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ASYMPTOTIC EXPECTED INFERIOR SAMPLE SIZE OF A SEQUENTIAL TEST INVOLVING TWO POPULATIONS

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Let $X_1, \cdots$ be i.i.d. $\sim N(\mu_1, 1)$ and $Y_1, \cdots$ be i.i.d.$\sim N(\mu_2, 1)$. A symmetric sequential procedure for $H_0 : \mu_1 > \mu_2$ vs. $H_1 : \mu_1 < \mu_2$ is proposed in this paper. The expected number of observations taken from the inferior population is given in an asymptotic form, which is optimum in Farrell's sense.

1. Introduction and Summary. Let $X_1, \cdots$ be i.i.d. $\sim N(\mu_1, 1)$ and $Y_1, \cdots$ be i.i.d. $\sim N(\mu_2, 1)$. A sequential procedure for testing $H_0 : \mu_1 > \mu_2$ vs. $H_1 : \mu_1 < \mu_2$ consists of a terminal decision rule for choosing H_0 or H_1, a stopping rule for deciding when to stop, and a sampling rule for deciding (before stopping) whether the next observation should be an X or a Y. Let us assume that the X population is inferior if H_1 is true and the Y population is inferior if H_0 is true. It is desirable to have a procedure not only with controlled error probabilities but with minimized expected number of observations taken from the inferior population. This kind of problem has been considered by Robbins and Siegmund [5].

A sequential procedure is proposed in Section 2 and its symmetric nature is discussed in Section 4. Section 5 gives other properties of the procedure. In Section 6 the expected number of observations taken from the inferior population is estimated asymptotically as $|\mu_2 - \mu_1| \to 0$. The optimum property of this procedure in the sense of Farrell [1] is discussed in Section 7.

2. The sampling rule, the stopping rule, and the terminal decision rule. Let $f(y) = (|y|\log|y|^{-1}(\log_2|y|^{-1})^{\frac{3}{2}})^{-1}$ for $0 < |y| < e^{-e}$ and $f(y) = 0$ elsewhere. We write $\log_2 x = \log\log x$, $\log_3 x = \log\log_2 x$, etc. Define $H(x,t) = \int_{-\infty}^{\infty} \exp(xy - (t/2)y^2)f(y)dy$. Given $0 < \alpha < \frac{1}{2}$, set

$$b = (2\alpha)^{-1} \tag{1}$$

$$A(t) = A(t,b) = \inf\{x > 0 : H(x,t) \geqslant b\}. \tag{2}$$

From (2) clearly

$$H(x,t) \geqslant b \quad \text{if and only if} \quad |x| \geqslant A(t). \tag{3}$$

By the dominated convergence theorem, $b = \int_{-\infty}^{\infty}\exp(A(t)y - (t/2)y^2)f(y)dy = \int_0^{\infty} \exp(A(t)y - (t/2)y^2)f(y)dy + o(1)$ as $t \to \infty$. Arguing as Section 4 of [3], we have

$$A(t) = \big(2t\log_2 t + 4t\log_3 t + \log b + \log(\pi)^{\frac{1}{2}} + o(1)\big)^{\frac{1}{2}} \text{ as } t \to \infty. \tag{4}$$

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LEMMA 1. *Let $W(t)$, $t \geqslant 0$ be the standard Wiener process; then $P(H(W(t),t) \geqslant b$, for some $t \geqslant 0) = b^{-1}$*

PROOF. Denote $z(t) = H(W(t), t)$. It can be verified that $\{z(t), \beta(W(s), s \leqslant t), t \geqslant 0\}$ is a martingale. (cf. [3], Section 3). For any $\delta > 0$, $P(z(t) > \delta) = P(|W(t)| > A(t,\delta)) \to 0$ as $t \to \infty$ by (3) and (4). Since $z(0) = 1$ and $b > 1$, this lemma is proven by Lemma 1 in [3].

