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ADMISSIBLE DESIGNS FOR POLYNOMIAL MONOSPLINE REGRESSION[1]

BY NORMAN T. BRUVOLD

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Admissible designs for polynomial monospline regression are shown to be a proper subset of the admissible designs for spline regression. Sufficient conditions are given for designs for polynomial monospline regression to be admissible, and examples are used to show that simply counting points in intervals does not determine admissibility for monospline functions.

1. Introduction. Let $f' = (f_0, f_1, \cdots, f_n)$ be a vector of linearly independent functions on a closed interval $[a, b]$. For each x or "level" in $[a, b]$ an experiment can be performed whose outcome is a random variable $Y(x)$ with mean value $\theta' f(x) = \Sigma\theta_i f_i(x)$ and variance σ^2, independent of x. The functions f_i: $i = 0, \cdots, n$ are called the regression functions and are assumed known to the experimenter while the vector of parameters $\theta' = (\theta_0, \cdots, \theta_n)$ and σ^2 are unknown. An experimental design is a probability measure μ concentrating mass $p_1, \cdots, p_r$ on the points $x_1, \cdots, x_r$ where $p_i N = n_i$: $i = 1, 2, \cdots, r$ are integers.

For an arbitrary probability measure on $[a, b]$, the covariance matrix of the least squares estimator of the parameters θ_i is given by $(\sigma^2/N)M^{-1}(\mu)$ where $M(\mu) = (m_{ij}(\mu))$, $m_{ij}(\mu) = \int_{[a,b]} f_i f_j d\mu(x)$, is the information matrix of the experimental design μ. For two probability measures μ and ν on $[a, b]$, we say $\nu \geqslant \mu$ or $M(\nu) \geqslant M(\mu)$ if the matrix $M(\nu) - M(\mu)$ is nonnegative definite and unequal to the zero matrix. A probability measure or design μ is said to be admissible if there is no design $\nu \geqslant \mu$. Otherwise, μ is inadmissible.

For the case of ordinary polynomial regression where $f' = (1, x, \cdots, x^n)$, Kiefer (1959, page 291) has shown that μ is admissible if and only if the spectrum of μ, $S(\mu)$, has at most $n - 1$ points in the open interval (a, b). Consider the interval $[a, b]$ and choose h fixed points $\xi_1, \xi_2, \cdots, \xi_h$ such that $a < \xi_1 < \xi_2 < \cdots < \xi_h < b$, with $\bar{f}(x)$ the vector of functions for $x \in [a, b]$

$$(1.1)\qquad 1, x, \cdots, x^n, (x-\xi_i)_+^{n-k_i}, (x-\xi_i)_+^{n-k_i+1}, \cdots, (x-\xi_i)_+^{n-l_i} \qquad i = 1, \cdots, h$$

where $n - 1 \geqslant k_i \geqslant l_i \geqslant 0$, and $(x - \xi)_+^p = (x - \xi)^p$ if $x \geqslant \xi$ and equals zero otherwise. When $l_i = 0$ for $i = 1, \cdots, h$, the admissible designs for polynomial spline regression have been completely characterized by Studden and Van Arman

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(1969). Their results show that a design μ is admissible if and only if the spectrum of μ, $S(\mu)$ has less than or equal to

$$n - 1 + \Sigma_{j=i+1}^{i+l}[(n + k_j + 1)/2] \tag{1.2}$$

points on the open interval (ξ_i, ξ_{i+l+1}) for $i = 0, 1, \cdots, h - l$; $l = 0, 1, \cdots, h$. Here $\xi_0 = a$, $\xi_{h+1} = b$ and $[x]$ denotes the greatest integer less than or equal to x. A linear function of the component functions of $\bar{f}(x)$ (1.1) when $l_i = 1$ for each i is called a polynomial monospline function of degree n. This concept was introduced by Schoenberg (1946) and plays an important role in the problems of curve fitting and the theory of approximation, Karlin and Studden (1966b). In this paper sufficient conditions are given for designs for polynomial monospline regression to be admissible and examples are given to show that counting points in intervals is not sufficient to determine admissibility for monospline functions. It is also shown that the class of admissible designs for monospline regression is a proper subset of the admissible designs for spline regression.

