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EXPECTED SAMPLE SIZE SAVINGS FROM CURTAILED PROCEDURES FOR THE t-TEST AND HOTELLING'S T^2

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Brown, Cohen and Strawderman propose curtailed procedures for the t-test and Hotelling's T^2. In this paper we present the exact forms of these procedures and examine the expected sample size savings under the null hypothesis. The sample size savings can be bounded by a constant which is independent of the sample size. Tables are given for the expected sample size savings and maximum sample size saving under the null hypothesis for a range of significance levels (α), dimensions (p) and sample sizes (n).

1. Introduction. Brown, Cohen and Strawderman (1979) propose a curtailed t-test procedure for testing the one-sided hypothesis $H_0 : \theta < 0$ vs. $H_1 : \theta > 0$, where X_i, $i = 1, 2, \cdots$ is a sequence of independent, identically distributed normal random variables with unknown mean θ and unknown variance σ^2. The curtailed procedure is shown to be better than the fixed sample size t-test and admissible under some restrictions on the critical value when the risk consists of two components, the probability of error and the expected sample size.

Similar results are obtained for Hotelling's T^2 to test $H_0 : \boldsymbol{\theta} = \mathbf{0}$ vs. $H_1 : \boldsymbol{\theta} \neq \mathbf{0}$, where $\mathbf{X}_i$, $i = 1, 2, \cdots$ is a sequence of p-dimensional independent, identically distributed normal random vectors with unknown p-dimensional mean $\boldsymbol{\theta}$ and unknown covariance matrix.

The object of this paper is to examine the savings in sample size which can be obtained using the proposed curtailed procedures rather than fixed sample size tests.

2. Curtailed procedures. The one-sided fixed sample size t-test based on a sample of size n is to reject if

$$T_{n-1} = ((n-1)/n)^{\frac{1}{2}} S_n \Big/ \left\{\Sigma_{i=1}^n \left(X_i - \bar{X}_n\right)^2\right\}^{\frac{1}{2}} > C, \tag{2.1}$$

where $S_n = \Sigma_{i=1}^n X_i$ and $\bar{X}_n = (1/n)\Sigma_{i=1}^n X_i$. If $C > 0$, then (2.1) is equivalent to rejecting if $S_n > 0$ and

$$S_n^2 / \left(\Sigma_{i=1}^n X_i^2\right) > nC^2 / \left\{(n-1) + C^2\right\} = K_n. \tag{2.2}$$

To obtain a test with significance level α, C is set equal to $t_{n-1}(1-\alpha)$, the $(1-\alpha)$ percentile of the t-distribution.

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The curtailed procedure improves on the fixed sample t-test for any critical value $|C| > 1$. We consider the case $C > 1$. Following Brown, Cohen and Strawderman, the curtailed procedure is defined by first determining the integer $k_n = 1, 2, \cdots$ for which the interval

$$\left(\{(n-1)(k_n-1)/(n-k_n+1)\}^{\frac{1}{2}}, \{(n-1)k_n/(n-k_n)\}^{\frac{1}{2}}\right]$$

contains C. Since

$$C = \{(n-1)K_n/(n-K_n)\}^{\frac{1}{2}}, \tag{2.3}$$

we have that $k_n - 1 < K_n \leqslant k_n$. The procedure specifies that we should stop and accept for the first m such that

(2.4)

$$((m-1)/m)^{\frac{1}{2}} S_m \Big/ \left\{\Sigma_{i=1}^m (X_i - \bar{X}_m)^2\right\}^{\frac{1}{2}} < \{(K_n + m - n)(m-1)/(n-K_n)\}^{\frac{1}{2}},$$

where $m = n + 1 - k_n, \cdots, n$. When $m = n$, stop and reject if (2.4) does not hold.