Let M_ν and N_ν denote the number of x's and the number of y's observed at stage ν of the experiment. Hence $M_\nu + N_\nu = \nu$. Set $\bar{X}_{M_\nu} = \Sigma_1^{M_\nu} X_i / M_\nu$, $\bar{Y}_{N_\nu} = \Sigma_1^{N_\nu} Y_i / N_\nu$, $t_\nu = M_\nu N_\nu / \nu$, and

(5) $$Z_\nu = Z_{M_\nu, N_\nu} = t_\nu\left(\bar{Y}_{N_\nu} - \bar{X}_{M_\nu}\right).$$

The stopping rule is defined:

(6) T = First $\nu \geqslant 2$ such that $|Z_\nu| \geqslant A(t_\nu)$; $T = \infty$ if no such ν occurs.

Set $\eta(\nu) = (1 + (\log_3 \nu - 1)^+)^{-1}$ for $\nu \geqslant 2$. Define the sampling rule: observe X_1, Y_1 first. When $T > \nu$ at stage ν, observe a Y if

(7) $$(N_\nu - M_\nu)/\nu \leqslant Z_\nu\left[A(t_\nu)(1 + \eta(\nu))\right]^{-1};$$

otherwise observe an X.

The terminal decision rule is defined on $(T < \infty)$: assert H_0 if $Z_T < 0$; assert H_1 if $Z_T > 0$.

(8) On $(T < \infty)$ set $M = M_T$ and $N = N_T$;
on $(T = \infty)$ set $M = \lim_{\nu\to\infty} M_\nu$ and $N = \lim_{\nu\to\infty} N_\nu$.

Set $\lambda(\nu) = \eta(\nu)/2(1 + \eta(\nu))$ for $\nu \geqslant 2$. Clearly $\lambda(\nu)\downarrow 0$ and $\nu\lambda(\nu)\uparrow\infty$ as $\nu \to \infty$.

LEMMA 2. $(T > \nu, M_\nu \leqslant \nu\lambda(\nu)) \subset (T > \nu, M_{\nu+1} > M_\nu) \subset (T > \nu, N_\nu > \nu\lambda(\nu))$.

PROOF. Cf. (5.4) and (5.13) in [3].

From Lemma 2 it is easily seen

(9) $$M_\nu \geqslant \frac{\nu}{2}\lambda(\nu) \text{ and } N_\nu \geqslant \frac{\nu}{2}\lambda(\nu) \text{ on } (T \geqslant \nu) \text{ for all } \nu \geqslant 2.$$

Hence $M_\nu N_\nu/\nu \geqslant (\nu/4)\lambda(\nu) \to \infty$ as $\nu \to \infty$ and $(T = \infty) \subset (\lim_{\nu\to\infty}(M_\nu N_\nu)/\nu = \infty)$. Since $\nu \geqslant M_\nu N_\nu/\nu$, $(\lim_{\nu\to\infty} M_\nu N_\nu/\nu = \infty) \subset (T = \infty)$. Denoting $\lim_{\nu\to\infty}(M_\nu N_\nu)/\nu$ by $MN/(M + N)$ on $(T = \infty)$, we have

LEMMA 3. $(T = \infty) = (MN/(M + N) = \infty)$.

3. Wiener process with drift *d*. Let $d = \mu_2 - \mu_1$. The definition of a Wiener process with drift d is in [5] and the following Lemma 4 is the Lemma 1 of [5].

LEMMA 4. *Let $X_1, \cdots; Y_1, \cdots$ be as in Section 1, $Z_{m,n} = (m\Sigma_1^n Y_i - n\Sigma_1^m X_j)/(m + n)$, $d = \mu_2 - \mu_1$, and $\{W_d(t), t \geqslant 0\}$ be a Wiener process with drift d.*

For any sequence of pairs (m, n) of positive integers which is nondecreasing in each coordinate, the sequences $\{Z_{m,n}\}$ and $\{W_d(mn/(m+n))\}$ have the same distribution.

Lemma 4 suggests that the sequential procedure proposed in Section 2 can be considered in terms of Wiener process with drift $d = \mu_2 - \mu_1$. To be precise, the following are defined: $\overline{M}_2 = \overline{N}_2 = 1$. For $\nu \geqslant 2$, define $\overline{N}_{\nu+1} = \overline{N}_\nu + 1$ if

$$(\overline{N}_\nu - \overline{M}_\nu)/\nu \leqslant \frac{W_d(\bar{t}_\nu)}{A(\bar{t}_\nu)(1+\eta(\nu))}; \tag{10}$$

otherwise $\overline{N}_{\nu+1} = \overline{N}_\nu$. Set $\overline{M}_\nu = \nu - \overline{N}_\nu$, $\bar{t}_\nu = \overline{M}_\nu \overline{N}_\nu / \nu$.