2. Background results. Elfving (1959, page 71) states that any measure concentrated on a subset of the spectrum of an admissible spectrum is admissible, or a subspectrum of an admissible spectrum is admissible. He also shows that if μ is inadmissible, a measure whose spectrum contains that of μ is also inadmissible. From these results we can classify admissible experimental design by their spectra. Karlin and Studden (1966a, page 797) allow us to restrict attention to those probability measures concentrating their mass on a finite number of points in the sense that for each probability measure μ there exists a μ' concentrating on a finite number of points x_i such that $M(\mu) = M(\mu')$. Since we may classify experimental designs by their spectra we restrict our attention to those spectra with a finite number of points. We shall need:

LEMMA. 2.1. *Let μ be an inadmissible design. There is an admissible design ν such that $\nu \geqslant \mu$. (Van Arman, 1968).*

This lemma also tells us that we get best linear unbiased estimation results by staying in the admissible design class.

The following theorem gives sufficient moment conditions for admissibility. It was motivated by and gives a slight generalization of a theorem of Studden and Van Arman (1969, page 1559–1561). All the integrals in the following will be over $[a, b]$ unless specified otherwise.

THEOREM 2.1. *Let $\bar{f}(x)$ consist of the vector of regression function* (1.1) *and let $\bar{g}(x)$ consist of the vector of regression functions*

$$\begin{aligned} &1, x, \cdots, x^{2n-1} && \textit{where } \delta_i = l_i \textit{ when } l_i \neq 0 \\ &(x - \xi_i)_+^{n-k_i}, \cdots, (x - \xi_i)_+^{2n-\delta_i} && \qquad = 1 \textit{ when } l_i = 0 \qquad (2.1) \\ &i = 1, 2, \cdots, h;\ k_i, l_i, h, \xi_i && \textit{same as in } \bar{f}(x). \end{aligned}$$

Then $\nu \geqslant \mu$ (or $M(\nu) \geqslant M(\mu)$, ν and μ designs for $\bar{f}(x)$) if and only if (1) $\int \bar{g}(x) d(\nu -$

$\mu) = 0$ and (2) $0 \neq \int x^{2n} d(\nu - \mu) \geq \int (x - \xi_{r_1})_+^{2n} d(\nu - \mu) \geq \cdots \geq \int (x - \xi_{r_m})_+^{2n} d(\nu - \mu) \geq 0$ where $r_j, j = 1, \cdots, m$ is the ordered set of i's for which $l_i = 0, 0 \leq m \leq h$.

PROOF. In proving sufficiency first, let $M = M(\nu) - M(\mu)$. Following the argument of Studden and Van Arman, we see that M has all diagonal elements $= 0$ except possibly the elements $\int x^{2n} d(\nu - \mu)$ and $\int (x - \xi_i)_+^{2n} d(\nu - \mu)$ when $l_i = 0$, $i = 1, \cdots, h$. Let $r =$ smallest i for which $l_i = 0$, $r_2 =$ next smallest i for which $l_i = 0, \cdots, r_m =$ largest i for which $l_i = 0$ and define A_{r_l} as $A_{r_l} = \int (x - \xi_{r_l})_+^{2n} d(\nu - \mu)$, for $l = 1, \cdots, m$ $A_{\xi_0} = \int x^{2n} d(\nu - \mu)$, for $l = 0$. The element corresponding to the r_s row and r_t column of M, $s < t$, is $\int (x - \xi_{r_s})_+^n (x - \xi_{r_t})_+^n d(\nu - \mu) = \int x^n (x - \xi_{r_t})_+^n d(\nu - \mu) = \int (x - \xi_{r_t})_+^{2n} d(\nu - \mu) = A_{r_t}$. Thus the conditions of Lemma 2.2 of Studden and Van Arman (1969) are satisfied which implies condition (2).

In order to prove necessity, we note that if conditions (1) and (2) hold and $M = M(\nu) - M(\mu)$, then $M \geq 0$ by Lemma 2.2 of Studden and Van Arman (1969).

3. Admissible designs. In this section we will be concerned with classifying the admissible experimental designs relative to regression on the functions $\bar{f}(x)$, given by (1.1). First the class of admissible designs for $\bar{f}(x)$ are restricted to a subclass of admissible designs for regressions on the functions $\bar{f}_1(x)$ which are defined the same as $\bar{f}(x)$ when all $l_i = 0$, $i = 1, \cdots, h$. $\bar{f}(x)$ is a general form of monosplines and $\bar{f}_1(x)$ are the polynomial splines whose admissible designs are given by (1.2).

