The Annals of Statistics
1980, Vol. 8, No. 2, 322–346

ASYMPTOTIC DISTRIBUTION OF L_2 NORMS OF THE DEVIATIONS OF DENSITY FUNCTION ESTIMATES[1]

BY ALDO JOSÉ VIOLLAZ

Universidad Nacional de Tucumán, Argentina

Let $(\phi_\nu)_{\nu=0}^{\infty}$ be a complete orthonormal system of functions on a interval $[a, b]$ and let w be a function defined on $\mathbf{R}$ with support $[a, b]$ and strictly positive on (a, b). Let $(x_j)_{j=1}^{\infty}$ be i.i.d.rv's absolutely continuous with density function f with respect to Lebesgue measure μ on $\mathbf{R}$. Let $\hat{f}_n$ be the estimate of f defined for $x \in (a, b)$ by $\hat{f}_n(x) = \sum_{\nu=0}^{m} a_\nu(m)\hat{d}_\nu\phi_\nu(x)/w(x)$, where $a_\nu(m)$, $(m = 0, 1, \cdots, \nu = 0, 1, \cdots, m)$ is a sequence in $\mathbf{R}$ and $\hat{d}_\nu = n^{-1}\sum_{j=1}^{n}\phi_\nu(X_j)w(X_j)$. In this work the asymptotic distributions of the functionals $T_n = n(m+1)^{-1}\int_a^b(f_n - Ef_n)^2w^2\,dx$ and $T_n^* = n(m+1)^{-1}\int_a^b(f_n - f)^2w^2\,dx$ are found. These results are used to construct tests of goodness-of-fit analogous to those proposed by Bickel and Rosenblatt. The basic idea in obtaining the results consists in finding the asymptotic distribution of $T_n(T_n^*)$ with f_n replaced by a conveniently chosen Gaussian process and showing that the two functionals converge to the same law. For this the normalized and centered sample distribution function is approximated by an appropriate Brownian motion process by using a Skorohod-like imbedding due to Brillinger and Breiman.

1. Introduction. Let $(X_j)_{j=1}^{n}$ be independent and identically distributed random variables, absolutely continuous with common probability density function (p. d. function) f with respect to Lebesgue measure on $\mathbb{R}$. Let $[a, b]$ be a closed interval of $\mathbf{R}$ and $(\phi_\nu)_{\nu=0}^{\infty}$ be an orthonormal basis for $L_2[a, b]$.

In this work we find the asymptotic distribution of L_2 norms of the deviations of estimators of $f(x)$ for $x \in [a, b]$ constructed from orthogonal expansions of $f|_{[a,b]}$, where $f|_{[a,b]}$ stands for the restriction of f to $[a, b]$.

In order to avoid some technical difficulties caused by the end points a and b it is convenient, previous to the estimation of $f(x)$, to multiply f by a function w such that $f(x)w(x)$ tends to zero at an appropriate rate as x tends to a^+ and b^-. We call w a weight function.

DEFINITION 1.1. A weight function is a Borel function w: $\mathbb{R} \to \mathbb{R}$ such that $w(x) > 0$ for $x \in (a, b)$ and $w(x) = 0$ for $x \notin [a, b]$. Throughout this work we use the symbol w to represent a weight function and the symbol g to denote

$$g = fw \tag{1.1}$$

where f is the common p. d. function of the random variables X_j, $j = 1, 2, \cdots, n$.

Received August 1977, revised December 1978.

[1]This paper was prepared with the partial support of the Consejo Nacional de Investigaciones Cientificas y Técnicas de Argentina and the National Science Foundation Grant MPS73-08698A01. The paper is part of the author's Ph. D. thesis.

AMS 1970 *subject classifications.* Primary 62G05; secondary 62E20.

Key words and phrases. Asymptotic distribution, quadratic deviations of density function estimates, Skorohod-like imbedding, orthonormal expansions, goodness-of-fit.

The orthonormal expansion of g is

$$g \sim \Sigma_{\nu=0}^{\infty} d_\nu \phi_\nu \tag{1.2}$$

where

$$d_\nu = \int_a^b g(s)\phi_\nu(s)\,ds = \int_a^b f(s)w(s)\phi_\nu(s)\,ds. \tag{1.3}$$

Therefore it is natural to estimate d_ν by (see Kronmal and Tarter (1968), Rosenblatt (1971) or Viollaz (1976)),

$$\hat{d}_\nu = \frac{1}{n}\Sigma_{j=1}^{n}\phi_\nu(X_j)w(X_j) \tag{1.4}$$

and then to estimate $g(x)$ by

$$g_n(x) = \Sigma_{\nu=0}^{m} a_\nu(m)\hat{d}_\nu\phi_\nu(x) \tag{1.5}$$

where $m = m(n)$ is a monotone increasing sequence of integers and $a_\nu\ (m)$ is a sequence of real numbers. Since we have assumed that w is positive on the open interval (a, b) we have $f(x) = g(x)/w(x)$ if $x \in (a, b)$, and therefore replacing $g(x)$ by its estimator $g_n(x)$ we obtain the following estimator of $f(x)$ for $x \in (a, b)$:

$$f_n(x) = \Sigma_{\nu=0}^{m} a_\nu(m)\hat{d}_\nu\phi_\nu(x)/w(x). \tag{1.6}$$

After inserting (1.4) in (1.6) it follows that

$$f_n(x) = \frac{1}{n}\Sigma_{j=1}^{n} k_m(x, X_j)/w(x) \tag{1.7}$$

where

$$k_m(x, s) = \Sigma_{\nu=0}^{m} a_\nu(m)\phi_\nu(x)\phi_\nu(s)w(s). \tag{1.8}$$

In this paper we study the asymptotic distribution of the functionals

$$T_n = \frac{n}{m+1}\int_a^b [f_n(x) - Ef_n(x)]^2 w^2(x)\,dx, \tag{1.9}$$

and

$$T_n^* = \frac{n}{m+1}\int_a^b [f_n(x) - f(x)]^2 w^2(x)\,dx. \tag{1.10}$$

Let us observe that T_n and T_n^* can also be written as

$$T_n = \frac{n}{m+1}\int_a^b [g_n(x) - Eg_n(x)]^2\,dx \tag{1.11}$$

$$T_n^* = \frac{n}{m+1}\int_a^b [g_n(x) - g(x)]^2\,dx. \tag{1.12}$$

The asymptotic distribution of the functionals T_n and T_n^* was found by Bickel and Rosenblatt (1973) for density estimators of the form

$$f_n(x) = \frac{1}{nb(n)}\Sigma_{j=1}^{n} K\left(\frac{x - X_j}{b(n)}\right). \tag{1.13}$$

A central idea in the paper by Bickel and Rosenblatt consists in finding the asymptotic distribution of (1.9) and (1.10) with f_n replaced by a conveniently chosen Gaussian process, and showing that the two functionals converge to the same law. The basic technique in obtaining the results consists of approximating the normalized and centered distribution function by an appropriate Brownian motion process on convenient probability space, by using a Skorohod-like imbedding due to Brillinger and Breiman. In this paper we find the asymptotic distribution of the functionals T_n and T_n^* following closely Bickel and Rosenblatt's approach.

The reader is invited to read their paper for the parallels between the results for estimators of the form (1.13) and results obtained in this paper for estimators constructed using the trigonometric and Legendre orthonormal systems.

This paper is organized as follows. In Section 2 we construct approximations for T_n and T_n^* using a Skorohod-like imbedding. In Section 3 we find the asymptotic distribution of T_n for the case that the orthonormal system is the trigonometric one. In Section 4 we particularize the results of Section 3 for the Dirichlet and Fejér estimators. Section 5 deals with the asymptotic distribution of T_n for estimators constructed from Legendre orthonormal system. In Section 6 we find asymptotic distribution of T_n^* for both trigonometric and Legendre estimators. In Section 7 we study some tests of goodness-of-fit analogous to those proposed by Bickel and Rosenblatt.

2. Approximations. Since for the cases we are interested in, the variance of $(g_n(x) - Eg_n(x))$ is asymptotically equal to $C(x)(m+1)n^{-1}$, where $C(x)$ is a function of x independent of m and n (see Viollaz (1976)), it is convenient to normalize g_n by defining

$$Y_n(x) = \left(\frac{n}{m+1}\right)^{\frac{1}{2}}(g_n(x) - Eg_n(x)). \tag{2.1}$$

Let

$$Z_n^0(s) = n^{\frac{1}{2}}(F_n(F^{-1}(s)) - s), \tag{2.2}$$

where F is the distribution function of a random variable X with p.d. function f, and F_n is the empirical distribution function corresponding to a sample $(X_j)_{j=1}^n$ of n independent observations of X. Then (2.1) can be written as

$$Y_n(x) = (m+1)^{-\frac{1}{2}}\textstyle\int_a^b k_m(x, s)\, dZ_n^0(F(s)). \tag{2.3}$$

Approximations ${}_0Y_n$ and ${}_1Y_n$ for Y_n are obtained by defining

$${}_0Y_n(x) = (m+1)^{-\frac{1}{2}}\textstyle\int_a^b k_m(x, s)\, dZ^0(F(s)) \tag{2.4}$$

$${}_1Y_n(x) = (m+1)^{-\frac{1}{2}}\textstyle\int_a^b k_m(x, s)\, dZ(F(s)), \tag{2.5}$$

where Z stands for the standard Brownian motion process, and Z^0 is the Brownian bridge process, i.e.,

$$Z^0(t) = Z(t) - tZ(1), \qquad 0 \leqslant t \leqslant 1. \tag{2.6}$$

The processes ${}_0Y_n$ and ${}_1Y_n$ are well defined if

$$\int_a^b k_m^2(x, s) f(s)\, ds < \infty. \tag{2.7}$$

Condition (2.7) holds if f and w are bounded on $[a, b]$.

For convenience, let us suppose that all our processes are realized as random elements taking their values in the space $D[0, 1]$ or in the space $D[a, b]$. For $y \in D[a, b]$, let $\|y\| = \sup\{|y(s)| : a \leq s \leq b\}$. Our approximations rest on a result obtained independently by Brillinger (1969) and Breiman (1969) which we now state.

THEOREM 2.1. *There exists a probability space* $(\Omega, \mathcal{Q}, P)$ *on which one can construct versions of* Z_n^0 *and* Z^0 *such that*

$$\|Z_n^0 - Z^0\| = 0_p\big(n^{-\frac{1}{4}}(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{4}}\big). \tag{2.8}$$

We will study first the asymptotic distribution of the functional

$$T_n = \frac{n}{m+1}\int_a^b (g_n(x) - Eg_n(x))^2\, dx = \int_a^b Y_n^2(x)\, dx \tag{2.9}$$

as $n \to \infty$ and $m = m(n) \to \infty$.