(11) $\overline{T}$ = First $\nu \geqslant 2$ such that $|W_d(\bar{t}_\nu)| \geqslant A(\bar{t}_\nu)$; $\overline{T} = \infty$ otherwise.

$\overline{M}$ and $\overline{N}$ are similarly defined as in (8). Lemma 3 becomes

$$(\overline{T} = \infty) = (\bar{t}_{\overline{T}} = \infty). \tag{12}$$

The M_ν, N_ν, $\overline{M}_\nu$, $\overline{N}_\nu$ are random variables; the m, n in Lemma 4 are constants. By sampling rule (7), $M_{\nu+1}$ and $N_{\nu+1}$ are $\beta(Z_i, i \leqslant \nu)$-measurable; by (10) $\overline{M}_{\nu+1}$ and $\overline{N}_{\nu+1}$ are $\beta(W_d(\bar{t}_i), i \leqslant \nu)$-measurable. Using Lemma 4, it is straightforward to justify the following

LEMMA 5. *$\{Z_\nu, \nu \geqslant 2\}$ and $\{W_d(\bar{t}_\nu), \nu \geqslant 2\}$ have the same distribution.*

4. The symmetry of the procedure. Let $1 = m_2 \leqslant m_3 \cdots$, and $1 = n_2 \leqslant n_3 \cdots$ be any sequences of positive integers such that $m_\nu + n_\nu = \nu$. Let $d = \mu_2 - \mu_1$ and $\delta > 0$ be any positive number. By Lemma 4 $\{Z_{m_\nu, n_\nu}\}$ in the case $d = \delta$ has the same distribution as $\{-Z_{n_\nu, m_\nu}\}$ in the case $d = -\delta$. By (7) it is direct to justify $P_\delta(M_i = m_i, N_i = n_i, i \leqslant \nu;\ |Z_{m_i, n_i}| < A(s_i), i \leqslant \nu - 1;\ Z_{m_\nu, n_\nu} \geqslant A(s_\nu)) = P_{-\delta}(M_i = n_i, N_i = m_i, i \leqslant \nu;\ |Z_{n_i, m_i}| < A(s_i), i \leqslant \nu - 1;\ Z_{n_\nu, m_\nu} \leqslant -A(s_\nu))$, where $s_i = m_i n_i / i$. From the above equality the next lemma follows easily.

LEMMA 6. $P_\delta(M = k) = P_{-\delta}(N = k)$, $E_\delta(M) = E_{-\delta}(N)$;
$P_\delta(T = \nu) = P_{-\delta}(T = \nu)$, $P_\delta(T < \infty) = P_{-\delta}(T < \infty)$;
$P_\delta(\textit{reject } H_1) = P_{-\delta}(\textit{reject } H_0)$.

In the following sections, only $d = \mu_2 - \mu_1 = \delta > 0$ and $d = 0$ are considered.

5. Other properties of the procedure. By the strong law of large numbers $P(\lim_{t\to\infty} W_\delta(t)/t = \delta) = 1$. By (4) $\lim_{t\to\infty} A(t)/t = 0$. With Lemma 5 and Lemma 6 we have

LEMMA 7. $P_\delta(T < \infty) = 1$.

Note $P_\delta(\overline{T} < \infty) = 1$ by Lemma 5. These kinds of paralleled results will not be stated explicitly in the following discussion.

LEMMA 8. $P_0(T = \infty) > 0$; $P_0(\bar{t}_{\overline{T}} = \infty) > 0$.

PROOF. By (12), $P_0(T = \infty) = P_0(\bar{T} = \infty) = P_0(t_{\bar{T}} = \infty) = P(|W_0(t_\nu)| < A(t_\nu)$, for all $\nu \geqslant 2) \geqslant P(H(W_0(t), t) < b$, for all $t \geqslant 0) = 1 - 1/b = 1 - 2\alpha > 0$, by (3) and Lemma 1.

LEMMA 9. *P_δ(reject H_1) $\leqslant \alpha$.*

PROOF. This follows from Lemmas 1 and 5 and symmetry.

