# TREND-FREE BLOCK DESIGNS: THEORY[1]

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A common polynomial trend in one or more dimensions is assumed to exist over the plots in each block of a classical experimental design. An analysis of covariance model is assumed with trend components represented through use of orthogonal polynomials. The objective is to construct new designs through the assignment of treatments to plots within blocks in such a way that sums of squares for treatments and blocks are calculated as though there were no trend and sums of squares for trend components and error are calculated easily. Such designs are called *trend-free* and a necessary and sufficient condition for a trend-free design is developed. It is shown that these designs satisfy optimality criteria among the class of connected designs with the same incidence matrix. The analysis of variance for trend-free designs is developed. The paper concludes with two examples of trend-free designs.

1. **Motivation, introduction and summary.** Various experimental designs have been considered for the two-way elimination of heterogeneity, the latin square being the best known. Many experimental situations arise in which response may be affected by the spatial or temporal position of the experimental unit or plot within a block and, frequently, an assumption of a common polynomial trend of specified degree over plots within blocks may be appropriate. The $v \times v$ latin square may be used to eliminate the effects of a common, one-dimensional, polynomial trend to degree $v - 1$ over the plots in rows of the square. Questions of criteria for and the optimality of other block designs, complete or incomplete, that eliminate the effects of common polynomial trends of specified degree over the plots within blocks are investigated. Examination of the existence and construction of the desired designs is deferred to a subsequent paper.

Experimental designs to be used in the presence of trends to avoid the complications of analysis of covariance and to increase design efficiencies have been developed by others. Cox (1951, 1952, 1958) considered the assignment of treatments to plots ordered in space or time without blocking and with a trend extending over the entire sequence of plots. Box (1952) and Box and Hay (1953) in similar experimental sequences investigated choices of levels of quantitative factors. Hill (1960) combined the designs of Cox and Box to form new designs to study the effects of both qualitative and quantitative factors in the presence of trends. Daniel and Wilcoxon (1966) and Daniel (1976, Chapter 15) provided methods of sequencing the assignments of factorial treatment combinations to experimental units to

achieve better estimation of specified factorial effects, again in the presence of a trend in time or distance. Phillips (1964, 1968a, 1968b) illustrated the use of magic figures (magic squares, magic rectangles, etc.) for the elimination of trend effects in certain classes of one-way, factorial, latin-square, and graeco-latin-square designs; a single trend was assumed again to affect all of the observations.

We consider block designs to compare $v$ treatments in $b$ blocks of equal size $k \leqslant v$ such that each treatment occurs at most once in each block. A common polynomial trend is assumed to exist over the plots in each block. The trend may be in one or several dimensions and is expressed in terms of $p \leqslant (k - 1)$ orthogonal components. The appropriate classical fixed-effects model for the general block design and its assumptions are used with the addition of orthogonal terms representing the assumed trend. The problem is to assign treatments to plots within blocks so that the known properties of the ordinary analysis of variance for treatment and block sums of squares are preserved and variation due to the trend may be removed from the error sum of squares. When the desired designs exist, we shall call them *trend-free* designs. We shall abbreviate and, for example, use $TF_p CB$ for a complete block design free of a common, one-dimensional trend of degree $p$ within blocks and $TF_p BIB$ and $TF_p PBIB$ for similar balanced and partially balanced incomplete block designs. Additional subscripts may be added if the trend is in several dimensions.

In this paper, necessary and sufficient conditions for the existence of trend-free block (TFB) designs are obtained, necessary analysis of variance is developed, and optimality properties are demonstrated. Several examples of TFB designs are included but results on design construction are not included.

**2. Notation, definitions and model.** Let $v$ treatments be applied to plots arranged in $b$ blocks, each of size $k$, $k \leqslant v$. Each plot receives only one treatment and each treatment occurs at most once in a block. The plot positions in a block are indexed by $m$-dimensional vectors of positive integers, $\mathbf{t} = (t_1, \cdots, t_m)$, $t_u = 1, \cdots, s_u$, $u = 1, \cdots, m$, where $s_u$ is the number of plot positions in the $u$th dimension, $\Pi_u s_u = k$. The polynomial trend extending over plots in each block is a function of the plot position $\mathbf{t}$ and is expressed as a linear function of $m$-dimensional orthogonal polynomials of the form, $\phi_\alpha(\mathbf{t}) \equiv \phi_{(\alpha_1, \cdots, \alpha_m)}(t_1, \cdots, t_m) = \Pi_{u=1}^m \phi_{\alpha_u}(t_u)$, where $\phi_{\alpha_u}(t_u)$ is a one-dimensional orthogonal polynomial of degree $\alpha_u$ satisfying the orthogonality conditions, for $u = 1, \cdots, m$,