Let ${}_0T_n$ and ${}_1T_n$ be the approximations to T_n defined by

$${}_0T_n = \int_a^b {}_0Y_n^2(x)\, dx \tag{2.10}$$

$${}_1T_n = \int_a^b {}_1Y_n^2(x)\, dx. \tag{2.11}$$

LEMMA 2.1. *Let f be a* p.d. *function and w a weight function both bounded on $[a, b]$. Let us assume that the system (ϕ_ν) and the function w are such that*:

(i) *for every ν, $\phi_\nu w$ is absolutely continuous on $[a, b]$ with derivative $(d/ds)(\phi_\nu(s)w(s))$ square integrable on $[a, b]$, and*

(ii)

$$\int_a^b \left|\frac{\partial}{\partial s} k_m(x, s)\right| ds = r(x)0(m \log m) \tag{2.12}$$

where $0(m \log m)$ *is uniform in x on $[a, b]$ and r is a positive function on (a, b) which is assumed to be square integrable (this is needed later). Then*

$$\left\|\frac{Y_n - {}_0Y_n}{r}\right\| = 0_p\big(m^{\frac{1}{2}}(\log m) n^{-\frac{1}{4}}(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{4}}\big). \tag{2.13}$$

PROOF. Since for every ν, $(\phi_\nu w)$ is absolutely continuous, with a square integrable derivative and, since f and w are bounded, integrating by parts we obtain

$$\begin{aligned}\int_a^b \phi_\nu(s) w(s)\, dZ_n^0(F(s)) &= \phi_\nu(b)w(b)Z_n^0(F(b)) - \phi_\nu(a)w(a)Z_n^0(F(a)) \\ &\quad - \int_a^b \frac{d}{ds}(\phi_\nu(s)w(s))Z_n^0(F(s))\, ds,\end{aligned} \tag{2.14}$$

and a similar expression holds for $\int_a^b \phi_\nu(s)w(s)\, dZ^0(F(s))$. Because

$$Z_n^0(F(b)) = Z_n^0(1) = 0, \qquad Z_n^0(F(a)) = Z_n^0(0) = 0 \tag{2.15}$$

we obtain from (2.14)

$$\int_a^b \phi_\nu(s)w(s)\, dZ_n^0(F(s)) = -\int_a^b \frac{d}{ds}(\phi_\nu(s)w(s))Z_n^0(F(s))\, ds \tag{2.16}$$

and similarly

$$\int_a^b \phi_\nu(s)w(s)\, dZ^0(F(s)) = -\int_a^b \frac{d}{ds}(\phi_\nu(s)w(s))Z^0(F(s))\, ds. \tag{2.17}$$

From (2.3), (2.4), (2.16) and (2.17) it follows that

$$\begin{aligned} |Y_n(x) - {}_0Y_n(x)| &= (m+1)^{-\frac{1}{2}}\left|\int_a^b \frac{\partial}{\partial s}k_m(x,s)(Z_n^0(F(s)) - Z^0(F(s)))\, ds\right| \\ &\leq (m+1)^{-\frac{1}{2}}\|Z_n^0 - Z^0\|\int_a^b\left|\frac{\partial}{\partial s}k_m(x,s)\right|ds. \end{aligned} \tag{2.18}$$

From Theorem 2.1, (2.12) and (2.18) the conclusion of the lemma follows. □

LEMMA 2.2. *Let the* p.d. *function f and the weight function w be bounded on $[a, b]$ such that $g = fw$ has Fourier coefficients d_ν satisfying the condition*

$$d_\nu = \int_a^b \phi_\nu(s)g(s)\, ds = O(\nu^{-\frac{1}{2}-\varepsilon}), \qquad \varepsilon > 0. \tag{2.19}$$

Let us assume that the numbers $a_\nu(m)$ are such that

$$\sup_m \sup_{0<\nu<m}|a_\nu(m)| < \infty. \tag{2.20}$$

Then

$$|{}_0T_n - {}_1T_n| = o_p(m^{-\frac{1}{2}}) \qquad \text{as} \quad m = m(n) \to \infty. \tag{2.21}$$

For the proof of the lemma we need the following two propositions.

PROPOSITION 2.1. *If on $[a, b]$, f is a bounded* p.d. *function and w is a bounded function, then*

$$\max_{0<\nu<m}\left|\int_a^b \phi_\nu(s)w(s)\, dZ(F(s))\right| = O_p((\log m)^{\frac{1}{2}}). \tag{2.22}$$

PROOF.

$$\begin{aligned} &P\left(\max_{0<\nu<m}\left|\int_a^b \phi_\nu(s)w(s)\, dZ(F(s))\right| > x\right) \\ &\qquad \leq \sum_{\nu=0}^m P\left(\left|\int_a^b \phi_\nu(s)w(s)\, dZ(F(s))\right| > x\right). \end{aligned} \tag{2.23}$$

Now $\int_a^b \phi_\nu(s)w(s)\, dZ(F(s))$ is a normal random variable with zero expectation and variance equal to $\int_a^b \phi_\nu^2(s)w^2(s)f(s)\, ds$ which is bounded by $\|f\|\|w^2\|$. Therefore, using a known bound for the tails of the normal distribution function, from (2.23) it follows that

$$\begin{aligned} &P\left(\max_{0<\nu<m}\left|\int_a^b \phi_\nu(s)w(s)\, dZ(F(s))\right| > x\right) \\ &\qquad \leq \frac{2(m+1)(\|f\|\|w^2\|)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}x}\exp\left(-\frac{x^2}{2\|f\|\|w^2\|}\right) \end{aligned} \tag{2.24}$$

from which the conclusion of the proposition follows. □

Proposition 2.2. *If on* $[a, b]$, f *is a bounded* p.d. *function and* w *is a bounded function, then*

$$\max_{0<\nu<m}\left|\int_a^b \phi_\nu(s)w(s)\, dZ^0(F(s))\right| = 0_p\big((\log m)^{\frac{1}{2}}\big). \tag{2.25}$$

Proof. From the definitions of Z and Z^0 it follows that

$$\int_a^b \phi_\nu(s)w(s)\, dZ^0(F(s)) = \int_a^b \phi_\nu(s)w(s)\, dZ(F(s)) - Z(1)\int_a^b \phi_\nu(s)w(s)f(s)\, ds. \tag{2.26}$$

This random variable is normal with zero expectation and variance equal to

$$\int_a^b \phi_\nu^2(s)w^2(s)f(s)\, ds - \left(\int_a^b \phi_\nu(s)w(s)f(s)\, ds\right)^2$$

which is uniformly bounded in ν, so that using the same arguments used to prove Proposition 2.1 the conclusion follows. □

Proof of Lemma 2.2.

$$\begin{aligned} |{}_0T_n - {}_1T_n| &= \frac{1}{m+1}\left|\int_a^b \left\{\left(\int_a^b k_m(x, s)\, dZ^0(F(s))\right)^2 - \left(\int_a^b k_m(x, s)\, dZ(F(s))\right)^2\right\} dx\right| \\ &= \frac{1}{m+1}\left|\int_a^b \left[\Sigma_{\nu=0}^m a_\nu(m)\phi_\nu(x)\int_a^b \phi_\nu(s)w(s)\, dZ^0(F(s))\right]^2 dx\right. \\ &\qquad \left. - \int_a^b \left[\Sigma_{\nu=0}^m a_\nu(m)\phi_\nu(x)\int_a^b \phi_\nu(s)w(s)\, dZ(F(s))\right]^2 dx\right|. \end{aligned}$$

By Parseval's relation we have that

$$\begin{aligned} |{}_0T_n - {}_1T_n| &= \frac{1}{m+1}\left|\Sigma_{\nu=0}^m a_\nu^2(m)\left(\int_a^b \phi_\nu(s)w(s)\, dZ^0(F(s))\right)^2\right. \\ &\qquad \left. - \Sigma_{\nu=0}^m a_\nu^2(m)\left(\int_a^b \phi_\nu(s)w(s)\, dZ(F(s))\right)^2\right| \\ &= \frac{1}{m+1}\left|\Sigma_{\nu=0}^m a_\nu^2(m)\int_a^b \phi_\nu(s)w(s)\, d\left[Z^0(F(s)) - Z(F(s))\right]\right. \\ &\qquad \left. \times\left[\int_a^b \phi_\nu(t)w(t)\, dZ^0(F(t)) + \int_a^b \phi_\nu(t)w(t)\, dZ(F(t))\right]\right| \\ &= \frac{1}{m+1}\left|Z(1)\Sigma_{\nu=0}^m a_\nu^2(m)\int_a^b \phi_\nu(s)w(s)f(s)\, ds\right. \\ &\qquad \left. \times\left[\int_a^b \phi_\nu(t)w(t)\, dZ^0(F(t)) + \int_a^b \phi_\nu(t)w(t)\, dZ(F(t))\right]\right|. \end{aligned} \tag{2.27}$$

Using Propositions 2.1 and 2.2 and since $d_\nu = 0(\nu^{-\frac{1}{2}-\varepsilon})$, $\varepsilon > 0$, we have that

$$\begin{aligned} |{}_0T_n - {}_1T_n| &< \frac{1}{m+1}|Z(1)|0_p\big((\log m)^{\frac{1}{2}}\big)\Sigma_{\nu=0}^m\left|\int_a^b \phi_\nu(s)w(s)f(s)\right| ds \\ &< |Z(1)|0_p\big(m^{-1}(\log m)^{\frac{1}{2}}\big)\Sigma_{\nu=1}^m\frac{1}{\nu^{\frac{1}{2}+\varepsilon}} \\ &< o_p\big(m^{-\frac{1}{2}}\big). \end{aligned}$$

□

To apply Lemmas 2.1 and 2.2 we make the elementary

REMARK. If (g_n) is a sequence of functionals on $D[0, 1]$ satisfying the Lipschitz condition

(2.28) $$|g_n(x) - g_n(y)| \leq M_n\|x - y\|$$

and (A_n), (B_n) are sequences of stochastic processes realizable in D such that $\|A_n - B_n\| = o_p(1/M_n)$, then $g_n(A_n)$ converges in law if and only if $g_n(B_n)$ does, and to the same limit. The proof of this statement is straightforward.

3. The trigonometric case. In this section we apply the results of Section 2 to the estimators constructed using the orthonormal system on $[-\pi, \pi)$,

(3.1) $$\phi_0(x) = (2\pi)^{-\frac{1}{2}},\ \phi_\nu(x) = \pi^{-\frac{1}{2}} \cos \nu x, \qquad \nu = 1, 2, \cdots,$$
$$\phi_\nu(x) = \pi^{-\frac{1}{2}} \sin \nu x, \qquad \nu = -1, -2, \ldots.$$

We change a little the definition of T_n as follows

(3.2) $$T_n = \frac{n}{2m+1}\int_{-\pi}^{\pi}(g_n(x) - Eg(x))^2\,dx$$
$$= \frac{1}{2m+1}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}k_m(x, s)\,dZ_n^0(F(s))\right)^2 dx$$

and, of course, the same changes have to be done in Y_n, ${}_0Y_n$, ${}_1Y_n$, ${}_0T_n$ and ${}_1T_n$.