For all $\gamma > 0$, $(t_{\bar{T}} = \gamma) = \cup_{i \geqslant 2}(T = i, \bar{t}_i = \gamma) \in \beta(W_\delta(s), s \leqslant \gamma)$. Hence $t_{\bar{T}}$ is a stopping time.

LEMMA 10. $\int_{(t_{\bar{T}} \leqslant c)}(W_\delta(t_{\bar{T}}) - A(t_{\bar{T}}))\, dP \leqslant \delta + 1$.

PROOF. Cf. Lemma 2 of [6]. Note $\bar{t}_{i+1} - \bar{t}_i \leqslant 1$ for all $i \geqslant 2$.

LEMMA 11. $E_\delta(t_{\bar{T}}) < \infty$.

PROOF. The main idea appears in several places, e.g., [4]; the detailed proof is omitted.

LEMMA 12. $E_\delta(t_{\bar{T}}) \sim 2\delta^{-2}(\log_2 \delta^{-2}) P_0(t_{\bar{T}} = \infty)$ *as* $\delta \downarrow 0$.

PROOF. $t_{\bar{T}}$ is written as T to simplify notation. $E_\delta(T) \geqslant tP_\delta(T > t) = t\int_{(T>t)} \exp(\delta W_\delta(t) - t\delta^2/2)\, dP_0$ (cf. [4]). By Fatou's lemma,

$$\liminf_{\delta\downarrow 0} E_\delta(T) \geqslant t \liminf_{\delta\downarrow 0} P_\delta(T > t) \geqslant tP_0(T > t). \tag{13}$$

Letting $t \to \infty$, we have $\lim_{\delta\downarrow 0} E_\delta T = \infty$ by Lemma 8. Since $E_\delta(T) < \infty$, by Wald's lemma, $P_\delta(T < \infty) = 1$ and Lemma 10,

$$\begin{aligned} \delta E_\delta(T) &= EW_\delta(T) = E_\delta(W_\delta(T) - A(T)) + E_\delta(A(T)) \\ &\leqslant \delta + 1 + E_\delta(A(T)). \end{aligned}$$

Arguing exact as in Section 3 of [4] gives that for all $\varepsilon > 0$

$$E_\delta(T) \leqslant 2(1 + \varepsilon)\, \delta^{-2}(\log_2 \delta^{-1}) P_0(T = \infty) \text{ as } \delta > 0. \tag{14}$$

On the other hand, note that $(t_1 \leqslant T \leqslant t_2) \subset (|W_\delta(t)| \geqslant A(t)$ for some $t_1 \leqslant t \leqslant t_2)$ and $A(t) \sim (2t \log_2 t)^{\frac{1}{2}}$. Arguing as in [6] (cf. Lemma 9, Lemma 12, and (40) in [6]) gives that for all $\varepsilon > 0$

$$\liminf_{\delta \to \infty} P_\delta\big(T \geqslant 2(1 - \varepsilon)\, \delta^{-2} \log_2 \delta^{-1}\big) \geqslant P_0(T = \infty). \tag{15}$$

By (14) and (15), this lemma is proven.

THEOREM 1. $E_\delta(MN/(M + N)) \sim 2\delta^{-2}(\log_2 \delta^{-1}) P_0(T = \infty)$ *as* $\delta \downarrow 0$.

PROOF. By Lemma 5, Lemma 8, Lemma 11 and Lemma 12.

6. The asymptotic behavior of $E_\delta(M)$ as $\delta \downarrow 0$.

LEMMA 13. *Let $X_n \geqslant 0$, $n \geqslant 1$ be a sequence of random variables. If $\lim_{n\to\infty} EX_n = a > 0$ and for all $\gamma > 0$, $\liminf_{n\to\infty} P(X_n > 1 - \gamma) \geqslant a$, then for all $\varepsilon > 0$, $\lim_{n\to\infty} \int_{(X_n \geqslant 1+\varepsilon)} X_n\, dP = 0$.*

PROOF. Given $\varepsilon > 0$, let $q > 0$ be such that $1 > \varepsilon q$. Set $I_n = \int_{(X_n \geqslant 1+\varepsilon)} X_n \, dP$.