$$(2.1) \qquad \qquad \sum_{t=1}^{s_u} \phi_{\alpha_u}(t) = 0$$

and

$$(2.2) \qquad \sum_{t=1}^{s_u} \phi_{\alpha_u}(t) \phi_{\alpha_u'}(t) = 1 \quad \text{if} \quad \alpha_u = \alpha_u',$$
$$= 0 \quad \text{if} \quad \alpha_u \neq \alpha_u'.$$

Note that multidimensional orthogonal polynomials may be obtained as indicated from one-dimensional ones and that $\alpha_u = \alpha_u'$ implies that $\phi_{\alpha_u} = \phi_{\alpha_u'}$. Tables of

one-dimensional orthogonal polynomials for equally spaced variables are given by Fisher and Yates (1957).

The mathematical model is a simple extension of the classical model for general block designs, trend terms added, and is written

$$(2.3) \quad y_{jt} = \mu + \Sigma_{i=1}^{v} \delta_{jt}^{i} \tau_i + \beta_j + \Sigma_{\alpha \in \mathcal{Q}} \theta_\alpha \phi_\alpha(\mathbf{t}) + \varepsilon_{jt},$$

$$j = 1, \cdots, b, \mathbf{t} = (t_1, \cdots, t_m), t_u = 1, \cdots, s_u, u = 1, \cdots, m,$$

where $y_{jt}$ is the observation on plot position $\mathbf{t}$ of block $j$, $\mu$, $\tau_i$ and $\beta_j$ are respectively the usual mean, treatment and block parameters, $\mathcal{Q}$ is an index set of $p$, $m$-dimensional, nonzero vectors of the form $\alpha$, and $\theta_\alpha$ is the regression coefficient of $\phi_\alpha(\mathbf{t})$. The trend effect on plot position $\mathbf{t}$ is $\Sigma_{\alpha \in \mathcal{Q}} \theta_\alpha \phi_\alpha(\mathbf{t})$, not dependent on the particular block $j$. Designation of the treatment applied to plot $(j, \mathbf{t})$ is effected through the indicator function,

$$(2.4) \quad \delta_{jt}^{i} = 1 \quad \text{if treatment } i \text{ on plot } (j, \mathbf{t}),$$

$$= 0 \quad \text{otherwise.}$$

The errors $\varepsilon_{jt}$ are assumed to be i.i.d. with zero means and they will be taken to be normal, $N(0, \sigma^2)$, in discussions of distribution theory below. The model (2.3) will be regarded as a fixed-effects model unless specifically stated to be otherwise in particular remarks.

Choices of values of $\delta_{jt}^{i}$ specify particular block designs. The construction of TFB designs is the determination of values of the $\delta_{jt}^{i}$ to meet criteria to be developed.

Some matrix notation will be needed. Let $\mathbf{I}_n$ be the identity matrix of order $n$, $\mathbf{0}_{m \times n}$ be the $m \times n$ null matrix, $\mathbf{J}_{m \times n}$ be the $m \times n$ matrix with unit elements, $\mathbf{1}_n$ be the $n$-dimensional column vector with unit elements, and $\mathbf{B} \otimes \mathbf{C}$, the Kronecker product of matrices $\mathbf{B}$ and $\mathbf{C}$. Elements in vectors and matrices need to be ordered in a consistent way; for convenience, lexicographic order is used.

The model (2.3) in matrix notation is

$$(2.5) \quad \mathbf{Y} = \mathbf{X}\gamma + \mathcal{E} = \mathbf{X}_\mu \mu + \mathbf{X}_\tau \tau + \mathbf{X}_\beta \beta + \mathbf{X}_\theta \theta + \mathcal{E},$$