Let N be the random stopping time when using the curtailed procedure. Then N can take the values $n + 1 - k_n, \cdots, n$. Clearly, the maximum possible savings in sample size is $k_n - 1$. The expected savings in sample size is given by

$$E(n-N) = \Sigma_{m=n+1-k_n}^n (n-m)\Pr\{N = m\} = \Sigma_{m=n+1-k_n}^{n-1} \Pr\{N \leqslant m\}. \tag{2.5}$$

Note that

$$\{N \leqslant m\} = \left\{((m-1)/m)^{\frac{1}{2}} S_m \Big/ \left\{\Sigma_{i=1}^m (X_i - \bar{X}_m)^2\right\}^{\frac{1}{2}} < \{(K_n + m - n)(m-1)/(n-K_n)\}^{\frac{1}{2}}\right\}.$$

In addition, for fixed m,

$$T_{m-1} = ((m-1)/m)^{\frac{1}{2}} S_m \Big/ \left\{\Sigma_{i=1}^m (X_i - \bar{X}_m)^2\right\}^{\frac{1}{2}}$$

has a t-distribution with $m - 1$ degrees of freedom. Recall that for $C = t_{n-1}(1 - \alpha)$, we have

$$K_n = nt_{n-1}^2/(n - 1 + t_{n-1}^2).$$

Substituting into (2.5) yields that

(2.6)

$$E(n-N) = \Sigma_{m=n+1-k_n}^{n-1} \Pr\left\{T_{m-1} < \left(\frac{\{(m-n)(n-1) + mt_{n-1}^2\}(m-1)}{n(n-1)}\right)^{\frac{1}{2}}\right\}$$

and

$$k_n = \left[nt_{n-1}^2/(n - 1 + t_{n-1}^2)\right] + 1, \tag{2.7}$$

where $[\cdot]$ is the greatest integer function.

As noted in Brown, Cohen and Strawderman (1979), a similar procedure can be derived for Hotelling's T^2 by observing that for $\boldsymbol{\theta} = \mathbf{0}$,

$$T_n^2 = n \sup_{\mathbf{a} \in R^p} \left(\mathbf{a}'\overline{\mathbf{X}}_n\right)^2 / \mathbf{a}'\mathbf{Q}_n\mathbf{a}, \tag{2.8}$$

where $\overline{\mathbf{X}}_n = (1/n)\Sigma_{i=1}^n \mathbf{X}_i$, $\mathbf{Q}_n = \Sigma_{i=1}^n (\mathbf{X}_i - \overline{\mathbf{X}}_n)(\mathbf{X}_i - \overline{\mathbf{X}}_n)'/(n-1)$.

The fixed sample size procedure accepts when $T_n^2 < C$ or, equivalently, when

$$T_n^{*2} = \sup_{\mathbf{a} \in R^p}\{(\Sigma_{i=1}^n \mathbf{a}'\mathbf{X}_i)^2/\Sigma_{i=1}^n(\mathbf{a}'\mathbf{X}_i)^2\} < K_n = nC/(n-1+C). \tag{2.9}$$

Proceeding analogously to the t-test case, the curtailed procedure is defined by specifying that we should stop and accept at the first value m such that

$$T_m^2 < (K_n - n + m)(m-1)/(n - K_n)$$

for $m = m^*, \cdots, n$, where

$$m^* = \max\{p+1, n+1-k_n\};$$

k_n is an integer with $k_n - 1 < K_n \leqslant k_n$; and T_m^2 is Hotelling's T^2 based on the first m observations. Again we note that the expected sample size savings is given by

$$\begin{aligned} E(n-N) &= \Sigma_{m=m^*}^{n-1} \Pr\{N \leqslant m\} \\ &= \Sigma_{m=m^*}^{n-1} \Pr\{T_m^2 < (K_n - n + m)(m-1)/(n-K_n)\}, \end{aligned} \tag{2.10}$$

where N is the random stopping time. We recall that for fixed $m > p$,

$$(m-p)/\{(m-1)p\}T_m^2 \sim \mathcal{F}_{p, m-p}.$$

If we let the upper $(1-\alpha)$ percentile of the F distribution be denoted by $F_{p,n-p}$, then we have

$$C = \{p(n-1)/(n-p)\}F_{p,n-p}$$

and from (2.9)

$$K_n = npF_{p,n-p}/(n - p + pF_{p,n-p}). \tag{2.11}$$

Substituting into (2.10) yields

$$E(n-N) = \Sigma_{m=m^*}^{n-1} \Pr\left\{\mathcal{F}_{p,m-p} < \frac{(m-p)\{(n-p)(m-n) + mpF_{p,n-p}\}}{pn(n-p)}\right\}. \tag{2.12}$$

Note that the two sided t-test is a special case of Hotelling's T^2 when $p = 1$.