LEMMA 3.1. *If μ is admissible for $\bar{f}(x)$, then μ is admissible for $\bar{f}_1(x)$, given by* 1.1 *with $l_i = 0$, $i = 1, \cdots, h$.*

PROOF. Assume μ is admissible $\bar{f}(x)$ and inadmissible $\bar{f}_1(x)$. Since μ is inadmissible $\bar{f}_1(x)$, there exists a ν admissible $\bar{f}_1(x)$ such that $M(\nu) \geq M(\mu)$, by Lemma 2.1. Let $M'(\nu)$ and $M'(\mu)$ represent the submatrices of $M(\nu)$ and $M(\mu)$ corresponding to $\bar{f}(x)$. Since μ is admissible $\bar{f}(x)$, we have that $M'(\nu) \equiv M'(\mu)$. By Theorem 2.1 this implies that $\int x^{2n} d\nu = \int x^{2n} d\mu$ which in turn implies that $M(\nu) \equiv M(\mu)$, the desired contradiction.

This lemma tells us that if μ is inadmissible for $\bar{f}_1(x)$ then μ is also inadmissible for $\bar{f}(x)$. In order to completely classify the admissible designs for $\bar{f}(x)$, we need only list those designs that are (i) admissible $\bar{f}_1(x)$ and (ii) inadmissible $\bar{f}(x)$ since the admissible designs for $\bar{f}_1(x)$ are given by (1.2).

The remainder of the section is devoted to the solution for several general cases.

LEMMA 3.2. *Given a design μ such that* (1) *$S(\mu)$ has $\leq n - 1 + \sum_{j=i+1}^{i+l}[(n + k_j + 1)/2]$ points on the open interval (ξ_i, ξ_{i+l+1}) for $i = 0, 1, \cdots, h - l$, $l = 0, 1, \cdots, h$, we can always add a set B of points in $[a, b]$ such that* (2) *$B \cap S(\mu) = \phi$, and* (3) *$S(\mu) \cup B$ has $\leq n - 1 + \sum_{j=i+1}^{i+l}[(n + k_j + 1)/2]$ points on the open interval (ξ_i, ξ_{i+l+1}) for $i = 0, \cdots, h - l$, $l = 0, 1, \cdots, h$, where equality holds for $l = h$ when $i = 0$.*

PROOF. The proof will be by induction on the number of knots. Let μ be a design satisfying (1) for which the number of knots $h = 1$.

If there were $\leqslant [(n + k_1 + 1)/2]$ points in $[\xi_1, b)$, we would add distinct points to $[\xi_1, b)$ until equality would hold. If $k_1 = n - 1$, then one of the points in $[\xi_1, b)$ either contributed or present in $S(\mu)$ would be ξ_1. If in the remaining piece (a, ξ_1) there were less than $n - 1$ points, we would add distinct points until there were exactly $n - 1$ points in (a, ξ_1). Let B be the set of points added. It is easily seen that (2) and (3) hold.

If there were $r > [(n + k_1 + 1)/2]$ points in $[\xi_1, b)$, we would let $s = r - [(n + k_1 + 1)/2]$ and note that (1) requires that we have $\leqslant n - 1 - s$ points in (a, ξ_1). If there were $< (n - 1 - s)$ points in (a, ξ_1), we would add distinct points until equality held. Let B be the set of points added. We have now shown (2) and (3) for the case of one knot.

Let μ be a design for which the number of knots $h = m + 1$. If there were $\leqslant [(n + k_{m+1} + 1)/2]$ points in $[\xi_{m+1}, b)$, we would use the induction hypothesis to require $S(\mu) \cup B'$ to satisfy (2) and (3) for the interval (a, ξ_{m+1}) and add necessary points to the interval $[\xi_{m+1}, b)$ to have the interval total $= [(n + k_{m+1} + 1)/2]$. If $k_{m+1} = n - 1$, then ξ_{m+1} would be a counted point. Let B be the set of all points added. $B' \subset B$ and again (2) and (3) hold.