$\mathbf{Y}$ and $\mathcal{E}$ being $bk$-element column vectors with elements $y_{jt}$ and $\in_{jt}$ in subscript lexicographic order on $(j, \mathbf{t})$, $\tau' = (\tau_1, \cdots, \tau_v)$, $\beta' = (\beta_1, \cdots, \beta_b)$, $\theta = (\theta_{\alpha_1}, \cdots, \theta_{\alpha_p})$, $\gamma' = (\mu, \tau', \beta', \theta')$, $\mathbf{X} = (\mathbf{X}_\mu, \mathbf{X}_\tau, \mathbf{X}_\beta, \mathbf{X}_\theta)$, $\mathbf{X}_\mu = \mathbf{1}_{bk}$, $\mathbf{X}_\tau' = (\Delta_1', \cdots, \Delta_b')$, $\mathbf{X}_\beta = \mathbf{I}_b \otimes \mathbf{1}_k$, $\mathbf{X}_\theta = \mathbf{1}_b \otimes \phi$, where $\Delta_j$ is the $k \times v$ matrix with $\delta_{jt}^{i}$ as the $i$th element in row $\mathbf{t}$ and $\phi$ is the $k \times p$ matrix with $\phi_\alpha(\mathbf{t})$ in row $\mathbf{t}$ and column $\alpha$, $\alpha \in \mathcal{Q}$, and the rows of $\Delta_j$ and $\phi$ being in subscript lexicographic order on $\mathbf{t}$. Conditions (2.1), (2.2), (2.4) and the assumptions that each plot receives only one treatment and each treatment occurs at most once in a block imply:

$$(2.6) \quad \mathbf{1}_k' \phi = \mathbf{0}_{1 \times p},$$

$$(2.7) \quad \phi' \phi = \mathbf{I}_p,$$

$$(2.8) \quad \Delta_j \Delta_j' = \mathbf{I}_k, \qquad j = 1, \cdots, b.$$

Note that each $\mathbf{\Delta}_j$ is known as a permutation matrix in view of (2.8) and the fact that all elements of $\mathbf{\Delta}_j$ are 0 or 1.

REMARK 2.1.    It is necessary that $p < k$. From (2.7), since $\phi$ is $k \times p$, the rank of $\phi$ is $p$ and it is necessary that $k \geqslant p$. If $k = p$, $\phi$ would be nonsingular contradicted by (2.6). Hence $p < k$. The number of orthogonal polynomials defining the trend in (2.3) must be less than the number of plots per block.

3.    **A necessary and sufficient condition for a TFB design.**    The sums of squares for treatments (adjusted for blocks if necessary), blocks (unadjusted), and trend of the analysis of covariance for the design modelled by (2.5) or (2.3) are considered. The general approach of Searle (1975) is used. A design described by (2.5) is trend-free relative to the trend effects in that model when the indicated treatment and block sums of squares may be calculated as though the trend effects were omitted from the model.

Some additional notation is needed. Let the reduction from the sum of squares $\mathbf{Y'Y}$ to the appropriate residual sum of squares due to a fitting of the general linear model, $\mathbf{Y} = \mathbf{X}\gamma + \mathcal{E}$, by least squares be denoted by $R(\gamma)$. Given an appropriate partition of $\mathbf{X}$ and $\gamma$, $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ and $\gamma' = (\gamma'_1, \gamma'_2)$, the model becomes $\mathbf{Y} = \mathbf{X}_1\gamma_1 + \mathbf{X}_2\gamma_2 + \mathcal{E}$ and $R(\gamma) = R(\gamma_1, \gamma_2)$. Let

$$(3.1) \qquad R(\gamma_2|\gamma_1) = R(\gamma_1, \gamma_2) - R(\gamma_1)$$

represent the difference in reduction of sum of squares due to fittings of the models, $\mathbf{Y} = \mathbf{X}\gamma + \mathcal{E}$ and $\mathbf{Y} = \mathbf{X}_1\gamma_1 + \mathcal{E}$. In our problem, $R(\tau|\mu, \beta, \theta)$ and $R(\beta|\mu, \theta)$ represent the desired treatment and block sums of squares and $R(\tau|\mu, \beta)$ and $R(\beta|\mu)$ represent the corresponding sums of squares for the model like (2.5) with trend effects deleted. The circumstances under which

$$(3.2) \qquad R(\tau|\mu, \beta, \theta) = R(\tau|\mu, \beta)$$

and

$$(3.3) \qquad R(\beta|\mu, \theta) = R(\beta|\mu)$$

are investigated.