THEOREM 3.1. *Let the* p.d. *function f be of bounded variation on $[-\pi, \pi)$ and the weight function w be bounded, absolutely continuous with derivative w' square integrable on $[-\pi, \pi)$. Let us assume that the kernel k_m satisfies conditions* (2.12) *and* (2.20), *and also that,*

(3.3) *for all* μ, $a_{\nu+\mu}(m) - a_\nu(m) \to 0$ *uniformly in* ν:
$$0 \leq |\nu| \leq m \qquad \text{as } m \to \infty,$$

and $a_\nu(m) = a_{-\nu}(m)$. *Finally, let us suppose that there exists the limits*

(3.3a) $$a^{(2)} \equiv \lim_{m\to\infty}\frac{1}{2m+1}\sum_{\nu=-m}^{m}a_\nu^2(m)$$

(3.3b) $$a^{(4)} \equiv \lim_{m\to\infty}\frac{1}{2m+1}\sum_{\nu=-m}^{m}a_\nu^4(m).$$

Then

(3.4) $$(2m+1)^{\frac{1}{2}}\left(T_n - \frac{1}{2\pi}\frac{1}{2m+1}\int_{-\pi}^{\pi}f(s)w^2(s)\,ds\sum_{\nu=-m}^{m}a_\nu^2(m)\right)$$

is asymptotically normally distributed with mean zero and variance

(3.5) $$a^{(4)}\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(s)w^4(s)\,ds$$

provided that $m = m(n)$ *is chosen such that* $2m + 1 = Kn^\delta$, $0 < \delta < \frac{1}{4}$, *K constant.*

PROOF. Since the functions ϕ_ν and w are bounded and absolutely continuous with derivatives ϕ_ν' and w' which are square integrable, $(\phi_\nu w)$ is absolutely continuous with square integrable derivative. Hence condition (i) of Lemma 2.1 holds.

Condition (2.19) follows upon integrating by parts and applying the Riemann-Lebesgue lemma to g'. Since the other hypotheses of Lemma 2.1 and Lemma 2.2 are clearly satisfied, we can apply these two lemmas. Hence,

$$
\begin{aligned}
|T_n - {}_0T_n| &= \left|\int_{-\pi}^{\pi} Y_n^2(x)\,dx - \int_{-\pi}^{\pi} {}_0Y_n^2(x)\,dx\right| \\
&\le \|(Y_n - {}_0Y_n)/r\| \int_{-\pi}^{\pi} |Y_n(x) + {}_0Y_n(x)| r(x)\,dx \\
&\le 0_p\left(m^{\frac{1}{2}} (\log m) n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}} (\log\log n)^{\frac{1}{4}}\right) \\
&\quad \times \int_{-\pi}^{\pi} |Y_n(x) + {}_0Y_n(x)| r(x)\,dx.
\end{aligned} \tag{3.6}
$$

Now, $\int_{-\pi}^{\pi} |{}_0Y_n(x)| r(x)\,dx$ is a sequence of random variables with second moments uniformly bounded. In effect, using Schwarz inequality and Parseval's relation it follows that

$$
\begin{aligned}
&E\left(\int_{-\pi}^{\pi} |{}_0Y_n(x)| r(x)\,dx\right)^2 \\
&\quad \le \int_{-\pi}^{\pi} r^2(x)\,dx\, E\left(\int_{-\pi}^{\pi} {}_0Y_n^2(x)\,dx\right) \\
&\quad = \frac{1}{2m+1} \int_{-\pi}^{\pi} r^2(x)\,dx\, E\left(\sum_{\nu=-m}^{m} a_\nu^2(m) \left(\int_{-\pi}^{\pi} \phi_\nu(s) w(s)\,dZ^0(F(s))\right)^2\right) \\
&\quad = \frac{1}{2m+1} \int_{-\pi}^{\pi} r^2(x)\,dx \sum_{\nu=-m}^{m} a_\nu^2(m) \int_{-\pi}^{\pi} \phi_\nu^2(s) w^2(s) f(s)\,ds \\
&\quad \le \frac{1}{2m+1} \int_{-\pi}^{\pi} r^2(x)\,dx \sum_{\nu=-m}^{m} a_\nu^2(m) \|fw^2\| \int_{-\pi}^{\pi} \phi_\nu^2(s)\,ds \\
&\quad = 0(1).
\end{aligned} \tag{3.7}
$$

Therefore, $\int_{-\pi}^{\pi} |{}_0Y_n(x)| r(x)\,dx = 0_p(1)$ and since

$$
\begin{aligned}
\left|\int_{-\pi}^{\pi} |Y_n(x)| r(x)\,dx - \int_{-\pi}^{\pi} |{}_0Y_n(x)| r(x)\,dx\right| &\le \int_{-\pi}^{\pi} |Y_n(x) - {}_0Y_n(x)| r(x)\,dx \\
&= \int_{-\pi}^{\pi} \left|\frac{Y_n(x) - {}_0Y_n(x)}{r(x)}\right| r^2(x)\,dx \\
&\le \left\|\frac{Y_n - {}_0Y_n}{r}\right\| \int_{-\pi}^{\pi} r^2(x)\,dx
\end{aligned} \tag{3.8}
$$

it follows from (2.13) that

$$\int_{-\pi}^{\pi} |Y_n(x) - {}_0Y_n(x)| r(x)\,dx = 0_p(1) \quad \text{as} \quad n \to \infty \tag{3.9}$$

provided that $m = m(n)$ is chosen such that the right-hand side of (2.13) is bounded, and this will hold if $2m(n) + 1 = Kn^\delta$ with $0 < \delta < \frac{1}{4}$, which is one of the hypotheses of the theorem. Hence from (3.8) and (3.9) it follows that

$$\int_{-\pi}^{\pi} |Y_n(x)| r(x)\,dx = 0_p(1). \tag{3.10}$$

Finally from (3.6), (3.7) and (3.10) it follows that

$$|T_n - {}_0T_n| = 0_p\left(m^{\frac{1}{2}} (\log m) n^{-\frac{1}{4}} (\log n)^{\frac{1}{2}} (\log\log n)^{\frac{1}{4}}\right)$$

as $n \to \infty$, $2m + 1 = Kn^\delta$, with $0 < \delta < \frac{1}{4}$.

Because of this result and the remark in Section 2, we can replace T_n by ${}_0T_n$ if

$$m(\log m)n^{-\frac{1}{4}}(\log n)^{\frac{1}{2}}(\log\log n)^{\frac{1}{4}} = o(1) \tag{3.11}$$

and this condition will be satisfied if $2m + 1 = Kn^{\delta}$, $0 < \delta < \frac{1}{4}$. Also according to the same remark and Lemma 2.2 we can use ${}_1T_n$ in place of ${}_0T_n$. Therefore, for the proof of the theorem it suffices to find the asymptotic distribution of ${}_1T_n$.

By Parseval's relation we have that

$$\begin{aligned} {}_1T_n = \textstyle\int_{-\pi}^{\pi} {}_1Y_n^2(x)\,dx &= \frac{1}{2m+1}\textstyle\int_{-\pi}^{\pi} \\ &\quad\times\left[\textstyle\sum_{\nu=-m}^{m} a_\nu(m)\phi_\nu(x)\int_{-\pi}^{\pi}\phi_\nu(s)w(s)\,dZ(F(s))\right]^2 dx \\ &= \frac{1}{2m+1}\textstyle\sum_{\nu=-m}^{m} a_\nu^2(m)\left[\int_{-\pi}^{\pi}\phi_\nu(s)w(s)\,dZ(F(s))\right]^2 \\ &= \frac{1}{2m+1}\textstyle\sum_{\nu=-m}^{m} Z_\nu^2 \end{aligned}$$

where

$$Z_\nu = a_\nu(m)\textstyle\int_{-\pi}^{\pi}\phi_\nu(s)w(s)\,dZ(F(s)). \tag{3.12}$$

Let us write for short $a_\nu(m) = a_\nu$, $g^* = w^2 f$. We see that ${}_1T_n$ is a quadratic form. The vector $(Z_\nu)_{\nu=-m}^{m}$ has a normal distribution with $EZ_\nu = 0$ and covariance matrix B_m with components b_{ij} given by

$$b_{ij} = E(Z_iZ_j) = a_ia_j\textstyle\int_{-\pi}^{\pi}\phi_i(s)\phi_j(s)g^*(s)\,ds.$$

The characteristic function of ${}_1T_n$ is

$${}_1\phi_m(t) = \Pi_{j=-m}^{m}\left(1 - \frac{2\lambda_j it}{2m+1}\right)^{-\frac{1}{2}} = \exp\left[\textstyle\sum_{j=-m}^{m}\left(-\frac{1}{2}\right)\log\left(1 - \frac{2\lambda_j it}{2m+1}\right)\right]$$

where λ_j are the eigenvalues of B_m. Using Taylor's theorem we write

$${}_1\phi_m(t) = \exp\left[it\frac{1}{2m+1}\textstyle\sum_{j=-m}^{m}\lambda_j + \frac{(it)^2}{(2m+1)^2}\sum_{j=-m}^{m}\lambda_j^2 + 4(it)^3\frac{\sum_{j=-m}^{m}(\lambda_j\theta_j)^3}{3(2m+1)^3}\right]$$

where $0 < \theta_j < 1$. By Lemma A.2 of the Appendix

$$\frac{1}{2m+1}\textstyle\sum_{j=-m}^{m}\lambda_j = \frac{1}{2\pi}\frac{1}{2m+1}\int_{-\pi}^{\pi} g^*\,ds\sum_{j=-m}^{m}a_j^2(m).$$

Therefore the rv (3.4) has characteristic function given by

$$\phi_m(t) = \exp\left[\frac{(it)^2}{2m+1}\textstyle\sum_{j=-m}^{m}\lambda_j^2 + 4(it)^3\frac{\sum_{j=-m}^{m}(\lambda_j\theta_j)^3}{3(2m+1)^{\frac{3}{2}}}\right]. \tag{3.13}$$

Because of Lemma A.2 of the Appendix

$$\lim_{m\to\infty}\frac{1}{2m+1}\Sigma_{j=-m}^{m}\lambda_j^2=\lim_{m\to\infty}\frac{1}{2m+1}\operatorname{tr}(B_m^2)=\frac{1}{2\pi}a^{(4)}\textstyle\int_{-\pi}^{\pi}f^2(s)w^4(s)\,ds$$

$$\begin{aligned}\lim_{m\to\infty}(2m+1)^{-\frac{3}{2}}\Sigma_{j=-m}^{m}(\lambda_j\theta_j)^3 &\leqslant \lim_{m\to\infty}(2m+1)^{-\frac{3}{2}}\Sigma_{j=-m}^{m}\lambda_j^3\\ &= \lim_{m\to\infty}(2m+1)^{-\frac{3}{2}}\operatorname{tr}(B_m^3)\\ &= 0.\end{aligned}$$

Hence the conclusion of the theorem follows passing to the limit in (3.13). □

REMARK. Under the additional condition

$$\frac{1}{2m+1}\Sigma_{\nu=-m}^{m}a_\nu^2(m)=a^{(2)}+o\big(m^{-\frac{1}{2}}\big)$$

it is easy to show that

$$(2m+1)^{\frac{1}{2}}\left[T_n-a^{(2)}\frac{1}{2\pi}\textstyle\int_{-\pi}^{\pi}f(s)w^2(s)\,ds\right]$$

has the asymptotic distribution stated in Theorem 3.1. Since in general it will be easier to find the limit of $(2m+1)^{-1}\Sigma_{\nu=-m}^{m}a_\nu^2(m)$ than the value of this expression for a finite m, this remark is useful.