3. Expected sample size savings. Values for the expected sample size savings under the null hypothesis have been calculated using equations (2.6) and (2.12). In Table I we present the expected sample size savings, $E(n-N)$, under the null hypothesis and the maximum possible savings, $\max(n-N)$, for the significance levels $\alpha = 0.05$ and 0.01 for the one-sided curtailed t-test. Similar results are provided in Table II for the curtailed Hotelling's T^2 procedure for various dimensions (p). As n increases, there is relatively little change in $E(n-N)$ or

TABLE I.

$E(n - N)$, $(\max(n - N))$ Expected and maximum sample size savings for the one-sided curtailed t-test procedure under the null hypothesis.

$\alpha \setminus n$	10	20	40	120
.05	1.68 (2)	1.70 (2)	1.70 (2)	1.70 (2)
.01	3.59 (4)	4.30 (5)	4.42 (5)	4.49 (5)

TABLE II.

$E(n - N)$, $(\max(n - N))$ Expected and maximum sample size savings for the curtailed Hotelling's T^2 procedure under the null hypothesis.

p	n / α	10	20	40	60	120	500
1	.05	2.04 (3)	2.25 (3)	2.32 (3)	2.34 (3)	2.36 (3)	2.37 (3)
	.01	3.68 (5)	4.41 (6)	4.90 (6)	5.00 (6)	5.09 (6)	5.15 (6)
2	.05	2.77 (5)	3.26 (5)	3.44 (5)	3.49 (5)	3.53 (5)	3.57 (5)
	.01	4.20 (6)	5.47 (8)	6.14 (8)	6.33 (8)	6.50 (9)	6.68 (9)
6	.05	1.90 (3)	4.44 (10)	5.43 (11)	5.72 (12)	5.99 (12)	6.19 (12)
	.01	2.37 (3)	6.43 (13)	8.45 (14)	9.09 (15)	9.72 (16)	10.2 (16)
10	.05		4.12 (9)	6.31 (16)	6.91 (17)	7.47 (17)	7.86 (18)
	.01		5.53 (9)	9.34 (19)	10.5 (21)	11.6 (22)	12.5 (22)
20	.05			6.45 (19)	8.24 (28)	9.78 (30)	10.8 (31)
	.01			8.92 (19)	11.9 (32)	14.6 (35)	16.5 (36)
40	.05				7.01 (19)	11.9 (52)	14.7 (54)
	.01				9.47 (19)	14.2 (54)	21.8 (62)
100	.05					7.44 (19)	21.2 (121)
	.01					9.90 (19)	30.6 (131)

$\max(n - N)$ for fixed α and p. As p increases, for fixed α and n, $E(n - N)$ and $\max(n - N)$ increase until the constraint that at least $p + 1$ out of n points must be observed before any possible curtailment occurs restricts the value of $\max(n - N)$. From (2.7), letting $n \to \infty$ we see that the maximum savings in sample size is given by

$$\lim_{n\to\infty}\max(n - N) = \lim_{n\to\infty} k_n - 1 = \left[z_{1-\alpha}^2\right],$$

for the one-sided curtailed t-test, where $z_{1-\alpha}$ is the $(1 - \alpha)$ percentile of the standard normal. Similarly, for the curtailed Hotelling's T^2 procedure, the maximum savings in sample size is given by letting $n \to \infty$ in (2.11) yielding

$$\lim_{n\to\infty}\max(n - N) = \lim_{n\to\infty} k_n - 1 = \left[\chi_p^2(1 - \alpha)\right],$$

where $\chi_p^2(1 - \alpha)$ is the $(1 - \alpha)$ percentile of the chi-squared distribution with p degrees of freedom.

In summary, we note that while the curtailed procedures considered in this paper represent improvements over the noncurtailed or standard fixed sample tests, they have two discouraging properties. First, the maximum sample size savings, and thus the expected sample size savings under the null hypothesis, can be bounded independently of the sample size. Secondly, an early decision implies that the null hypothesis has been accepted, i.e., one cannot make an early decision in which one rejects the null hypothesis. In contrast, for cases where a univariate nonparametric one-sample procedure is appropriate, Herrmann and Szatrowski (1978) show that the curtailed form of the Fraser test has the property that the expected sample size savings under the null hypothesis is unbounded as $n \to \infty$ and the curtailed Fraser test can result in an early decision which accepts the null hypothesis or rejects the null hypothesis.

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