$$\begin{aligned} EX_n &\geqslant I_n + \int_{(1+\varepsilon > X_n > 1-\varepsilon q)} X_n \, dP \\ &\geqslant I_n + (1-\varepsilon q)(P(X_n > 1-\varepsilon q) - P(X_n \geqslant 1+\varepsilon)) \\ &\geqslant I_n + (1-\varepsilon q)P(X_n > 1-\varepsilon q) - \frac{1-\varepsilon q}{1+\varepsilon} I_n. \end{aligned}$$

As $n \to \infty$, $a \geqslant \varepsilon(1+q)/(1+\varepsilon) \limsup I_n + (1-\varepsilon q)a$. Hence $qa \geqslant ((1+q)/(1+\varepsilon)) \limsup I_n$. As $q \downarrow 0$, $\lim_{n\to\infty} I_n = 0$. To simplify notation, denote $f(\delta) = 2\delta^{-2} \log\log \delta^{-1}$ from now on.

LEMMA 14. *For all* $\varepsilon > 0$,

$$\int_{(MN > (1+\varepsilon)f(\delta)(M+N))} \frac{MN}{(M+N)f(\delta)} dP_\delta \to 0 \quad \text{as} \quad \delta \to 0.$$

PROOF. By Theorem 1, (15) and Lemma 13.

LEMMA 15. *If* $\lim_{\delta\downarrow 0} P_\delta(A) = 0$, *then* $\int_A MN/((M+N)f(\delta))\, dP_\delta \to 0$ *as* $\delta \to 0$.

PROOF. Note $A \subset (MN > 2(M+N)f(\delta)) \cup (A(MN \leqslant 2(M+N)f(\delta)))$ and Lemma 14.

LEMMA 16. *For all* $\varepsilon > 0$ $\lim_{\delta\downarrow 0} P_\delta(M > (1+\varepsilon)f(\delta)) = 0$.

PROOF. Cf. (5.10) to (5.14) in [5] and (7) of [3].

LEMMA 17. *If* $\lim_{\delta\downarrow 0} P_\delta(A) = 0$, *then* $\int_A M/f(\delta)\, dP_\delta \to 0$ *as* $\delta \downarrow 0$, *and* $E_\delta(M) < \infty$.

PROOF. Set $M_1 = N_1 = 0$.

$$\begin{aligned} \int_A \frac{MN}{M+N} dP_\delta &= \Sigma_{\nu=2}^{\infty} \int_{A(T \geqslant \nu)} \{M_\nu N_\nu/\nu - M_{\nu-1}N_{\nu-1}/(\nu-1)\}\, dP_\delta \\ &\geqslant \Sigma_{\nu=2}^{\infty} \int_{A(T \geqslant \nu,\, M_\nu > M_{\nu-1})} N_\nu^2/\nu(\nu-1)\, dP_\delta \qquad \text{(Cf(5.16) of [5])} \\ &\geqslant \tfrac{1}{18} \Sigma_{\nu=2}^{\infty} P_\delta(A, T \geqslant \nu, M_\nu > M_{\nu-1}, N_{\nu-1} > (\nu-1)/3) \\ &= \tfrac{1}{18}\{\Sigma_{\nu=2}^{\infty} P_\delta(A, T \geqslant \nu, M_\nu > M_{\nu-1}) - \Sigma_{\nu=1}^{\infty} P_\delta(A, T > \nu, M_{\nu+1} \\ &\qquad > M_\nu, N_\nu \leqslant \nu/3)\} \\ &= \tfrac{1}{18}\{\textstyle\int_A M\, dP_\delta - \Sigma_\nu P_\nu\}, \text{ say.} \end{aligned}$$

Set $\nu_k = [kf(\delta)]$.