4. The Dirichlet and Fejér estimators. In this section we discuss two particular cases, namely, the Dirichlet and Fejér estimators.

The Dirichlet estimator arises when we take $a_\nu(m)=1$ for $m=0,\pm 1,\cdots,|\nu|\leqslant m$ and zero otherwise. The corresponding kernel is $k_m(x,s)=D_m(x-s)w(s)$, where

$$D_m(t)=\frac{1}{2\pi}\frac{\sin((m+1/2)t)}{\sin(t/2)}. \tag{4.1}$$

The Fejér estimator arises when we take $a_\nu(m)=(1-(|\nu|/(m+1)))$ for $m=0,\pm 1,\cdots,|\nu|\leqslant m$ and zero otherwise and the corresponding kernel is $k_m(x,s)=F_m(x-s)w(s)$, where

$$F_m(t)=\frac{1}{2\pi(m+1)}\left\{\frac{\sin((m+1)t/2)}{\sin(t/2)}\right\}^2. \tag{4.2}$$

In order to apply the results of Section 3 to these estimators we have to show that they satisfy the hypotheses of Lemmas 2.1 and 2.2; specifically we have to show that (3.3), (3.3a), (3.3b), (3.12) and (2.20) hold. That (3.3) and (2.20) hold is obvious, so that we prove (2.12), (3.3a) and (3.3b). Let us consider first the Dirichlet kernel. The weight function w is assumed bounded and with a square integrable derivative. Hence, and after applying Schwarz inequality, we obtain

$$\begin{aligned}\textstyle\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial s}k_m(x,s)\right|ds &\leqslant \textstyle\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial s}D_m(x-s)\right|w(s)\,ds+\int_{-\pi}^{\pi}|D_m(x-s)w'(s)|\,ds\\ &\leqslant \|w\|\textstyle\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial s}D_m(x-s)\right|ds+\left(\int_{-\pi}^{\pi}(w'(s))^2\,ds\right)^{\frac{1}{2}}\left(\int_{-\pi}^{\pi}D_m^2(x-s)\,ds\right)^{\frac{1}{2}}.\end{aligned}$$

Because of the periodicity of D_m we have that

$$(4.3)\qquad \int_{-\pi}^{\pi}\left|\frac{\partial}{\partial s}k_m(x,s)\right|ds \leq \|w\|\int_{-\pi}^{\pi}\left|\frac{d}{ds}D_m(s)\right|ds + \left(\int_{-\pi}^{\pi}(w'(s))^2\,ds\right)^{\frac{1}{2}}\left(\int_{-\pi}^{\pi}D_m^2(s)\,ds\right)^{\frac{1}{2}}.$$

Since $D_m(s) = 0(m)$ it follows that the second term in the right-hand side of (4.3) is $0(m)$. Now,

$$(4.4)\qquad \int_{-\pi}^{\pi}\left|\frac{d}{ds}D_m(s)\right|ds = \int_{-1/m}^{1/m}\left|\frac{d}{ds}D_m(s)\right|ds + \left(\int_{-\pi}^{-1/m} + \int_{1/m}^{\pi}\right)\left|\frac{d}{ds}D_m(s)\right|ds = I_1 + I_2.$$

It is known that $\pi D_m(s) = \frac{1}{2} + \cos s + \cos 2s + \cdots + \cos ms$, so that $\pi(d/ds)D_m(s) = 0(m^2)$; hence

$$(4.5)\qquad I_1 = 0(m).$$

Now,

$$\frac{d}{ds}D_m(s) = \frac{m+\frac{1}{2}}{2\pi}\,\frac{\cos\left[\left(m+\frac{1}{2}\right)s\right]}{\sin\frac{s}{2}} - \frac{1}{4\pi}\,\frac{\sin\left[\left(m+\frac{1}{2}\right)s\right]\cos\frac{s}{2}}{\sin^2\frac{s}{2}}.$$

Therefore,

$$\left|\frac{d}{ds}D_m(s)\right| = 0(m)\frac{1}{|s|} + 0(1)\frac{1}{s^2}, \qquad \text{as } s \to 0,$$

and then

$$(4.6)\qquad I_2 = 0(m\log m) + 0(m) = 0(m\log m).$$

From (4.4), (4.5) and (4.6) it follows that (2.12) holds.

Using a similar argument for the Fejér kernel, it can be shown that

$$\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial s}k_m(x,s)\right|ds = 0(m),$$

whence (2.12) follows.

Now we write explicitly Theorem 3.1 for the Dirichlet and Fejér estimators. Let us consider first the Dirichlet estimator. In this case

$$a^{(2)} = \frac{1}{2m+1}\sum_{\nu=-m}^{m}a_\nu^2(m) = 1, \qquad a^{(4)} = \lim_{m\to\infty}\frac{1}{2m+1}\sum_{\nu=-m}^{m}a_\nu^4(m) = 1;$$

hence conditions (3.3a) and (3.3b) also hold. Therefore we can state the following

THEOREM 4.1. *Let f be a* p.d. *function of bounded variation on $[-\pi,\pi)$ and w be a weight function absolutely continuous with a square integrable derivative on $[-\pi,\pi)$. Let $g_n(x)$ be defined by*

$$g_n(x) = \frac{1}{n}\sum_{j=1}^{n}D_m(x-X_j)w(X_j).$$

If T_n is defined as in (3.2), *then*

$$(2m+1)^{\frac{1}{2}}\left[T_n - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)w^2(s)\,ds\right] \tag{4.7}$$

is asymptotically normally distributed with 0 *mean and variance*

$$\pi^{-1}\int_{-\pi}^{\pi} f^2(s)w^4(s)\,ds \tag{4.8}$$

as $n \to \infty$ *and* $2m(n)+1 = Kn^{\delta}$, $0<\delta<\frac{1}{4}$, *K constant.*

For the Fejér estimator simple integral estimation gives that

$$a^{(2)} = \lim_{m\to\infty} \frac{1}{2m+1}\Sigma_{\nu=-m}^{m}\left(1 - \frac{|\nu|}{m+1}\right)^2 = \tfrac{1}{3}$$

$$a^{(4)} = \lim_{m\to\infty} \frac{1}{2m+1}\Sigma_{\nu=-m}^{m}\left(1 - \frac{|\nu|}{m+1}\right)^4 = \tfrac{1}{5}.$$

Therefore we have proved the following

THEOREM 4.2. *Let f be a* p.d. *function of bounded variation on* $[-\pi,\pi)$ *and w a weight function absolutely continuous with square integrable derivative on* $[-\pi,\pi)$. *Let* $g_n(x)$ *be the estimator*

$$g_n(x) = \frac{1}{n}\Sigma_{j=1}^{n} F_m(x - X_j)w(X_j).$$

If T_n is defined as in (3.2), *then*

$$(2m+1)^{\frac{1}{2}}\left[T_n - \frac{1}{6\pi}\int_{-\pi}^{\pi} f(s)w^2(s)\,ds\right]$$

is asymptotically normally distributed with mean 0 *and variance*

$$(5\pi)^{-1}\int_{-\pi}^{\pi} f^2(s)w^4(s)\,ds.$$

5. The Legendre case. In this section we apply the results of Section 2 to estimators constructed using the orthonormal system of Legendre polynomials. Let $(P_\nu(x))_{\nu=0}^{\infty}$ and $(p_\nu(x))_{\nu=0}^{\infty}$ be the unnormed and normed Legendre polynomials, respectively, as defined in Sansone (1959). Let $g_n(x)$ be the estimator

$$g_n(x) = \frac{1}{n}\Sigma_{j=1}^{n} k_m(x, X_j) = \frac{1}{n}\Sigma_{j=1}^{n}\Sigma_{\nu=0}^{m} p_\nu(x)p_\nu(X_j)w(X_j) \tag{5.1}$$

where w is a weight function absolutely continuous on $[-1, 1]$ and such that $w(s) = 0[(1-s^2)^{\eta}]$, $w'(s) = 0[(1-s^2)^{\eta-1}]$, $\eta \geq \frac{3}{4}$, as s tends to -1^+ and 1^-. Clearly condition (i) of Lemma 2.1 holds here. We prove now that (2.12) also holds with $r(x) = (1-x^2)^{-\frac{1}{4}}$.

PROPOSITION 5.1. *If w has the properties above stated, then*

$$\int_{-1}^{1}\left|\frac{\partial}{\partial s}k_m(x,s)\right|ds = (1-x^2)^{-\frac{1}{4}}0(m\log m). \tag{5.2}$$

PROOF. Let $K_m(x,s) = \Sigma_{\nu=0}^{m} p_\nu(x)p_\nu(s)$. From Christoffel-Darboux's formula (see Szegö (1939), page 41) we have

$$K_m(x,s) = 0(1)\frac{p_{m+1}(s)p_m(x) - p_m(s)p_{m+1}(x)}{s-x}. \tag{5.3}$$

Then $k_m(x, s) = K(x, s)w(s)$ and

$$(5.4)\quad \int_{-1}^{1}\left|\frac{\partial}{\partial s}k_m(x, s)\right|ds \leq \int_{-1}^{1}\left|\frac{\partial}{\partial s}K_m(x, s)\right|w(s)\,ds + \int_{-1}^{1}|K_m(x, s)w'(s)|ds$$
$$\leq A + B.$$

Let $\Delta_m(x) = \{s : |x - s| \leq 1/m\}$, and $\Delta_m^*(x)$ be its complement with respect to $[-1, 1]$. Then

$$(5.5)\qquad A = \int_{\Delta_m(x)}\left|\frac{\partial}{\partial s}K_m(x, s)\right|w(s)\,ds + \int_{\Delta_m^*(x)}\left|\frac{\partial}{\partial s}K_m(x, s)\right|w(s)\,ds$$
$$= I_1 + I_2.$$

Since (cf. Szegö (1939))

$$(5.6)\qquad p_\nu(x) = 0(1)(1 - x^2)^{-\frac{1}{4}}$$
$$p_\nu'(x) = 0(\nu)(1 - x^2)^{-\frac{3}{4}}$$

we have that

$$(5.7)\qquad I_1 = \int_{\Delta_m(x)}|\Sigma_{\nu=0}^{m}p_\nu(x)p_\nu'(s)|w(s)\,ds$$
$$= 0(m^2)(1 - x^2)^{-\frac{1}{4}}\int_{\Delta_m(x)}(1 - s^2)^{-\frac{3}{4}}w(s)\,ds.$$