The sums of squares in (3.2) and (3.3) are obtained by the Searle method. We have

$$(3.4) \qquad R(\tau|\mu, \beta, \theta) = \mathbf{Y'D}_1\mathbf{X}_\tau\mathbf{Q}_1^-\mathbf{X}'_\tau\mathbf{D}_1\mathbf{Y},$$

$$(3.5) \qquad R(\tau|\mu, \beta) = \mathbf{Y'D}_2\mathbf{X}_\tau\mathbf{Q}_2^-\mathbf{X}'_\tau\mathbf{D}_2\mathbf{Y},$$

and

$$(3.6) \qquad R(\beta|\mu, \theta) = R(\beta|\mu) = \frac{1}{k}\mathbf{Y'X}_\beta\mathbf{X}'_\beta\mathbf{Y} - \frac{1}{bk}(\mathbf{X}'_\mu\mathbf{Y})^2,$$

where

$$(3.7) \qquad \mathbf{D}_i = \mathbf{I}_{bk} - \mathbf{X}_i(\mathbf{X}_i'\mathbf{X}_i)^- \mathbf{X}_i',$$

$$(3.8) \qquad \mathbf{Q}_i = \mathbf{X}_\tau'\mathbf{D}_i\mathbf{X}_\tau, \qquad\qquad i = 1, 2,$$

$$(3.9) \qquad \mathbf{X}_1 = (\mathbf{X}_\mu, \mathbf{X}_\beta, \mathbf{X}_\theta) \quad \text{and} \quad \mathbf{X}_2 = (\mathbf{X}_\mu, \mathbf{X}_\beta).$$

Note that $\mathbf{A}^-$ is a generalized inverse of a matrix $\mathbf{A}$. It is obvious from (3.6) that the trend effects in the model (2.5) do not affect calculation of the unadjusted block sum of squares nor do the values of the $\delta_{jt}^i$ in $\mathbf{\Delta}_j$. This is intuitively clear since trend effects have zero sums over plots within blocks from (2.1) and the definition of $\phi_\alpha(t)$. The ordinary block sum of squares calculated from block totals results from (3.6). Both (3.4) and (3.5) depend on $\mathbf{X}_\tau$ and hence might depend on the $\delta_{jt}^i$ when the model (2.5) applies. But we know that (3.5) does not depend on the $\delta_{jt}^i$, since it is the usual treatment sum of squares for the block design.

Consider $\mathbf{X}_i(\mathbf{X}_i'\mathbf{X}_i)^-\mathbf{X}_i'$, $i = 1, 2$, in (3.7). Searle (1971, page 20) shows that this quantity is invariant to the choice of the generalized inverse and it may be verified that we may take

$$(3.10) \qquad (\mathbf{X}_1'\mathbf{X}_1)^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{k}\mathbf{I}_b & 0 \\ 0 & 0 & \frac{1}{b}\mathbf{I}_p \end{bmatrix} \quad \text{and} \quad (\mathbf{X}_2'\mathbf{X}_2)^- = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{k}\mathbf{I}_b \end{bmatrix}.$$

Then

$$(3.11) \qquad \mathbf{D}_1 = \mathbf{I}_{bk} - \frac{1}{k}\mathbf{X}_\beta\mathbf{X}_\beta' - \frac{1}{b}\mathbf{X}_\theta\mathbf{X}_\theta' \quad \text{and} \quad \mathbf{D}_2 = \mathbf{I}_{bk} - \frac{1}{k}\mathbf{X}_\beta\mathbf{X}_\beta'.$$

In additional summarization and notation, $\mathbf{X}_\tau'\mathbf{X}_\tau = \mathbf{D}_R$, the diagonal matrix with $i$th diagonal element $r_i$, the number of replications of treatment $i$, $\mathbf{X}_\tau'\mathbf{X}_\beta = \mathbf{N}$, the $v \times b$ incidence matrix with elements $n_{ij} = 1$ or $0$ as treatment $i$ is or is not in block $j$, $\mathbf{X}_\tau'\mathbf{Y} = \mathbf{T}$, the column vector of treatment totals, and $\mathbf{X}_\tau'\mathbf{X}_\mu = \mathbf{R}$, the column vector with $i$th element $r_i$.

We continue with a definition of a trend-free design and develop a necessary and sufficient condition.

DEFINITION 3.1. A block design modelled by (2.5) is trend-free relative to the trend in the model if $R(\tau| \mu, \beta, \theta) = R(\tau| \mu, \beta)$.