If there were $r > [(n + k_{m+1} + 1)/2]$ points on (ξ_{m+1}, b), we would use the induction hypothesis to require $S(\mu) \cup B'$ to satisfy (2) and (3) on (a, ξ_{m+1}). Let $s = r - [(n + k_{m+1} + 1)/2]$ and note that B' has at least s points, otherwise assumption (1) would be contradicted. We now remove the largest s points of B' and call the remaining set B. All that remains is to check the requirement (2) on subintervals that contain $[\xi_{m+1}, b)$. Let (ξ_t, b) be any interval that contains points in B. Since (ξ_t, ξ_{m+1}) has $\leqslant n - 1 + \sum_{j=t+1}^{m}[(n + k_j + 1)/2) - s$ points, we have that (ξ_t, b) has $\leqslant n - 1 + \sum_{j=t+1}^{m+1}[(n + k_j + 1)/2]$ points. If (ξ_t, b) does not contain points of B, the subinterval requirement is a part of our assumption (1).

This completes the discussion since (2) and (3) hold. Remark: We can delete any number of points from B and condition (1) would hold for $S(\mu) \cup$ (B deleted).

In the next two lemmas we develop properties of spectra that when used with the preceding lemmas and Theorem 2.1 will give a large class of admissible designs. Essentially we can classify as admissible those designs for which the moments $\int \bar{g}(x)\, d\mu$ prohibit the existence of a ν admissible $\bar{f}_1(x)$ such that $\nu \geqslant \mu$. The results will be stated in Theorems 3.1 and 3.2.

LEMMA 3.3. *If a design μ is such that $S(\mu)$ has $\leqslant n - 1 + \sum_{j=i+1}^{i+l}[(n + k_j + 1)/2]$ points on the open interval (ξ_i, ξ_{i+l+1}) for $i = 0, 1, \cdots, h - l$, $l = 0, 1, \cdots, h$, where equality holds for $l = h$ when $i = 0$, and p is such that $k_p = n - 1$, $1 \leqslant p \leqslant h$, then $\xi_p \in S(\mu)$.*

PROOF. The number of points in (a, ξ_p) is $\leqslant n - 1 + \sum_{j=1}^{p-1}[(n + k_j + 1)/2]$. The number of points in (ξ_p, b) is $\leqslant n - 1 + \sum_{j=p+1}^{h}[(n + k_j + 1)/2]$. The number of points in $(a, \xi_p) \cup (\xi_p, b)$ is $\leqslant 2(n-1) + \Sigma_{j=1}^{h}[(n + k_j + 1)/2] - [(n + k_p + 1)/2] = n - 2 + \Sigma_{j=1}^{h}[(n + k_j + 1)/2]$ since $[(n + k_p + 1)/2] = n$. The number of points in $(a, b) - [(a, \xi_p) \cup (\xi_p, b)] = 1$. This implies that $\xi_p \in S(\mu)$.

LEMMA 3.4. *Let $\bar{f}_2(x)$ consist of the vector of regression functions (a subset of those in $\bar{f}_1(x)$); $\bar{f}_2(x) = 1, x, \cdots, x^n, (x - \xi_i)_+^{n-k_i}, \cdots, (x - \xi_i)_+^n$; $i = 1, 2, \cdots, h$ where for each i, k_i is such that $n + k_i$ is even or $k_i = n - 1$. Let $\bar{g}_1(x)$ consist of the vector of regression functions $\bar{g}_1(x) = 1, x, \cdots, x^{2n-1}, (x - \xi_i)_+^{n-k_i}, \cdots, (x - \xi_i)_+^{2n-1}$ $i = 1, \cdots, h$ where the ξ_i and k_i are the same for $\bar{f}_2(x)$ above.*

If μ and ν are admissible designs relative to $f_2(x)$ with supports $S(\mu)$ and $S(\nu)$, then any design relative to $\bar{g}_1(x)$ with support $S(\mu) \cup S(\nu)$ is admissible $\bar{g}_1(x)$.

(In applying this lemma, we are more concerned with the placement of points in their spectra than with admissibility with respect to $g_1(x)$.)

PROOF. $S(\mu)$ and $S(\nu)$ each have $\leqslant n - 1 + \Sigma_{j=i+1}^{i+l}[(n + k_j + 1)/2]$ points in the interval (ξ_i, ξ_{i+l+1}) for $i = 0, 1, \cdots, h - l$; $l = 0, 1, \cdots, h$, where we may assume equality holds for $l = h$ when $i = 0$ for $S'(\mu)$ and $S'(\nu)$. $S'(\mu) \equiv S(\mu) \cup B$ from Lemma 3.2 and $S'(\nu)$ is defined similarly. An admissible design for $g_1(x)$ would have

$$\leqslant 2n - 2 + \Sigma_{j=i+1}^{i+l}\left[(2n - 1 + n + k_j + 1)/2\right] \tag{3.1}$$