But since $w(s) = 0((1 - s^2)^\eta)$ with $\eta \geq \frac{3}{4}$, it follows that $(1 - s^2)^{-\frac{3}{4}}w(s)$ is bounded. Hence the integral in the right-hand side of (5.7) is $0(m^{-1})$, and then

$$(5.8)\qquad I_1 = 0(m)(1 - x^2)^{-\frac{1}{4}}.$$

Now, by differentiation of Christoffel-Darboux's formula we have

$$(5.9)\qquad \frac{\partial}{\partial s}K_m(x, s) = 0(1)\frac{p_{m+1}'(s)p_m(x) - p_m'(s)p_{m+1}(x)}{s - x}$$
$$-0(1)\frac{p_{m+1}(s)p_m(x) - p_m(s)p_{m+1}(x)}{(s - x)^2},$$

$$\int_{\Delta_m^*(x)}\left|\frac{p_{m+1}'(s)p_m(x) - p_m'(s)p_{m+1}(x)}{s - x}\right|w(s)\,ds$$
$$= 0(m)(1 - x^2)^{-\frac{1}{4}}\int_{\Delta_m^*(x)}\frac{w(s)}{(1 - s^2)^{\frac{3}{4}}}\frac{ds}{|s - x|}$$

$$(5.10)\qquad = (1 - x^2)^{-\frac{1}{4}}0(m\log m),$$

$$\int_{\Delta_m^*(x)}\left|\frac{p_{m+1}(s)p_m(x) - p_m(s)p_{m+1}(x)}{(s - x)^2}\right|w(s)\,ds$$

$$(5.11)\qquad = 0(1)(1 - x^2)^{-\frac{1}{4}}\int_{\Delta_m^*(x)}\frac{w(s)}{(1 - s^2)^{\frac{1}{4}}}\frac{ds}{(s - x)^2}.$$

Since $s \in \Delta_m^*(x)$ implies that $|s - x| \geqslant 1/m$, and since $w(s) = 0((1 - s^2)^\eta)$, $\eta \geqslant \frac{3}{4}$, it follows that (5.11) is bounded by

$$0(m)(1 - x^2)^{-\frac{1}{4}} \int_{\Delta_m^*(x)} \frac{ds}{|s - x|} = (1 - x^2)^{-\frac{1}{4}} 0(m \log m). \tag{5.12}$$

From (5.10), (5.11) and (5.12) it follows that

$$I_2 = (1 - x^2)^{-\frac{1}{4}} 0(m \log m).$$

Considering B, we obtain

$$\begin{aligned} B &= \int_{-1}^1 |K_m(x, s) w'(s)| ds = 0(m)(1 - x^2)^{-\frac{1}{4}} \int_{-1}^1 (1 - s^2)^{-\frac{1}{4}} |w'(s)| ds \\ &= 0(m)(1 - x^2)^{-\frac{1}{4}} \int_{-1}^1 (1 - s^2)^{-\frac{1}{4} + \eta - 1} ds \\ &= 0(m)(1 - x^2)^{-\frac{1}{4}}. \end{aligned} \tag{5.13}$$

□

THEOREM 5.1. *Let the* p.d. *function f be of bounded variation on* $[-1, 1]$; *let the weight function w be absolutely continuous with square integrable derivative both on* $[-1, 1]$, *and such that* $w(s) = 0[(1 - s^2)^\eta]$, $w'(s) = 0[(1 - s^2)^{\eta - 1}]$, $\eta \geqslant \frac{3}{4}$ *as s tends to* -1^+, 1^-. *Let g_n be defined by* (5.1) *and T_n by*

$$T_n = \frac{n}{m + 1} \int_{-1}^1 [g_n(x) - Eg_n(x)]^2 \, dx. \tag{5.14}$$

Then

$$(m + 1)^{\frac{1}{2}} \left[T_n - \frac{1}{\pi} \int_{-1}^1 f(s) w^2(s) (1 - s^2)^{-\frac{1}{2}} \, ds \right] \tag{5.15}$$

is asymptotically normally distributed with 0 *mean and variance*

$$\frac{2}{\pi} \int_{-1}^1 \frac{f^2(s) w^4(s)}{(1 - s^2)^{\frac{1}{2}}} ds \tag{5.16}$$

provided that $m + 1 = m(n) + 1 = Kn^\delta$, $0 < \delta < \frac{1}{4}$, *K constant.*

PROOF. From the hypotheses of the theorem and from Proposition 5.1 it follows that all the hypotheses of Lemma 2.1 and Lemma 2.2 are satisfied. Condition (2.19) follows from Sansone (1959), page 200 and the second theorem of the mean value. Since all the arguments used in the trigonometric case to prove that ${}_1T_n$ can be used in place of T_n are independent of the particular orthonormal system (ϕ_ν), provided that for every ν, ϕ_ν is bounded and absolutely continuous with square integrable derivative ϕ'_ν, we conclude that in the present situation we can use ${}_1T_n$ in place of T_n.

The rest of the proof goes exactly the same as the proof for the trigonometric estimators, so it is omitted. □

6. The asymptotic distribution of T_n^*. The statistic T_n^* defined by (1.10) is probably of greater interest than T_n. In this section we find the asymptotic distribution of T_n^* in the trigonometric and Legendre cases. Let us consider first the trigonometric case.

Let us expand T_n^* in the form

(6.1) $$T_n^* = T_n + \frac{n}{2m+1}\Big[\Sigma_{\nu=-m}^{m}(1 - a_\nu(m))^2 d_\nu^2 + \Sigma_{|\nu|>m} d_\nu^2 - 2\Sigma_{\nu=-m}^{m} a_\nu(m)(1 - a_\nu(m))\, d_\nu(\hat{d}_\nu - d_\nu)\Big].$$

For the Dirichlet estimator we have

(6.2) $$T_n^* = T_n + \frac{n}{2m+1}\Sigma_{|\nu|>m} d_\nu^2$$

and, therefore, T_n^* will have the same asymptotic distribution as T_n if

(6.3) $$|T_n^* - T_n| = \frac{n}{2m+1}\left|\Sigma_{|\nu|>m} d_\nu^2\right| = o_p(m^{-\frac{1}{2}}).$$

The following theorem holds.

THEOREM 6.1. *Let f be a* p.d. *function such that $f|_{[-\pi,\pi)}$ has a second derivative of bounded variation. Let w be a weight function with a second derivative of bounded variation on $[-\pi, \pi)$, and such that $w(x)$, $w'(x)$, $w''(x)$ are $o(1)$ as x tends to $-\pi^+, \pi^-$. Then T_n^* has the asymptotic distribution given by Theorem 4.1, when n tends to infinity and $2m + 1 = Kn^\delta$ where $\frac{2}{11} < \delta < \frac{1}{4}$.*

PROOF. We have to show that (6.3) holds. As before, let us write $g = fw$ and let g^e, f^e and w^e be the periodical extensions of $g|_{[-\pi,\pi)}$, $f|_{[-\pi,\pi)}$ and $w|_{[-\pi,\pi)}$ respectively. Since $(g^e)'' = (f^e)''w^e + 2(f^e)'(w^e)' + f^e(w^e)''$ it follows from the hypotheses that $(g^e)''$ exists for every $x \in \mathbb{R}$ and it is a function of bounded variation. Therefore the Fourier coefficients d_ν satisfy the equality

$$d_\nu = \int_{-\pi}^{\pi} g(x)\phi_\nu(x)\,dx = \int_{-\pi}^{\pi} g^e(x)\phi_\nu(x)\,dx = 0(\nu^{-3})$$

and

(6.4) $$\frac{n}{2m+1}\Sigma_{|\nu|>m} d_\nu^2 = \frac{n}{2m+1}0(m^{-5}) = 0(nm^{-6}).$$

From (6.4) it follows that (6.3) holds provided that $2m + 1 = Kn^\delta$ with $\delta > \frac{2}{11}$. □

Let us consider now the Fejér estimator. Let us expand T_n^* in the form (6.1) and let us consider the fourth term in the right-hand side of (6.1):

$$U_n = \frac{2n}{2m+1}\Sigma_{\nu=-m}^{m}\left(1 - \frac{|\nu|}{m+1}\right)\frac{|\nu|}{m+1} d_\nu(\hat{d}_\nu - d_\nu).$$

We will prove that (6.1) cannot be used to find the asymptotic distribution of T_n^* because in general U_n does not tend to 0 fast enough. Clearly $E(U_n) = 0$. Let us take the p.d. function

$$f(x) = \frac{1}{2\pi}(1 + \cos x)I_{[-\pi,\pi)}(x)$$

and

$$\begin{aligned} w(x) = I_{[-\pi,\pi)}(x) &= 1, \qquad x \in [-\pi, \pi) \\ &= 0, \qquad x \notin [-\pi, \pi). \end{aligned}$$

The function $g = fw$ and its periodical extension g^e are very smooth since they are infinitely differentiable.

Since $d_\nu = 0$ for $|\nu| > 1$ we have

$$\begin{aligned}\operatorname{Var}(U_n) &= \frac{4n^2}{(2m+1)^2(m+1)^2}\Sigma^1_{\nu,\mu=-1}\left(1-\frac{|\nu|}{m+1}\right)\left(1-\frac{|\mu|}{m+1}\right)\\ &\quad\times|\nu||\mu|d_\nu d_\mu \operatorname{Cov}(\hat{d}_\nu, \hat{d}_\mu)\\ &= C(n,m)nm^{-4}\end{aligned}$$

where $C(n, m)$ is bounded away from 0 uniformly in n and m. Therefore

$$U_n = 0_p(n^{\frac{1}{2}}m^{-2}).$$

In order to use (6.1) to find the asymptotic distribution of T_n^* we need to have $m^{\frac{1}{2}}U_n = 0_p(1)$ and this will hold if we take $m = Kn^\delta$ with $\delta > \frac{1}{3}$. On the other hand we have that $\delta < \frac{1}{4}$; hence it is impossible to find the asymptotic distribution of T_n^* using the approach of this work which is based on a Skorohod-like imbedding from which the bound $\delta < \frac{1}{4}$ comes.

It should be of interest to study estimators constructed using other kernels, like the de la Vallée Poussin and Jackson's kernels, but we do not consider them here.

Let us discuss now the Legendre estimator.