THEOREM 3.1. *A necessary and sufficient condition for a block design to be trend-free is that*

$$(3.12) \qquad \mathbf{X}_\tau'\mathbf{X}_\theta = \mathbf{0}.$$

PROOF. To prove necessity, equate (3.4) and (3.5), substitute from (3.7) and (3.8), and obtain the equality,

$$(3.13) \qquad \mathbf{D}_1\mathbf{X}_\tau\mathbf{Q}_1^-\mathbf{X}_\tau'\mathbf{D}_1 = \mathbf{D}_2\mathbf{X}_\tau\mathbf{Q}_2^-\mathbf{X}_\tau'\mathbf{D}_2.$$

Pre- and post-multiplication of both sides of (3.13) by $X'_\tau$ and $X_\tau$ respectively reduce the equality to $Q_1 = Q_2$ or, equivalently (see 3.8), to

$$(3.14) \qquad\qquad X'_\tau(D_2 - D_1)X_\tau = 0.$$

Use of (3.11) in (3.14) yields (3.12) to complete the proof of necessity.

To prove sufficiency, (3.12) is used with (3.11) to show that $Q_1 = Q_2$ and that $X'_\tau D_1 = X'_\tau D_2$ and the equality of (3.4) and (3.5) follows.

REMARK. 3.1. The design construction problem may be reformulated. Note that $X'_\tau = (\Delta'_1, \cdots, \Delta'_b)$, where each $\Delta_j$ is a permutation matrix. Then (3.12) is equivalent to

$$(3.15) \qquad\qquad \Sigma_j \Delta'_j \phi = \Delta'_+ \phi = 0,$$

where $\Delta_+ = \Sigma_j \Delta_j$. The matrix $\Delta_+$ has nonnegative integer elements such that all row sums are $b$ and column sums are $r_i$, $i = 1, \cdots, v$. The TFB design is constructed if $\Delta_+$ satisfying (3.15) is found first and then if $b$ permutation matrices $\Delta_j$ are found such that $\Sigma_j \Delta_j = \Delta_+$.

Notice that (3.15) implies that

$$(3.16) \qquad\qquad \Sigma_{j=1}^b \Sigma_{t_1=1}^{s_1} \cdots \Sigma_{t_m=1}^{s_m} \delta_{jt}^i \phi_\alpha(t) = 0,$$

for all $i = 1, \cdots, v$ and all $\alpha \in \mathcal{C}$. In a TFB design, the total effect of each trend component over the plots assigned to any treatment is zero.

**4. Optimality properties of TFB designs.** Let us suppose that the primary purpose of an experiment is to compare treatment effects. Then it is desirable in most experiments that the treatment design matrix $X_\tau$ be chosen such that each possible treatment difference, $\tau_i - \tau_{i'}$, $i \neq i'$, $i, i' = 1, \cdots, v$, be estimable under model (2.5) as stated or with trend components omitted. Designs with this property are called *connected*. The selection of $X_\tau$ may be viewed as a two-stage process. The first stage is the determination of a way of blocking specified by the incidence matrix $N = X'_\tau X_\beta$ and the second stage is the allocation of treatments to plots within blocks. In this section we decompose the class of connected designs into subclasses with identical incidence matrices. Given (2.5), optimality properties possessed by TFB designs within these subclasses are considered.

4.1. *Estimability and connectedness.* Consider the model (2.5), first in its general form and then with trend terms omitted.

The least-squares normal equation associated with (2.5) is $X'X\hat{\gamma} = X'Y$, where $\hat{\gamma}$ is the estimator of $\gamma$. Since $N'1_v = k1_b$, $N1_b = R$, and $1'_v X'_\tau X_\theta = 1'_{bk} X_\theta = 0_{1 \times p}$, it reduces to

$$(4.1) \qquad \begin{bmatrix} bk & R' & k1'_b & 0_{1 \times p} \\ R & D_R & N & X'_\tau X_\theta \\ k1_b & N' & kI_b & 0_{b \times p} \\ 0_{p \times 1} & X'_\theta X_\tau & 0_{p \times b} & bI_p \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\tau} \\ \hat{\beta} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} G \\ T \\ B \\ W \end{bmatrix},$$

where $G = \mathbf{X}_\mu' \mathbf{Y}$, $\mathbf{T} = \mathbf{X}_\tau' \mathbf{Y}$, $\mathbf{B} = \mathbf{X}_\beta' \mathbf{Y}$, and $\mathbf{W} = \mathbf{X}_\theta' \mathbf{Y}$, are the observation total and vectors of treatment, block and trend totals respectively. Let