points in the interval (ξ_i, ξ_{i+l+1}) for $i = 0, 1, \cdots, h - l$; $l = 0, 1, \cdots, h$. $S'(\mu) \cup S'(\nu)$ has

$$\leqslant 2\left(n - 1 + \Sigma_{j=i+1}^{i+l}\left[(n + k_j + 1)/2\right]\right) - r_{il} \tag{3.2}$$

distinct points in (ξ_i, ξ_{i+l+1}). r_{il} is the number of indexes j such that $i + 1 \leqslant j \leqslant i + l$ for which $k_j = n - 1$. To see this we note that by Lemma 3.3, $\xi_j \in S'(\mu)$ and $\xi_j \in S'(\nu)$ $k_j = n - 1$. The subtraction of r_{il} eliminates the counting of ξ_j twice in $S(\mu) \cup S(\nu)$. It is easily seen that $(3.2) = 2n - 2 + \Sigma_{j=i+1}^{i+l}(n + k_j)$ with the restrictions on k_i. Since $(3.2) \leqslant 2n - 2 + \Sigma_{j=i+1}^{i+l}(n + [(n + k_i)/2]) = (3.1)$, we have the $S(\mu) \cup S(\nu)$ is admissible $g_1(x)$.

THEOREM 3.1. *Let $\bar{f}_3(x)$ consist of the vector of regression functions in $\bar{f}(x)$ with $l_i = 0$ or 1 for each i. A design μ is admissible $\bar{f}_3(x)$ if $S(\mu)$ has $\leqslant n - 1 + \Sigma_{j=i+1}^{i+l}[(n + k_j)/2]$ points on (ξ_i, ξ_{i+l+1}) for $i = 0, 1, \cdots, h - l$, $l = 0, 1, \cdots, h$.*

PROOF. Assume μ is inadmissible $\bar{f}_3(x)$. Then after consideration of Lemmas 3.1 and 2.1, there exists a ν admissible with respect to $\bar{f}_1(x)$ (with the same k_i, ξ_i, h, a and b as in $\bar{f}_3(x)$ above) such that $\nu \geqslant \mu$.

Now $S(\nu)$ has $< n - 1 + \Sigma_{j=i+1}^{i+l}[(n + k_j + 1)/2]$ points on (ξ_i, ξ_{i+l+1}) for $i = 0, \cdots, h - l$; $l = 0, 1, \cdots, h$. And $S(\nu) \cup S(\mu)$ has

$$\begin{aligned}(3.3)\qquad &\leqslant 2(n - 1 + \Sigma_{j\in[i+1,i+l],\, n+k_j \text{ even}}(n - k_j))\\ &\quad + \tfrac{1}{2}(\Sigma_{j\in[i+1,i+l],\, n+k_j \text{ odd}}(n + k_j - 1))\\ &\quad + \tfrac{1}{2}(\Sigma_{j\in[i+1,i+l],\, n+k_j \text{ odd}}(n + k_j + 1))\\ &= 2n + 2 + \Sigma_{j=i+1}^{i+l}(n + k_j) \qquad \text{points on } (\xi_i, \xi_{i+l+1})\\ &\qquad\qquad \text{for } i = 0, 1, \cdots, h - l;\ l = 0, 1, \cdots, h.\end{aligned}$$

Now $S(\mu) \cup S(\nu)$ is admissible with respect to $\bar{g}_2(x) = 1, x, \cdots, x^{2n-1}, (x - \xi_i)_+^{n-k_i}, \cdots, (x - \xi_i)_+^{2n-1}$; n, ξ_i, k_i the same as in $\bar{f}_3(x)$ above, $i = 1, \cdots, h$, by the previous lemma. Without loss of generality, we may assume the equality holds in (3.3) for $l = h$ when $i = 0$ by Lemma 3.2. Note that the exact number of functions in $\bar{g}_2(x)$ is $2n + \Sigma_{j=1}^{h}(n + k_j)$.