Theorem 6.2. *Let f be a* p.d. *function with a second derivative of bounded variation on* $[-1, 1]$. *Let w be a weight function with a second derivative of bounded variation on* $[-1, 1]$. *Then T_n^* has the asymptotic distribution given by Theorem* 5.1 *when n tends to infinity and $m = Kn^\delta$ with $\frac{2}{11} < \delta < \frac{1}{4}$.*

Proof. Clearly all the hypotheses of Theorem 5.1 hold. Since f and w have second derivatives of bounded variation it follows that $g = fw$ has a second derivative of bounded variation, and then $d_\nu = 0(\nu^{-3})$. Hence $|n(m+1)^{-1}\Sigma_{\nu>m}d_\nu^2| = 0(nm^{-6})$ and (6.3) holds provided that $m = Kn^\delta$ with $\delta > \frac{2}{11}$. □

7. Applications. An explicit confidence band is impossible to obtain from the theorems of Sections 3-6. However we can test $H: f = f_0$ at (approximate) level α by calculating T_n for $f = f_0$ and rejecting when $T_n \geqslant d(\alpha)$, where, by Theorem 5.1 in the Legendre case,

$$d(\alpha) = (m+1)^{-\frac{1}{2}}\phi^{-1}(1-\alpha)\left(\frac{2}{\pi}\int_{-1}^{1}\frac{f^2w^4}{(1-s^2)^{\frac{1}{2}}}ds\right)^{\frac{1}{2}} + \frac{1}{\pi}\int_{-1}^{1}\frac{fw^2}{(1-s^2)^{\frac{1}{2}}}ds; \tag{7.1}$$

by Theorem 4.1 in the Dirichlet case

$$(7.2)\qquad d(\alpha) = (2m+1)^{-\frac{1}{2}}\phi^{-1}(1-\alpha)\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f^2 w^4\, ds\right)^{\frac{1}{2}} + \frac{1}{2\pi}\int_{-\pi}^{\pi} f w^2\, ds$$

and by Theorem 4.2 in the Fejér case

$$(7.3)\qquad d(\alpha) = (2m+1)^{-\frac{1}{2}}\phi^{-1}(1-\alpha)\left(\frac{1}{5\pi}\int_{-\pi}^{\pi} f^2 w^4\, ds\right)^{\frac{1}{2}} + \frac{1}{6\pi}\int_{-\pi}^{\pi} f w^2\, ds$$

where $\phi(x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}\exp(-t^2/2)\, dt$.

We also can test a composite hypothesis $H : f = f(., \theta)$ where θ is an unknown vector parameter. For this if $\hat{\theta}$ is an estimator of θ we may use T_n with f replaced by $f(., \hat{\theta})$ and $d(\alpha)$ with f_0 replaced by $f_0(., \hat{\theta})$ provided that the following condition holds: for each θ_0, $\partial^2 f(x, \theta)/\partial_\theta(i)\partial_\theta(j)$ is bounded in absolute value for all θ in a neighborhood of θ_0 and all x, i, j. Moreover if θ_0 is true

$$(7.4)\qquad |\hat{\theta} - \theta_0| = 0_p(n^{-\frac{1}{2}} m^{\frac{1}{2}}).$$

The proof goes exactly as in Bickel and Rosenblatt (1973) and it is omitted.

To make local power calculations of the test we consider the behavior of T_n (calculated under f_0) for a sequence of alternatives of the form

$$(7.5)\qquad h_n(x) = f_0(x) + \gamma_n(\eta(x) + \delta_n(x)),$$

where $\eta(x)$ is of bounded variation, $\delta_n(x)$ is of bounded variation uniformly in n and $0(1)$ uniformly in n and $\gamma_n \downarrow 0$ at a suitable rate. Note that h_n is of bounded variation uniformly in n as required by Theorem 3.1.

THEOREM 7.1. *Let h_n as above, w and f_0 as in Theorem 3.1 and $\gamma_n = n^{-\frac{1}{2}+\delta/4}$. Suppose that $(a_\nu(m))$ satisfies conditions (2.20), (3.3), (3.3a), (3.3b), and, moreover, for every fixed ν, $a_\nu(m)$ tends to 1 as m tends to ∞. Define T_n in terms of f_0 by (3.2). Assume that h_n is the true* p.d. *function of the random variables (X_j). Then if $\delta < \frac{1}{4}$ and $2m + 1 = Kn^\delta$,*

$$(7.6)\qquad (2m+1)^{\frac{1}{2}}\left(T_n - \frac{1}{2\pi}\frac{1}{2m+1}\int_{-\pi}^{\pi} f_0(s) w^2(s)\, ds \sum_{\nu=-m}^{m} a_\nu^2(m)\right)$$

is asymptotically normally distributed with mean $K^{-\frac{1}{2}}\int_{-\pi}^{\pi}\eta^2(s) w^2(s)\, ds$ and variance given by $a^{(4)}\frac{1}{\pi}\int_{-\pi}^{\pi} f_0^2(s) w^4(s)\, ds$.

PROOF. Let

$$\hat{d}_\nu = \frac{1}{n}\sum_{j=1}^{n}\phi_\nu(X_j) w(X_j)$$

$$d_\nu = E\hat{d}_\nu = \int_{-\pi}^{\pi}\phi_\nu(s) h_n(s) w(s)\, ds$$

$$d_{0\nu} = \int_{-\pi}^{\pi}\phi_\nu(s) f_0(s) w(s)\, ds.$$

Let $E_0 g_n(x)$ be the expected value of $g_n(x)$ calculated under f_0. Then by Parseval's relation it follows that

$$T_n = \frac{n}{2m+1}\int_{-\pi}^{\pi}(g_n(x) - E_0(g_n(x))^2\,dx = \frac{n}{2m+1}\Sigma_{\nu=-m}^{m}a_\nu^2(m)(\hat{d}_\nu - d_{0\nu})^2$$

$$(7.7)\qquad = \frac{n}{2m+1}\Sigma_{\nu=-m}^{m}a_\nu^2(m)(\hat{d}_\nu - d_\nu)^2 + \frac{n}{2m+1}\Sigma_{\nu=-m}^{m}a_\nu^2(m)(d_\nu - d_{0\nu})^2$$

$$+ \frac{2n}{2m+1}\Sigma_{\nu=-m}^{m}a_\nu^2(m)(\hat{d}_\nu - d_\nu)(d_\nu - d_{0\nu}).$$

But

$$(7.8)\qquad \frac{n}{2m+1}\Sigma_{\nu=-m}^{m}a_\nu^2(m)(d_\nu - d_{0\nu})^2$$

$$= K^{-1}n^{-\delta/2}\Sigma_{\nu=-m}^{m}a_\nu^2(m)\left[\int_{-\pi}^{\pi}[\eta(s) + \delta_n(s)]w(s)\phi_\nu(s)\,ds\right]^2.$$

Since the total variation of $\eta(s) + \delta_n(s)$ is uniformly bounded in n, and w is of bounded variation, it follows that

$$\int_{-\pi}^{\pi}[\eta(s) + \delta_n(s)]w(s)\phi_\nu(s)\,ds = 0(\nu^{-1})$$

with $0(\nu^{-1})$ uniform in n. On the other hand $a_\nu^2(m)$ are uniformly bounded in ν and m. Hence that νth term of the series in the right-hand side of (7.8) is dominated by $0(\nu^{-2})$ and therefore, by the dominated convergence theorem applied to (7.8) we obtain

$$\lim \Sigma_{\nu=-m}^{m}a_\nu^2(m)\left[\int_{-\pi}^{\pi}(\eta(s) + \delta_n(s))w(s)\phi_\nu(s)\,ds\right]^2$$

$$(7.9)\qquad = \Sigma_{\nu=-\infty}^{\infty}\left[\int_{-\pi}^{\pi}\eta(s)w(s)\phi_\nu(s)\,ds\right]^2$$

$$= \int_{-\pi}^{\pi}\eta^2(s)w^2(s)\,ds.$$

Let us define

$$U_n = \frac{2n}{2m+1}\Sigma_{\nu=m}^{m}a_\nu^2(m)(\hat{d}_\nu - d_\nu)(d_\nu - d_{0\nu}).$$

Then $EU_n = 0$. Since $\mathrm{Cov}(\hat{d}_\nu, \hat{d}_\mu) = 0(n^{-1})$ with $0(n^{-1})$ uniform in ν and μ, and since

$$d_\nu - d_{0\nu} = \gamma_n\int_{-\pi}^{\pi}[\eta(s) + \delta_n(s)]w(s)\phi_\nu(s)\,ds = \gamma_n 0(\nu^{-1})$$

with $0(\nu^{-1})$ uniform in n, it follows that

$$\mathrm{Var}(U_n) < \gamma_n^2\frac{4n^2}{(2m+1)^2}0(1)\Sigma_{\nu=-m;\,\nu\neq 0}^{m}\Sigma_{\mu=-m;\,\mu\neq 0}^{m}\frac{1}{n}\frac{1}{\nu}\frac{1}{\mu}$$

$$< \gamma_n^2 0(m^{-2}n(\log m)^2) = 0(n^{-3\delta/2}(\log m)^2).$$

Therefore,

$$(7.10)\qquad U_n = 0_p(n^{-3\delta/4}\log m).$$

From (7.7), (7.9) and (7.10) the conclusion of the theorem follows. □

For the Legendre case we have the following

THEOREM 7.2. *Let h_n as above, f_0 and w as in Theorem 5.1 and $\gamma_n = n^{-\frac{1}{2}+\delta/4}$. Define T_n in terms of f_0 by (5.14). Then if $\delta < \frac{1}{4}$ and $m + 1 = Kn^{\delta}$*

$$(m+1)^{\frac{1}{2}}\left(T_n - \frac{1}{\pi}\int_{-1}^{1} f_0 w^2(1-s^2)^{-\frac{1}{2}}\, ds\right) \tag{7.11}$$

is asymptotically normally distributed with mean $K^{-\frac{1}{2}}\int_{-1}^{1}\eta^2(s)w^2(s)\, ds$ and variance $2\pi^{-1}\int_{-1}^{1} f_0^2(s)w^4(s)(1-s^2)^{-\frac{1}{2}}\, ds$.

The proof is the same as that of Theorem 7.1.

It is interesting to compare the tests based on different estimators. However the possibility of such comparison is seriously limited because f_n and T_n are in fact functions of n and m and we should denote them by $f_{n,m}$ and $T_{n,m}$, respectively. Theorems 7.1 and 7.2 show that the asymptotic power of the tests are decreasing functions of K which tend to one when $K \downarrow 0$. Therefore, choosing K small enough the asymptotic power can be done as near to one as we want.

The difficulty in comparing the asymptotic powers corresponding to two different estimators is due to the fact that the asymptotic powers are functions of the constants K, which we have no reason to assume equal, or assume that they are functionally related in some way. Given two estimators, with proper selection of the constants K to use for each of them we can obtain for the relative Pitman efficiency any prefixed number of the open interval $(0, \infty)$.

