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A COORDINATE-FREE APPROACH TO FINDING OPTIMAL PROCEDURES FOR REPEATED MEASURES DESIGNS

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A repeated measures design occurs in analysis of variance when a particular individual receives several treatments. Let $X_i = (x_{i1}, \cdots, x_{ip})'$ be the vector of observations on the ith individual. It is assumed that the X_i are independently normally distributed with mean μ_i and common covariance $\Sigma > 0$. The researcher wants to test hypotheses about the μ_i. Let $\varepsilon_i = (\varepsilon_{i1}, \cdots, \varepsilon_{ip})' = X_i - \mu_i$. For this paper, in order to get powerful tests, the simplifying assumption that the $\varepsilon_{i1}, \cdots, \varepsilon_{ip}$ are exchangeable is made. We assume that the design is given and use a coordinate-free approach to find optimal (i.e., UMP invariant, UMP unbiased, most stringent, etc.) procedures for testing a large class of hypotheses about the μ_i.

1. Introduction. A repeated measures design occurs in analysis of variance when a particular individual (person, rat, field, etc.) receives several treatments. Therefore, the observations cannot be assumed independent as they are assumed in the usual independent measures design. Let $X_i = (x_{i1}, \cdots, x_{ip})'$ be the vector of observations on the ith individual. It is assumed that the X_i are independently normally distributed with mean μ_i and common covariance $\Sigma > 0$. The researcher wants to test hypotheses about the μ_i. Ideally, he would make no assumption about Σ. If $\mu_i = \mu$, Giri (1977) finds the UMP invariant test for testing that $\mu \in W$ vs. $\mu \in V$, where W and V are specified subspaces of R_p. Unfortunately, the researcher usually does not have enough measurements to get a good estimator of Σ. Therefore, tests for this model are often not very powerful. Let $\varepsilon_i = (\varepsilon_{i1}, \cdots, \varepsilon_{ip})' = X_i - \mu_i$. In order to get more powerful tests the simplifying assumption that the $\varepsilon_{i1}, \cdots, \varepsilon_{ip}$ are exchangeable is often made. This is equivalent to assuming that

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}, \sigma^2 > 0, -\frac{1}{p-1} < \rho < 1. \tag{1.1}$$

This covariance structure can also be derived from assuming that the effect due to a particular individual is an additive random effect. However, under this model, ρ would be nonnegative. For the main body of this paper we assume only that ρ satisfies (1.1). For an extensive bibliography on repeated measures designs see Hedayat and Afsarinejad (1975).

Most previous work on optimality for repeated measures designs has been concerned with the design of the experiment. In this paper we assume that the

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design is given. We use a coordinate-free approach to find optimal (i.e., UMP invariant, UMP unbiased, most stringent, etc.) procedures for testing a large class of hypotheses about the μ_i. We first write $\mu_i = \theta_i e + \delta_i$ where $e' = (1, \cdots, 1)$, θ_i is a scalar and δ_i is orthogonal to e. (This factorization is always possible. See Section 3.) We consider testing hypotheses about θ_i (type A), and hypotheses about the δ_i (type B). We show that these two types of problems are analyzed differently. For either type of problem the mean square for the effect being tested is the same as it would be if the measures were independent. However, the mean square for variance is different for the two types of problems. If the treatment being tested is handled in such a way that each individual receives only one treatment, then the hypothesis is of type A. Most others are of type B. In the language of split plot designs, type A hypotheses involve whole plot effects while type B hypotheses involve subplot effects.

For example, consider a two-way analysis of variance in which each individual receives one α treatment level together with each β treatment level. (This model might occur if the α treatment represented sex, race or degree of illness, for example.) Hypotheses about the α treatment would be of type A, while those about the β treatment would be of type B, as would the hypotheses about the interactions. (See Example 3, Section 2.) We also consider testing the hypothesis that $\rho = 0$, i.e., that the repeated measures really are independent. This hypothesis is called type C.

In Section 3 the problem is defined and in Section 4 it is transformed to an easier problem. In Section 5 results about products of problems are used to derive optimal procedure for the three types of hypotheses. In Section 6 we show that, if we have the formulae for the equivalent independent measures model, we need to compute only one other statistic in order to test the hypotheses in question. These results can also be used to show that the F-tests that we derive are identical with those used in applications. (See, for example, Winer (1971) for some specific formulae.)