Since $\nu \geqslant \mu$ we have by Theorem 2.1 that $\int \bar{g}_2(x)d(\nu - \mu) = 0$. This can be written as $M'(\bar{t}, \bar{g}_2)\bar{\nu} = M'(\bar{t}, \bar{g}_2)\bar{\mu}$ where $\nu(t_p) = \nu_p$, $\mu(t_p) = \mu_p$ are the weights assigned to the vector $\bar{t}$ of the $m = 2n + \Sigma_{j=1}^{h}(n + k_j)$ ordered points of $S(\mu) \cup S(\nu) \cup \{a\} \cup \{b\}$. We define $M(\bar{t}, \bar{g}_2)$ to be the matrix with the vector $\bar{g}_2(t_i)$ in the ith row. If t_i values coincide, then the successive rows are replaced by successive derivatives taken from the right. $M'(\bar{t}, \bar{g}_2)$ is the transpose of the matrix $M(\bar{t}, \bar{g}_2)$ and is nonsingular by Lemma 3.1 of Studden and Van Arman (1969), since $t_{\gamma_i} < \xi_i < t_{2n+1+\gamma_{i-1}}$ where $\gamma_i = \Sigma_{j=1}^{i}(n + k_j)$. $M(\bar{t}, \bar{g}_2)$ being invertible implies $\nu = \mu$, and we have the desired contradiction. The following theorem is closely related to Theorem 3.1 but does describe some additional admissible designs.

THEOREM 3.2. *Let $\bar{f}_4(x)$ consist of the vector of regression functions in $\bar{f}_3(x)$ with the restriction that $n + k_i$ is even or $k_i = n - 1$ for each $i = 1, 2, \cdots, h$. If $S(\mu)$ has $\leqslant n - 1 + \Sigma_{j=i+1}^{i+p}[(n + k_i + 1)/2]$ points in (ξ_i, ξ_{i+p+1}) $i = 0, 1, \cdots, h - p$; $p = 0, 1, \cdots, h_i$, then μ is admissible $\bar{f}_4(x)$.*

PROOF. Note that $n + k_i$ is even if and only if $n - k_i$ is even. The "only if" part follows from Lemma 3.1. The "if" part follows that of Theorem 3.1 with some modification. We would have the ν and μ with similar assumptions and notice that (3.2) for this theorem equals $2n - 2 + \Sigma_{j=i+1}^{i+l}(n + k_j)$ which is the case in Theorem 3.1 for $S(\mu) \cup S(\nu)$. The remainder of the proof follows that of Theorem 3.1 word for word.

The following development leads to a relationship between admissible designs and the existence of a nontrivial polynomial in its regression functions.

Let $\phi(x)$ denote the set of functions $1, x, \cdots, x^{2n}, (x - \xi_i)_+^{n-k_i}, \cdots, (x - \xi_i)_+^{2n-1}$ $i = 1, \cdots, h$ where ξ_i and k_i are the same as in $\bar{f}_3(x)$ and $x \in [a, b]$. Also let $\phi_0(x) = 1, \phi_1(x) = x, \cdots, \phi_{2n}(x) = x^{2n}, \phi_{2n+1}(x) = (x - \xi_1)_+^{n-k_1}, \cdots, \phi_m =$

$(x - \xi_h)_+^{2n-1}$ where $m = 2n + \Sigma_{i=1}^h(n + k_i)$. Let

$$\mathfrak{M} = \{\bar{c} = (c_1, \cdots, c_m)|c_t = \int\phi_t(x)\, d\mu(x), \mu \in \mathcal{P}, t = 1, \cdots, m\}$$

where $\mathcal{P}$ is the set of probability measures on $[a, b]$. $\mathfrak{M}$ is a closed convex set in m-space since the functions in $\phi(x)$ are continuous and defined on a compact space. Theorem 2.1 states that a design μ is admissible $\bar{f}_5(x)$ ($\bar{f}(x)$ with $l_i = 1$ for all i) if and only if, for fixed c_t, $t = 1, \cdots, m$, $t \neq 2n$, μ maximizes $c_{2n} = \int x^{2n}\, d\nu(x)$ for all probability measures ν defined on $[a, b]$ with $c_t = \int\phi_t(x)\, d\mu(x) = \int\phi_t(x)\, d\nu(x)$ for all $t \neq 2n$. Roughly speaking, μ is admissible if and only if it corresponds to an "upper" boundary point of $\mathfrak{M}$. Since $\mathfrak{M}$ is closed and convex, there must be a nontrivial supporting hyperplane at any boundary point of $\mathfrak{M}$.