The situation is the same with kernel type estimators. If, in Theorem 4.2 of Bickel and Rosenblatt (1973), we take $b(n) = Kn^{-\delta}$, then under the hypotheses of that theorem

$$b^{-\frac{1}{2}}(n)\left[T_n - \left[\int f_0(s)a(s)\, ds\right]\left[\int v^2(z)\, dz\right]\right]$$

is asymptotically normally distributed with mean $K^{-\frac{1}{2}}\int \eta^2(s)a(s)\, ds$ and variance $2(v * \bar{v})^{(2)}\int a^2(s)f_0^2(s)\, ds$ where v is the kernel function named w in Bickel and Rosenblatt's paper. Therefore, the asymptotic power is again a decreasing function of the constant K.

From the above discussion we arrive at the conclusion that in general it is meaningless to compare the asymptotic powers obtained using different orthonormal systems, or using kernels, and it is so due to the fact that the asymptotic power depends on the undetermined constant K.

There are some cases where some comparison is possible.

Consider, for example, the following two kernels

$$v_1(s) = \tfrac{1}{2}I_{[-1,1]}(s), \qquad v_2(s) = I_{\left[-\frac{1}{2},\frac{1}{2}\right]}(s).$$

Here it is clear that $b_1(n)$ should be two times $b_2(n)$.

Bickel and Rosenblatt (1973) proved that their test with kernel function $v = I_{[-\frac{1}{2},\frac{1}{2}]}(x)$ is asymptotically more efficient than the classical χ^2 test Of goodness-of-fit. Here the asymptotic comparison has meaning since there are sound reasons to take the length of the cells of the χ^2 test equal to the bandwidth $b(n)$. Since there are no clear equivalences between the number of harmonics of our tests and the number of cells, it is difficult to obtain asymptotic relative efficiency figures with actual meaning. However we feel that the tests proposed in this paper have promise as competitors of the χ^2 test.

From a computational point of view, our tests are more convenient than those of Bickel and Rosenblatt. The latter involves the evaluation of an integral by numerical methods. On the other hand Bickel and Rosenblatt's tests are more flexible since they allow one to emphasize certain regions of the domain of f by choosing an appropriate weight function.

APPENDIX

On the traces of powers of certain matrices.

A.1. Let f and w be differentiable functions defined on $[-1, 1]$ such that $w(s) = 0((1 - s^2)^\eta)$, $w'(s) = 0((1 - s^2)^{\eta-1})$, $\eta > \frac{1}{4}$. Let $(p_\nu)_{\nu=0}^\infty$ be the orthonormal system of Legendre polynomials. Define $g^*(s) = f(s)w^2(s)$. Let B_m be the $(m + 1) \times (m + 1)$ matrix with entries b_{ij} given by

$$b_{ij} = \int_{-1}^{1} p_i(s)p_j(s)g^*(s)\, ds. \tag{1}$$

LEMMA A.1. *If B_m is defined as above, then*

$$\frac{\operatorname{tr}(B_m)}{m+1} = \frac{1}{\pi}\int_0^\pi g^*(\cos\theta)\,d\theta + o\left(m^{-\frac{1}{2}}\right), \qquad m \to \infty,$$

$$\lim_{m\to\infty} \frac{\operatorname{tr}\left(B_m^2\right)}{m+1} = \frac{1}{\pi}\int_{-1}^{1} \frac{f^2(s)w^4(s)}{(1-s^2)^{\frac{1}{2}}}\,ds,$$

$$\operatorname{tr}\left(B_m^3\right) = 0(m\log m) \qquad \textit{as } m \to \infty.$$

PROOF. From Szegö (1939), page 189, we have

$$p_\nu(\cos\theta) = \left(\frac{2\nu+1}{2}\right)^{\frac{1}{2}} \frac{2}{\pi}\, \frac{\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{3}{2}\right)}\, \frac{\cos\left[\left(\nu+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right]}{(2\sin\theta)^{\frac{1}{2}}} + 0(\nu^{-1})\frac{1}{(\sin\theta)^{\frac{3}{2}}}, \qquad 0 < \theta < \pi. \tag{2}$$

Then

$$b_{ij} = \int_0^\pi p_i(\cos\theta)p_j(\cos\theta)g^*(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{4}{\pi}\left(\frac{2i+1}{2}\right)^{\frac{1}{2}}\left(\frac{2j+1}{2}\right)^{\frac{1}{2}}\frac{\Gamma(i+1)\Gamma(j+1)}{\Gamma(i+\frac{3}{2})\Gamma(j+\frac{3}{2})}$$

$$\times\left\{\int_0^\pi \frac{\frac{1}{2}\cos[(j-i)\theta]}{2\sin\theta}g^*(\cos\theta)\sin\theta\, d\theta\right.$$

$$+\int_0^\pi \frac{\frac{1}{2}\cos\left[(i+j+1)\theta-\frac{\pi}{2}\right]}{2\sin\theta}\sin\theta\, g^*(\cos\theta)\, d\theta \tag{3}$$

$$+0(i^{-1})0(j^{-1})\int_0^\pi \frac{g^*(\cos\theta)}{\sin^2\theta}\, d\theta$$

$$+0(j^{-1})\int_0^\pi \cos\left[\left(i+\tfrac{1}{2}\right)\theta-\frac{\pi}{4}\right]\frac{g^*(\cos\theta)}{\sin\theta}\, d\theta$$

$$\left.+0(i^{-1})\int_0^\pi \cos\left[\left(j+\tfrac{1}{2}\right)\theta-\frac{\pi}{4}\right]\frac{g^*(\cos\theta)}{\sin\theta}\, d\theta\right\}.$$

Now, for $\eta > \frac{1}{4}$, $g^*(\cos\theta)/\sin^2\theta$ is integrable. Since f is assumed to be differentiable it follows that for $\eta > \frac{1}{4}$, $g^*(\cos\theta)/\sin\theta$ is differentiable with integrable derivative; hence it is of bounded variation. Therefore (3) can be written as

$$b_{ij} = \frac{1}{\pi}\left(\frac{2i+1}{2}\right)^{\frac{1}{2}}\left(\frac{2j+1}{2}\right)^{\frac{1}{2}}\frac{\Gamma(i+1)\Gamma(j+1)}{\Gamma(i+\frac{3}{2})\Gamma(j+\frac{3}{2})}$$

$$\times\left\{\int_0^\pi \cos[(j-i)\theta]\, g^*(\cos\theta)\, d\theta + \int_0^\pi \sin[(i+j+1)\theta]\, g^*(\cos\theta)\, d\theta\right\} \tag{4}$$

$$+0(i^{-1})0(j^{-1}).$$

From (4) it follows that

$$\operatorname{tr}(B_m) = \frac{1}{\pi}\Sigma_{i=0}^m \frac{2i+1}{2}\left(\frac{\Gamma(i+1)}{\Gamma(i+\frac{3}{2})}\right)^2\left[\int_0^\pi \cos\theta\, g^*(\cos\theta)\, d\theta\right.$$

$$\left.+\int_0^\pi \sin[(2i+1)\theta]\, g^*(\cos\theta)\, d\theta + 0(m^{-2})\right].$$

Hence

$$\frac{\operatorname{tr}(B_m)}{m+1} = \frac{1}{\pi}\int_0^\pi g^*(\cos\theta)\, d\theta \frac{1}{m+1}\Sigma_{i=0}^m \frac{2i+1}{2}\left[\frac{\Gamma(i+1)}{\Gamma(i+\frac{3}{2})}\right]^2 + 0(\log m/m). \tag{5}$$

Using Stirling's formula it can be proved that

$$(6) \qquad \frac{1}{m+1}\Sigma_{i=0}^{m}\frac{2i+1}{2}\left[\frac{\Gamma(i+1)}{\Gamma\left(i+\frac{3}{2}\right)}\right]^2 = 1 + o\left(m^{-\frac{1}{2}}\right)$$

and, therefore, from (5) and (6) it follows that

$$(7) \qquad \frac{\operatorname{tr}(B_m)}{m+1} = \frac{1}{\pi}\int_0^{\pi} g^*(\cos\theta)\,d\theta + 0\left(m^{-\frac{1}{2}}\right).$$

From (4) it follows that

$$\frac{\operatorname{tr}\left(B_m^2\right)}{m+1} = \frac{1}{\pi^2(m+1)}\Sigma_{i=0}^{m}\Sigma_{j=0}^{m}\frac{2i+1}{2}\,\frac{2j+1}{2}\left[\frac{\Gamma(i+1)\Gamma(j+1)}{\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}\right]^2$$

$$\times\left[\int_0^{\pi}\cos[(j-i)\theta]\,g^*(\cos\theta)\,d\theta\right]^2 + o(1)$$

$$(8) \qquad = \frac{1}{\pi^2(m+1)}\left\{\left[\int_0^{\pi} g^*(\cos\theta)\,d\theta\right]^2\Sigma_{i=0}^{m}\left(\frac{2i+1}{2}\right)^2\left[\frac{\Gamma(i+1)}{\Gamma\left(i+\frac{3}{2}\right)}\right]^4\right.$$

$$+2\Sigma_{k=1}^{m}\Sigma_{\nu=0}^{m-k}\frac{2(\nu+k)+1}{2}\,\frac{2\nu+1}{2}$$

$$\left.\times\left[\frac{\Gamma(\nu+k+1)\Gamma(\nu+1)}{\Gamma\left(\nu+k+\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}\right]^2\left[\int_0^{\pi} g^*(\cos\theta)\cos k\theta\,d\theta\right]^2\right\}$$

$$+o(1).$$

Since

$$a_{km} \equiv \Sigma_{\nu=0}^{m-k}\frac{2(\nu+k)+1}{2}\,\frac{2\nu+1}{2}\left[\frac{\Gamma(\nu+k+1)\Gamma(\nu+1)}{\Gamma\left(\nu+k+\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}\right]^2\frac{1}{m+1}$$

is uniformly bounded in k and m and for every fixed k $\lim a_{km} = 1$ as $m \to \infty$, from dominated convergence theorem applied to (8) it follows that

$$\lim_{m\to\infty}\frac{\operatorname{tr}\left(B_m^2\right)}{m+1} = \frac{1}{\pi^2}\left\{\left[\int_0^{\pi} g^*(\cos\theta)\,d\theta\right]^2 + 2\Sigma_{k=1}^{\infty}\left[\int_0^{\pi} g^*(\cos\theta)\cos k\theta\right]^2\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\int_{-\pi}^{\pi}(2\pi)^{-\frac{1}{2}}g^*(\cos\theta)\,d\theta\right]^2 + \Sigma_{k=1}^{\infty}\left[\int_{-\pi}^{\pi}\pi^{-\frac{1}{2}}g^*(\cos\theta)\cos k\theta\right]^2\right\}$$

$$(9) \qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[g^*(\cos\theta)\right]^2 d\theta$$

$$= \frac{1}{\pi}\int_{-1}^{1}\frac{g^{*2}(s)}{\left(1-s^2\right)^{\frac{1}{2}}}\,ds$$

$$= \frac{1}{\pi}\int_{-1}^{1}\frac{f^2(s)w^4(s)}{\left(1-s^2\right)^{\frac{1}{2}}}\,ds.$$