$$(4.2) \qquad \mathbf{H} = \left( \mathbf{0}_{v \times 1}, \mathbf{I}_v, -\frac{1}{k}\mathbf{N}, -\frac{1}{b}\mathbf{X}_\tau'\mathbf{X}_\theta \right),$$

premultiply both sides of (4.1) by $\mathbf{H}$, and obtain the reduced normal equation,

$$(4.3) \qquad \mathbf{C}\hat{\tau} = \mathbf{Q},$$

for the estimation of treatment parameters, where

$$(4.4) \qquad \mathbf{C} = \mathbf{D}_R - \frac{1}{k}\mathbf{N}\mathbf{N}' - \frac{1}{b}\mathbf{X}_\tau'\mathbf{X}_\theta\mathbf{X}_\theta'\mathbf{X}_\tau$$

and

$$(4.5) \qquad \mathbf{Q} = \mathbf{H}\mathbf{X}'\mathbf{Y} = \mathbf{T} - \frac{1}{k}\mathbf{N}\mathbf{B} - \frac{1}{b}\mathbf{X}_\tau'\mathbf{X}_\theta\mathbf{W}.$$

For a TFB design, $\mathbf{X}_\tau'\mathbf{X}_\theta = \mathbf{0}$ and $\mathbf{C}$ reduces to $\mathbf{C}_0 = \mathbf{D}_R - (1/k)\mathbf{N}\mathbf{N}'$, the matrix of the reduced normal equation for model (2.5) with trend terms omitted. Thus a TFB design is connected if the corresponding block design is connected. Use of Theorem 4.2.2 of Raghavarao (1971), with $rk\mathbf{A}$ being the rank of the matrix $\mathbf{A}$, leads to the following theorem:

THEOREM 4.1. *A* TFB *design under model* (2.5) *is connected if and only if* $rk\mathbf{C}_0 = rk(\mathbf{D}_R - (1/k)\mathbf{N}\mathbf{N}') = v - 1$. *If a* TFB *design exists, imposed on a block design connected for a model like* (2.5) *with trend effects absent, the* TFB *design is connected under* (2.5).

4.2 *Optimality properties.* Assume model (2.5) and the existence of a TFB design. The simple analysis of variance of Section 5 is used for the TFB design and analysis of covariance is assumed for the corresponding block design. Strong optimality properties may be shown for the TFB design relative to the corresponding block design with unrestricted randomization of treatments over plots within blocks. The matrices $\mathbf{C}$ in (4.4) and $\mathbf{C}_0$ provide the means, since $\mathbf{C}_0 - \mathbf{C} = (1/b)\mathbf{X}_\tau'\mathbf{X}_\theta\mathbf{X}_\theta'\mathbf{X}_\tau$ is nonnegative definite.

Consider the first form of model (2.5) and $\mathbf{u}'\gamma$, where $\mathbf{u}$ is a given nonnull vector of constants of proper dimension. It is well known that $\mathbf{u}'\gamma$ is estimable if and only if

$$(4.6) \qquad rk(\mathbf{u}, \mathbf{X}'\mathbf{X}) = rk(\mathbf{X}'\mathbf{X}).$$

A linear function $\mathbf{a}'\tau$ may be represented by $\mathbf{u}'\gamma$ with $\mathbf{u}' = (0, \mathbf{a}', \mathbf{0}_{1 \times b}, \mathbf{0}_{1 \times p})$, $\mathbf{a}$ being a $v$-element, nonnull vector of constants. By an argument similar to that of Chakrabarti (1962), it may be shown that $rk(\mathbf{u}, \mathbf{X}'\mathbf{X}) = b + p + rk(\mathbf{a}, \mathbf{C})$ and $rk(\mathbf{X}'\mathbf{X}) = b + p + rk\mathbf{C}$ under model (2.5). From substitution in (4.6), the necessary and sufficient condition for the estimability of $\mathbf{a}'\tau$ is that $rk(\mathbf{a}, \mathbf{C}) = rk\mathbf{C}$, or that there exist solutions $\rho$ to the equation,

$$(4.7) \qquad \mathbf{C}\rho = \mathbf{a}.$$

Part (a) of Lemma 2.1 of Ehrenfeld (1956) assures us of estimability of $\mathbf{a}'\tau$ for the TFB design, given solutions to (4.7), and solutions $\rho_0$ to

$$(4.8) \qquad\qquad\qquad \mathbf{C}_0\rho_0 = \mathbf{a},$$

since $\mathbf{C}_0 - \mathbf{C}$ is nonnegative definite.