LEMMA 3.5. *Any admissible design μ for $\bar{f}_5(x)$ ($\bar{f}(x)$ with $l_i = 1$ for all i) has an associated nontrivial polynomial $P(x)$ in the $\phi(x)$ such that*: (1) $P(x) = 0$ *for* $x \in S(\mu)$, (2) $P(x) \geqslant 0$ *for* $x \in [a, b]$, *and* (3) *the coefficient of* x^{2n} *in* $P(x)$ *is* $\leqslant 0$.

PROOF. Let c^0 be the point $(c_1^0, \cdots, c_m^0)$ in $\mathfrak{M}$ where $c_t^0 = \int\phi_t(x)\, d\mu(x)$ for $t = 1, \cdots, m$. In constructing a supporting hyperplane at c^0 there exist real constants $\{a_t\}_{t=0}^m$, not all zero, such that

$$(3.4)\quad \Sigma_{t=1}^m a_t c_t + a_0 \geqslant 0 \quad \text{and} \quad \Sigma_{t=1}^m a_t c_t^0 + a_0 = 0 \quad \text{for all } c \in \mathfrak{M}.$$

Now $\Sigma_{t=1}^m a_t c_t + a_0 = \Sigma_{t=0}^m a_t \int\phi_t(x)\, d\nu(x) = \int(\Sigma_{t=0}^m a_t\phi_t(x))\, d\nu(x) \geqslant 0$, for all $\nu \in \mathcal{P}$. Let $P(x) = \Sigma_{t=0}^m a_t\phi_t(x)$. Note that $P(x) \geqslant 0$ for $x \in [a, b]$ and thus $P(x) = 0$ for $x \in S(\mu)$. The point $c_\lambda = (c_1^0, \cdots, c_{2n-1}^0, c_{2n}^0 + \lambda, c_{2n+1}^0, \cdots, c_m^0)$ for all $\lambda > 0$ lies in the half space complementary to that of (3.4) so that $\Sigma_{t=0}^m a_t c_t^0 + \lambda a_{2n} \leqslant 0$ for all $\lambda > 0$. This requires that $a_{2n} \leqslant 0$. A lemma which is a partial converse of the preceding follows.

LEMMA 3.6. *A design μ is admissible for $\bar{f}_5(x)$ if there exists a nontrivial polynomial $P(x)$ in the $\phi(x)$ such that*: (1) $P(x) \geqslant 0$ *for* $x \in [a, b]$, (2) $P(x) = 0$ *for* $x \in S(\mu)$, *and* (3) *the coefficient of* x^{2n} *in* $P(x)$ *is negative.*

PROOF. Let ν be a probability measure on $[a, b]$ such that $M(\nu) \geqslant M(\mu)$. By Theorem 2.1 we have that $\int x^{2n} d(\nu - \mu) > 0$. Also by Theorem 2.1 we have that $\int P(x) d(\nu - \mu) = \int a_{2n} x^{2n} d(\nu - \mu)$ where a_{2n} is the coefficient of x^{2n} in $P(x)$. $\int P(x) d(\nu - \mu) = \int P(x)\, d\nu \geqslant 0$ by conditions (1) and (2) of the lemma. Combining the above inequalities, we have $\int a_{2n} x^{2n} d(\nu - \mu) = \int P(x) d(\nu - \mu) = \int P(x)\, d\nu \geqslant 0$. This implies that $\int x^{2n} d(\nu - \mu) \leqslant 0$ by condition (3). This is the desired contradiction.

4. Examples. The preceeding section provided sufficient conditions for polynomial monospline regression designs to be admissible. The following examples show that the conditions are not also necessary.

EXAMPLE 4.1. Consider the regression functions $\{1, x, x^2, x^3, x^4, (x - 1)_+, (x - 1)_+^2, (x - 1)_+^3, (x - 2)_+^3\}$ $x \in [0, 3]$. The following design is admissible by Theorem 3.2 but not covered by Theorem 3.1. The points $\{0, 1, 3\}$ with three points in both $(0, 1)$ and $(1, 2)$ and two points in $(2, 3)$.

EXAMPLE 4.2. For the regression functions $\{1, x, \cdots, x^6, (x-1)_+^3, (x-1)_+^4, (x-1)_+^5\}$ $x \in [0, 2]$ the following design is admissible by Theorem 3.1 but not covered by Theorem 3.2. The points $\{0, 2\}$ and 5 points in $(0, 1)$ and 4 points in $(1, 2)$.