$$\begin{aligned} \Sigma_\nu P_\nu &= \Sigma_{k=0}^{\infty} \Sigma_{\nu=1+\nu_k}^{\nu_{k+1}} P_\nu \\ &\leqslant \nu_1 P_\delta(A) + \Sigma_{k\geqslant 1} \Sigma_\nu P_\delta\Big(M_{\nu+1} > M_\nu, M_\nu \geqslant \frac{2\nu}{3}, \frac{\nu}{3} \geqslant N_\nu \geqslant \nu\lambda(\nu)\Big), \\ &\hspace{20em} \text{by Lemma 2} \\ &\leqslant o(f(\delta)) + \Sigma_{k\geqslant 1} \Sigma_\nu P_\delta\big(\overline{Y}_{N_\nu} - \overline{X}_{M_\nu} \leqslant 0, M_\nu N_\nu/\nu \geqslant \tfrac{2}{3}\nu\lambda(\nu)\big), \quad \text{by (7)} \\ &\leqslant o(f(\delta)) + \Sigma_{k\geqslant 1} f(\delta) P_\delta\big(W_\delta(t_\nu)/t_\nu \leqslant 0, t_\nu \geqslant \tfrac{2}{3}\nu_k\lambda(\nu_k)\big) \\ &\leqslant o(f(\delta)) + f(\delta)\Sigma_{k\geqslant 1} 2\Big(1 - \Phi\Big(\delta\big(\tfrac{2}{3}\nu_k\lambda(\nu_K)\big)^{\frac{1}{2}}\Big)\Big), \qquad \text{by (7) of [3].} \end{aligned}$$

Note $\nu_k \to \infty$ as $\delta \downarrow 0$ and $\nu\lambda(\nu) = \nu/(2\log_3 \nu)$ for large ν and $\delta(\frac{2}{3}\nu_k\lambda(\nu_k))^{\frac{1}{2}} \geqslant k^{\frac{1}{4}}((2\log\delta^{-1})/(3\log_3 f(\delta)))^{\frac{1}{2}} = k^{\frac{1}{4}}g(\delta)$, where $g(\delta) \to \infty$ as $\delta\downarrow 0$. $\Sigma_\nu P_\nu \leqslant o(f(\delta)) + f(\delta)\int_0^\infty (1 - \Phi(x^{\frac{1}{4}}g(\delta)))\,dx = o(f(\delta))$. Hence $\int_A M/f(\delta)\,dP_\delta \to 0$ as $\delta \to 0$ by Lemma 15. Let A be the whole space; it is easily seen $E_\delta(M) < \infty$.

LEMMA 18. *For all* $\varepsilon > 0$ $\limsup_{\delta\downarrow 0} E_\delta(M/f(\delta)) \leqslant (1+\varepsilon)P_0(T = \infty)$.

PROOF. $E_\delta(M/f) = \int_{(M \leqslant \gamma f)} M/f\,dP_\delta + \int_{(\gamma f < M \leqslant (1+\varepsilon)f)} M/f\,dP_\delta + \int_{((1+\varepsilon)f < M)} M/f\,dP_\delta$, where $f = f(\delta)$ and $\gamma > 0$ arbitrarily small.

$$\begin{aligned} E_\delta(M/f) &\leqslant \gamma + (1+\varepsilon)P_\delta(M > \gamma f) + o(1), \text{ by Lemma 16 and Lemma 17} \\ &\leqslant \gamma + (1+\varepsilon)P_\delta(MN/(M+N) \geqslant \gamma f/4), \text{ by (9)} \\ &\leqslant \gamma + (1+\varepsilon)P_\delta(\bar{t_T} > t_0), \text{ where } t_0 \leqslant \frac{\gamma}{4}f \end{aligned}$$

$$\limsup_{\delta\downarrow 0} E_\delta(M/f) \leqslant \gamma + (1+\varepsilon)P_0(\bar{t_T} > t_0) (\text{cf. (13) or (34) in [4]}).$$

Letting $t_0 \to \infty$, $\gamma\downarrow 0$, we have proved the lemma.

THEOREM 2. $E_\delta(M) \sim 2\delta^{-2}(\log\log\delta^{-1})P_0(T = \infty)$ *as* $\delta\downarrow 0$.

PROOF. Note $M \geqslant MN/(M+N)$ and Theorem 1 and Lemma 18.

7. The optimum property of the procedure.

For any sequential procedure with $P_0(M'N'/(M'+N') = \infty) > 0$ (M', N' similarly defined as in (8)), we have

$$\limsup_{\delta\to 0}\{E_\delta M'/2\delta^{-2}\log_2\delta^{-1}\} \geqslant P_0(M'N'/(M'+N') = \infty) \quad (\text{cf. [1]}).$$

The procedure in Section 2 is optimum in the sense that

$$\lim_{\delta\to 0}\{E_\delta M/2\delta^{-2}\log_2\delta^{-1}\} = P_0(MN/(M+N) = \infty).$$

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