Part (b) of Lemma 2.1 of Ehrenfeld leads to the strong result on variances of Theorem 4.2 below. Let $\hat{\tau}_0$ and $\hat{\tau}$ be the maximum likelihood (or least squares) estimators of $\tau$ under model (2.5) for the TFB design and the corresponding block design with within-block treatment randomization respectively. Then the estimators of $\mathbf{a}'\tau$ are $\mathbf{a}'\hat{\tau}_0$ and $\mathbf{a}'\hat{\tau}$ with variances, $\mathbf{a}'\mathbf{C}_0^-\mathbf{a}\sigma^2$ and $\mathbf{a}'\mathbf{C}^-\mathbf{a}\sigma^2$; see Raghavarao (1971, Section 4.2). Part (b) of the lemma results in the inequality, $\rho_0'\mathbf{C}_0\rho_0 \leqslant \rho'\mathbf{C}\rho$. Substitution for $\mathbf{a}$ from (4.7) and (4.8) yields $\mathbf{a}'\mathbf{C}_0^-\mathbf{a} = \rho_0'\mathbf{C}_0\rho_0$ and $\mathbf{a}'\mathbf{C}^-\mathbf{a} = \rho'\mathbf{C}\rho$. Let $V$ denote variance.

THEOREM 4.2. *Given model (2.5), a block design and any estimable $\mathbf{a}'\tau$, and a corresponding TFB design,*

(i) *$\mathbf{a}'\tau$ is estimable also for the TFB design, and*

(ii) *$V(\mathbf{a}'\hat{\tau}_0) \leqslant V(\mathbf{a}'\hat{\tau})$.*

Note that $\mathbf{a}'\tau$ may be estimable for the TFB design but not for the block design if an unfortunate realization of the within-block randomization occurred. Keifer (1950) considered optimality criteria. If $\psi$ is an optimality criterion such that $\psi(\mathbf{C}_1) \leqslant \psi(\mathbf{C}_2)$ whenever $\mathbf{C}_1 - \mathbf{C}_2$ is nonnegative definite, then the TFB design yields a smaller $\psi(\mathbf{C})$ than does any design for which the matrix $\mathbf{C}$ in (4.4), say $\overline{\mathbf{C}}$, has $\overline{\mathbf{C}} - \mathbf{C}_0$ nonnegative definite. The usual optimality criteria are of this nature. But Theorem 4.2 provides a stronger result for the limited comparison of the TFB design with the corresponding block design.

The practical effect of these optimality properties is to assure the user of TFB designs that they will be optimal as indicated relative to model (2.5) in comparison with the analysis of covariance for the corresponding block design with treatments randomized over plots within blocks. The user of TFB designs benefits also from the simple available analysis of variance calculations.

## 5. Analysis of variance of TFB designs.

Calculations and justifications for the analysis of variance of TFB designs may be developed by standard methods. A TFB design is derived from a standard block design for which the model is $\mathbf{Y} = \mathbf{X}_\mu\mu + \mathbf{X}_\beta\beta + \mathbf{X}_\tau\tau + \mathcal{E}$, model (2.5) with trend terms deleted, and an analysis of variance table is available. That standard analysis of variance is used to obtain sums of squares for treatments (adjusted for blocks if necessary) and blocks (unadjusted) and the corrected total sum of squares for the analysis of variance of the TFB design with model (2.5). The sum of squares for trend with $p$ degrees of freedom, when model (2.5) applies, is $\mathbf{W}'\mathbf{W}/b$, $\mathbf{W} = \mathbf{X}_\theta'\mathbf{Y}$. The error sum of squares is obtained by subtraction. The sum of squares for trend is partitioned easily into components, $W_i^2/b$, $i = 1, \cdots, p$, each with one degree of freedom.

While the paragraph above describes the necessary calculations, the general analysis of variance for a TFB design is displayed in Table 1. In that table, $G$, $\mathbf{T}$ and $\mathbf{B}$ are as defined for (4.1) and

$$F = \frac{1}{k}\gamma\mathbf{X}'\left(\mathbf{X}_\beta\mathbf{X}'_\beta - \frac{1}{b}\mathbf{X}_\mu\mathbf{X}'_\mu\right)\mathbf{X}\gamma, \quad \mathbf{C}_0 = \mathbf{D}_R - \frac{1}{k}\mathbf{N}\mathbf{N}'.$$

The sums of squares are independent and, when divided by $\sigma^2$, have chi-square distributions with degrees of freedom (d.f.) given in Table 1 and noncentrality parameters obtainable from the expected mean squares of the table, expected mean square divided by $\sigma^2$ less one.