EXAMPLE 4.3. Let $P(x)$ be the polynomial $-1(x+1)(x+\frac{3}{4})^2(x+\frac{2}{4})^2(x+\frac{1}{4})^2(x-\frac{3}{47})$ for $x \in [-1, 0]$ and the polynomial $-1(x-1)(x-\frac{3}{4})^2(x-\frac{2}{4})^2(x-\frac{1}{4})^2(x+\frac{3}{47})$ for $x \in [0, 1]$. $P(x)$ can be written in the form $\Sigma_{j=0}^{8} a_j x^j + \Sigma_{j=3}^{7} b_j x_+^j$ where $a_8 = -1$ and the $\frac{3}{47}$ has been chosen to make the coefficients of 1, x and x^2 identical in both $[-1, 0]$ and $[0, 1]$. If μ is such that $S(\mu) = \{-1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ then μ is admissible for the regression functions $\{1, x, x^2, x^3, x^4, x_+^3\}$ by Lemma 3.6 and is not covered by either Theorem 3.1 or 3.2.

EXAMPLE 4.4. Consider the regression function $\{1, x, x^2, x^3, x^4, x_+^3\}$. In this example a nontrivial polynomial in the functions $\{1, x, \cdots, x^8, x_+^3, \cdots, x_+^7\}$ does not exist for which $P(x) \geqslant 0$ on $[-1, 1]$ and $P(x) = 0$ for $x \in \{-1, -\frac{3}{4}, -\frac{2}{4}, -\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$. If we first assume that the coefficient of x^8 in $P(x)$ is nonzero then there is one remaining root for the polynomial part of $P(x)$ as defined on $[-1, 0)$ and one remaining root of the polynomial part of $P(x)$ as defined on $(0, 1]$. These roots must be chosen so that the coefficients of 1, x and x^2 agree on both $[-1, 0]$ and $[0, 1]$. This leads to a system of 3 linear equations in 2 unknowns. In order to have a solution the determinant of the augmented matrix must be zero but it is nonzero. If the coefficient of x^8 were zero, Lemma 3.1 of Studden and Van Arman (1969) implies that $P(x)$ must be trivial ($\equiv 0$). The design is inadmissible by Lemma 3.5.

Theorem 3.1 states for the preceding two examples that a design would be admissible if $S(\mu)$ has at most 3 points in $(-1, 0)$ and two in $(0, 1)$. Both the above designs have 3 points in each interval with one admissible and one inadmissible. Thus a simple counting argument cannot guarantee admissibility.

EXAMPLE 4.5. For the regression functions given in Example 4.4 it can be shown that symmetric designs $\{-1, -x_1, -x_2, -x_3, x_3, x_2, x_1, 1\}$ are admissible and that there are some nonsymmetric designs with 3 points in both $(-1, 0)$ and $(0, 1)$ that are admissible by Lemma 3.6.

EXAMPLE 4.6. For the regression functions $\{1, x, \cdots, x^4, x_+^3, (x-1)_+^3\}$ the design with $S(\mu) = \{-1, -\frac{3}{4}, -\frac{2}{4}, -\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \frac{4}{3}, \frac{5}{3}\}$ is inadmissible since it is not subadmissible on $(-1, 1)$ by Example (4.4). However let $P(x) = \frac{4}{81}(x-1)_+^3 - \frac{4}{9}(x-1)_+^4 + \frac{13}{9}(x-1)_+^5 - 2(x-1)_+^6 + (x-1)_+^7$ which is equal to $(x-1)^3$

$(x - \frac{4}{3})(x - \frac{5}{3})^2$ for $x \in [1, 2]$. The conditions of Lemma 3.5 on $P(x)$ are satisfied but μ is inadmissible so the converse of Lemma 3.5 is not true.

5. Conclusion. Polynomial monosplines that share the same fixed points as polynomial splines have as their set of admissible designs a proper subset of the admissible designs for the polynomial splines. An admissible design for a monospline ($l_i = 0$ in Theorem 3.1) differs from one of splines by at most l points on (ξ_i, ξ_{i+l+1}). Also, if $n + k_i$ is even or $k_i = n - 1$ for all i, then a design admissible for splines is also admissible for monosplines. For the monospline case $\{1, x, \cdots, x^4, x_+^3\}$ symmetric designs with three points in both $(-1, 0)$ and $(0, 1)$ are admissible. There exist nonsymmetric designs with three points in each open interval of which some are admissible and some are inadmissible. Thus, one must consider the placement of points in an admissible monospline spectra as well as their number, for a counting argument will not give a sufficient condition for admissibility.

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