To prove the third equality of the lemma let us write

$$\mathrm{tr}(B_m^3) = \Sigma_i \Sigma_j \Sigma_k b_{ij} b_{jk} b_{ki}. \tag{10}$$

Replacing (4) in (10) and since

$$\int_0^\pi g^*(\cos\theta)\sin[(i+j+1)\theta]\, d\theta = 0((i+j)^{-1}) < \frac{0(1)}{i^{\frac{1}{2}} j^{\frac{1}{2}}}$$

$$\int_0^\pi g^*(\cos\theta)\cos[(j-i)\theta]\, d\theta = 0((j-i)^{-1})$$

it follows after some computations that

$$\mathrm{tr}(B_m^3) = 0(m \log m). \qquad \square$$

A.2. Let f be of bounded variation on $[-\pi, \pi)$ and the weight function w be bounded, absolutely continuous with derivative w' square integrable on $[-\pi, \pi)$. Define $g^* = w^2 f$. Let (ϕ_i) be the orthonormal system defined by (3.1). Let B_m be the matrix with components b_{ij} given by

$$b_{ij} = a_i a_j \int_{-\pi}^{\pi} \phi_i(s)\phi_j(s) g^*(s)\, ds$$

where the $a_i = a_i(m)$ satisfy the conditions (2.20), (3.3), (3.3a) and (3.3b). Define $a^{(2)}$ and $a^{(4)}$ by (3.3a) and (3.3b), respectively,

LEMMA A.2. *If B_m is defined as above, then*

$$\frac{\mathrm{tr}(B_m)}{2m+1} = \frac{1}{2\pi}\frac{1}{2m+1}\Sigma_{i=-m}^{m} a_i^2 \int_{-\pi}^{\pi} g^*(s)\, ds$$

$$\lim_{m\to\infty} \frac{\mathrm{tr}(B_m^2)}{2m+1} = \frac{1}{2\pi} a^{(4)} \int_{-\pi}^{\pi} f^2(s) w^4(s)\, ds$$

$$\mathrm{tr}(B_m^3) = 0(m \log m) \text{ as } m \to \infty$$

where tr(·) *stands for the trace.*

PROOF.

$$\mathrm{tr}(B_m) = \Sigma_{i=-m}^{-1} a_i^2 \int_{-\pi}^{\pi} \frac{1}{\pi}\sin^2(is)\, g^*\, ds + a_0^2 \int_{-\pi}^{\pi} \frac{1}{2\pi} g^*\, ds$$

$$+ \Sigma_{i=1}^{m} a_i^2 \int_{-\pi}^{\pi} \frac{1}{\pi}\cos^2(is)\, g^*\, ds$$

$$= \frac{1}{\pi}\Sigma_{i=-m}^{-1} a_i^2 \int_{-\pi}^{\pi}\left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2is)\, g^*\right) ds + \frac{a_0^2}{2\pi}\int_{-\pi}^{\pi} g^*\, ds$$

$$+ \frac{1}{\pi}\Sigma_{i=1}^{m} a_i^2 \int_{-\pi}^{\pi}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2is)\, g^*\right) ds$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g^*\, ds\, \Sigma_{i=-m}^{m} a_i^2.$$

$$\operatorname{tr}(B_m^2) = \Sigma_{i=-m}^m \Sigma_{j=-m}^m a_i^2 a_j^2 \left[\int_{-\pi}^{\pi} \phi_i \phi_j g^* \, ds\right]^2$$

$$= \frac{1}{\pi^2}\Big\{\Sigma_{i=1}^m \Sigma_{j=1}^m a_i^2 a_j^2\left[\tfrac{1}{2}\int_{-\pi}^{\pi}\cos[(i-j)s]\, g^*\, ds + \tfrac{1}{2}\int_{\pi}^{\pi}\cos[(i+j)s]\, g^*\, ds\right]^2$$

$$+\Sigma_{i=-m}^{-1}\Sigma_{j=-m}^{-1} a_i^2 a_j^2\left[\tfrac{1}{2}\int_{-\pi}^{\pi}\cos[(i-j)s]\, g^*\, ds - \tfrac{1}{2}\int_{-\pi}^{\pi}\cos[(i+j)s]\, g^*\, ds\right]^2$$

$$+\Sigma_{i=1}^{m}\Sigma_{j=-m}^{-1} a_i^2 a_j^2\left[\tfrac{1}{2}\int_{-\pi}^{\pi}\sin[(i+j)s]\, g^*\, ds - \tfrac{1}{2}\int_{-\pi}^{\pi}\sin[(i-j)s]\, g^*\, ds\right]^2$$

$$+\Sigma_{i=-m}^{-1}\Sigma_{j=1}^{m} a_i^2 a_j^2\left[\tfrac{1}{2}\int_{-\pi}^{\pi}\sin[(i+j)s]\, g^*\, ds + \tfrac{1}{2}\int_{-\pi}^{\pi}\sin[(i-j)s]\, g^*\, ds\right]^2$$

$$+2\frac{a_0^2}{2\pi}\Sigma_{j=-m}^{m}\left[\int_{-\pi}^{\pi}\phi_j g^*\, ds\right]^2.$$

Using the condition $a_i = a_{-i}$ we can write

$$\operatorname{tr}(B_m^2) = \frac{1}{2\pi^2}\Big\{\Sigma_{i=1}^m\Sigma_{j=1}^m a_i^2 a_j^2\Big[\left(\int_{-\pi}^{\pi}\cos[(i-j)s]\, g^*\, ds\right)^2 + \left(\int_{-\pi}^{\pi}\cos[(i+j)s]\, g^*\, ds\right)^2$$

$$+ \left(\int_{-\pi}^{\pi}\sin[(i-j)s]\, g^*\, ds\right)^2 + \left(\int_{-\pi}^{\pi}\sin[(i+j)s]\, g^*\, ds\right)^2\Big]\Big\}$$

$$+\frac{a_0^2}{\pi}\Sigma_{j=-m}^{m}\left[\int_{-\pi}^{\pi}\phi_j g^*\, ds\right]^2.$$

Since g^* is of bounded variation

$$\left(\int_{-\pi}^{\pi}\cos[(i+j)s]\, g^*\, ds\right)^2 = 0((i+j)^{-2}) = \frac{0(1)}{ij}$$

$$\left(\int_{-\pi}^{\pi}\sin[(i+j)s]\, g^*\, ds\right)^2 = 0((i+j)^{-2}) = \frac{0(1)}{ij}.$$

Therefore, using this in the expression for $\operatorname{tr}(B_m^2)$ we have

$$\operatorname{tr}(B_m^2) = \frac{1}{2\pi^2}\Sigma_{i=1}^m\Sigma_{j=1}^m a_i^2 a_j^2\Big[\left(\int_{-\pi}^{\pi}\cos[(i-j)s]\, g^*\, ds\right)^2$$

$$(1) \qquad + \left(\int_{-\pi}^{\pi}\sin[(i-j)s]\, g^*\, ds\right)^2\Big] + 0((\log m)^2)$$

$$= \frac{1}{2\pi^2}\Sigma_{i=1}^m\Sigma_{j=1-i}^{m-i} a_{i+j}^2 a_j^2\Big[\left(\int_{-\pi}^{\pi}\cos(is) g^*\, ds\right)^2 + \left(\int_{-\pi}^{\pi}\sin(is) g^*\, ds\right)^2\Big]$$

$$+0((\log m)^2).$$

Using (3.3) it follows that

$$(2) \qquad \lim_{m\to\infty}\frac{1}{2m+1}\Sigma_{j=1-i}^{m-i} a_{i+j}^2 a_j^2 = \lim_{m\to\infty}\left[\Sigma_{j=1-i}^{m-i} a_j^4 + o(1)\Sigma_{j=1-i}^{m-i} a_j^2\right]$$

$$= \tfrac{1}{2}a^{(4)}.$$

Since by hypotheses g^* is of bounded variation and since (2.20) holds the series in the right-hand side of (1) is dominated term by term by the convergent series whose

mth term is

$$(\sup_m \sup_{0<i<m} |a_i(m)|)^4 \left[(\int_{-\pi}^{\pi} \cos(is) g^* \, ds)^2 + (\int_{-\pi}^{\pi} \sin(is) g^* \, ds)^2 \right].$$

Hence, applying the dominated convergence theorem to (1) and using (2) we obtain

$$\lim \frac{\operatorname{tr}(B_m^2)}{2m+1} = \frac{1}{2\pi} a^{(4)} \int_{-\pi}^{\pi} (g^*)^2 \, ds.$$

Let us now consider $\operatorname{tr}(B_m^3)$.

$$\begin{aligned} |\operatorname{tr}(B_m^3)| &= \Sigma_{i,j,k=-m}^{m} a_i^2 a_j^2 a_k^2 \int_{-\pi}^{\pi} \phi_i \phi_j g^* \, ds \int_{-\pi}^{\pi} \phi_j \phi_k g^* \, ds \int_{-\pi}^{\pi} \phi_k \phi_i g^* \, ds \\ &= 0(1) \Sigma_{i \neq j \neq k} \frac{1}{|j-i|} \frac{1}{|k-j|} \frac{1}{|i-k|} \\ &= 0(m \log m). \end{aligned}$$

□

Acknowledgments. This work is based on part of a (University of California, Berkeley) Ph. D. thesis written under the guidance of Professor Peter J. Bickel. The author is deeply indebted to him for his generous advice and encouragement. The author's thanks to the National Science Foundation, grant MPS73-08698A01 and to the Consejo Nacional de Investigaciones Cientificas y Tecnicas de Argentina, for their support.

REFERENCES

[1] Bela, Sz-Nagy (1965). *Introduction to Real Functions and Orthogonal Expansions*. Oxford Univ. Press, New York.

[2] Bickel, P. and Rosenblatt, M. (1973). On some global measures of the deviations of density functions estimates. *Ann. Statist.* 1 1071–1095.

[3] Breiman, L. (1969). *Probability*. Addison-Wesley.

[4] Brillinger, D. (1969). An asymptotic representation of the sample distribution function. *Bull. Amer. Math. Soc.* 75 545–547.

[5] Cencov, N. N. (1962). Evaluation of an unknown distribution density from observations. *Soviet Math.* 3 1559–1562.

[6] Kronmal, R. and Tarter, M. (1968). The estimation of probability densities and cumulatives by Fourier series methods. *J. Amer. Statist. Assoc.* 63 925–952.

[7] Rosenblatt, M. (1971). Curve estimates. *Ann. Math. Statist.* 42 1815–1842.

[8] Sansone, G. (1959). *Orthogonal Functions*. Interscience Publishers, New York.

[9] Szegö, G. (1939). *Orthogonal Polynomials*. Amer. Math. Soc. Colloq. Publications, Vol. XXIII.

[10] Viollaz, A. (1976). Nonparametric estimation of probability density functions using orthogonal expansions. Ph. D. thesis, University of California, Berkeley.

Las Piedras 731
4000–San Miguel de Tucuman
Tucuman, Argentina