Table 1 is simplified in well known simple situations. For a TFCB design, $k = v$ and $r_i = b$, $i = 1, \cdots, v$, and then

$$F = v\beta'\left(\mathbf{I}_b - \frac{1}{b}\mathbf{1}_b\mathbf{1}'_b\right)\beta, \quad \mathbf{C}_0 = b\left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}'_v\right),$$

$$\mathbf{Q}'\mathbf{C}_0^-\mathbf{Q} = \frac{1}{b}\left(\mathbf{T}'\mathbf{T} - \frac{1}{v}G^2\right), \quad \tau'\mathbf{C}_0\tau = b\tau'\left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}'_v\right)\tau,$$

and $\quad rk\mathbf{C}_0 = v - 1.$

For a TFBIB design with parameters $v$, $b$, $k$, $r$ and $\lambda$,

$$\mathbf{C}_0 = \frac{\lambda v}{k}\left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}'_v\right), \quad \mathbf{Q}'\mathbf{C}_0^-\mathbf{Q} = \frac{k}{\lambda v}\mathbf{Q}'\mathbf{Q},$$

$$\tau'\mathbf{C}_0\tau = \frac{\lambda v}{k}\tau'\left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}'_v\right)\tau, \text{ and } rk\mathbf{C}_0 = v - 1.$$

REMARK 5.1. The TFB designs developed for model (2.5) are trend-free also if treatment or block effects are random rather than fixed. Theorem 3.1 holds for random or mixed effects models since condition (3.12) does not depend on distributional assumptions on $\mathbf{Y}$. Expected mean squares in Table 1 would require minor reexpression for a model with random treatment or block effects. When block effects are random and treatment effects are fixed, treatment comparisons

TABLE 1
*General analysis of variance for a TFB design, model* (2.5).

| Source of variation | d.f. | Sum of squares | Expected mean square |
|---|---|---|---|
| Blocks (unadjusted) | $b - 1$ | $(\mathbf{B}'\mathbf{B} - \frac{1}{b}G^2)/k$ | $\sigma^2 + F/(b - 1)$ |
| Treatments (adjusted) | $rk\mathbf{C}_0$ | $\mathbf{Q}'\mathbf{C}_0^-\mathbf{Q}$ | $\sigma^2 + (\tau'\mathbf{C}_0\tau)/rk\mathbf{C}_0$ |
| Trend term 1 | 1 | $W_1^2/b$ | $\sigma^2 + b\theta_1^2$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| Trend term $p$ | 1 | $W_p^2/b$ | $\sigma^2 + b\theta_p^2$ |
| Error | $bk - b - p - rk\mathbf{C}_0$ | by subtraction | $\sigma^2$ |
| Total | $bk - 1$ | $\mathbf{Y}'\mathbf{Y} - G^2/bk$ | $\cdots$ |

may still be made and the demonstrated optimality properties of TFB designs apply.

**6. Examples.** Given an incidence matrix for a block design and a polynomial trend, a TFB design may or may not exist. Existence theorems and design construction will be discussed in a subsequent paper. Two examples are given below, one a $TF_2CB$ design and one a $TF_1BIB$ design. In the arrays below, letters represent treatments and rows represent blocks. Orthogonal trend components, without normalization, are given in the upper rows of the arrays.

EXAMPLE 1.   A $TF_2CB$ design, $v = 8, b = 4, p = 2$:

| $-7$ | $-5$ | $-3$ | $-1$ | $1$ | $3$ | $5$ | $7$ |
|------|------|------|------|-----|-----|-----|-----|
| $7$  | $1$  | $-3$ | $-5$ | $-5$ ， | $-3$ | $1$ | $7$ |
| A | B | C | D | E | F | G | H |
| G | H | B | A | C | D | F | E |
| F | E | H | G | B | A | D | C |
| D | C | E | F | H | G | A | B |

EXAMPLE 2.   A $TF_1BIB$ design, $v = 5, b = 10, k = 3, r = 6, \lambda = 3, p = 1$:

| $-1$ | $0$ | $1$ |
|------|-----|-----|
| A | B | C |
| A | B | D |
| A | B | E |
| C | D | A |
| C | E | A |
| D | E | A |
| D | B | C |
| E | B | C |
| E | B | D |
| C | D | E |

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