $\pi_1, \pi_2, \cdots, \pi_{m-h}$ defined in Section 4 and get the corresponding equation of type (21). We shall consider the equation thus obtained for a specified $m - h$ values of $\mathbf{C}$. We write

$$A\mathbf{x} = \mathbf{C}^{(i)}, \qquad i = 1, \cdots, m - h, \tag{25}$$

where $\mathbf{C}^{(i)}$ is a $(m - h \times 1)$ vector whose ith coordinate is 0 and the other coordinates are 1. Notice that the $m - h$ factorial effects $\pi_i (i = 1, \cdots, m - h)$ are independent, i.e., Rank $(A) = m - h$ and, moreover, the weight of any vector (except the null) in the row space of A is greater than or equal to 3. For $i = 1, \cdots, m - h$, we get matrices $T_{0i}(h + 1 \times m)$ as described in Section 5. We now form $T_2 = T_{01} + T_{02} + \cdots + T_{0m}$ with $N_2 = (m - h) \times (h + 1)$ treatments. It follows from Theorems 4 and 6 that the design $T = T_1 + T_2$ is a MEP.1 plan with $N = m + 1 + (m - h) \times (h + 1)$ treatments. We illustrate the above method with the following example.

EXAMPLE. Consider $m = 7 = 2^3 - 1$, $h = 3$. The main effect plan $T_1(8 \times 7)$ for a 2^7 factorial is

$$T_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Here, the number of π's $= 2^{m-h} - 1 = 2^4 - 1 = 15$, and $\pi_1 = F_{124}$, $\pi_2 = F_{135}$, $\pi_3 = F_{236}$, and $\pi_4 = F_{1234567}$. Thus we have the following equations over $GF(2)$.

(26)

$$\begin{aligned} x_1 + x_2 \quad\quad + x_4 \quad\quad\quad\quad\quad &= C_1, \\ x_1 \quad\quad + x_3 \quad\quad + x_5 \quad\quad\quad &= C_2, \\ x_2 + x_3 \quad\quad\quad\quad + x_6 \quad &= C_3, \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 &= C_4, \end{aligned}$$

where for $i = 1, 2, 3, 4$, $C_i = 0$ or 1. The (4×7) matrices T_{0i} $(i = 1, 2, 3, 4)$ are obtained by interchanging the columns 4 and 7 of the matrices $T^*_{0i} (i = 1, 2, 3, 4)$, where

$$T^*_{0i} = [I_4 : U_i],$$

$$U_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$U_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad U_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Thus, $T_2 = T_{01} + T_{02} + T_{03} + T_{04}$ is a design with $4 \times 4 = 16$ treatments. The design $T = T_1 + T_2$ with $8 + 16 = 24$ treatments is a MEP.1 plan for a 2^7 factorial experiment.

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