2. Preliminaries.

2.1. *Products of problems.* Let P be the testing problem in which we observe X_1 and X_2 independent random vectors, X_i having distribution $F_i(x_i, \theta_i)$. We want to test the hypothesis that $\theta_1 \in A_1$, $\theta_2 \in A_2$ vs. the alternative that $\theta_1 \in B_1$, $\theta_2 \in B_2$. Note that there is no relationship between θ_1 and θ_2 under either hypothesis. We then say that P is a *product* of the problems P_1 and P_2, where P_i is the problem in which we observe X_i having distribution $F_i(x_i, \theta_i)$, and are testing $\theta_i \in A_i$ vs. the alternative that $\theta_i \in B_i$. If $A_2 = B_2$, we say that the problem P_2 is *trivial*. In Section 5 we show that hypotheses of types A and B are each a product of two problems P_1 and P_2, in which P_2 is trivial. In this situation it seems evident that any sensible procedure would ignore X_2, since the hypotheses really only involve θ_1, and the distribution of X_2 is independent of both X_1 and θ_1. Theorem B of Arnold (1973) (see Arnold (1970) for a proof) makes this idea rigorous by establishing that if a critical function $\phi(X_1)$ has one of several optimal properties as a procedure for P_1, it has that property for the product of P_1 and a trivial problem. The optimal properties include: UMP, UMP unbiased, UMP invariant, most stringent, Bayes and admissible. In Arnold (1973) it is also stated that a likelihood ratio test (LRT) for P_1 is a LRT for P. A counterexample to this result is given in Arnold (1978). It is true however, that if $\phi(X_1)$ is a LRT for P_1, and there exist maximum likelihood estimators (MLE's) for P_2, then $\phi(X_1)$ is the LRT for P.

2.2. *Notation.* In this section we define the notation used in this paper. Let V be a subspace of R^m, and let $y \in R^m$. Then $P_V y$ is the orthogonal projection of y on V, and $V\perp$ is the orthogonal complement of V. If W is a subspace of V, then $V|W = V \cap W\perp$. If $W \subset R^m$, $V \subset R^n$ are w and v dimensional subspaces, define $W \times V$ to be the $v + w$ dimensional subspace of R^{m+n} defined by $(y_1', y_2')' \in W \times V$ iff $y_1 \in W, y_2 \in V (y_1 \in R^m, y_2 \in R^n)$. Define V^s recursively by $V^s = V^{s-1} \times V$, $V^1 = V$. It is easily verified that

$$(V_1 \times V_2)|(W_1 \times W_2) = (V_1|W_1) \times (V_2|W_2), (V \times W)\perp = V\perp \times W\perp,$$

$$\|P_{V\times W}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\|^2 = \|P_V y_1\|^2 + \|P_W y_2\|^2$$

provided all the expressions are meaningful.

The following special matrices are used. I_n is the $n \times n$ identity matrix, E_n is the $n \times n$ matrix of 1's, and e_n is the $n \times 1$ vector of 1's. Also A^*B is the Kronecker product of A and B. That is, if $B = (b_{ij})$ then

$$A^*B = \begin{bmatrix} Ab_{11} & \cdots & Ab_{1c} \\ \vdots & & \\ Ab_{r1} & \cdots & Ab_{rc} \end{bmatrix}.$$

We use the following notation. We write $X \sim N_p(\mu, \Sigma)$ to mean that X has a multivariate normal distribution with mean vector μ and covariance matrix Σ. We write that $y \sim F_{a,b}(\delta)$ to mean that y has a noncentral F distribution with a and b degrees of freedom and noncentrality parameter δ.

3. Setting up the model.

Let $X_1, \cdots, X_n$ be independent p-dimensional normal random vectors such that

(3.1)

$$X_k \sim N_p(\mu_k, \Sigma), \Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix} = \sigma^2(1-\rho)I_p + \sigma^2 \rho E_p,$$

$$\mu_k = \theta_k e_p + \delta_k$$

where θ_k is a real number and $\delta_k \perp e_p$. Let U be the 1 dimensional subspace spanned by e_p. Then $\mu_k = P_U \mu_k + P_{U\perp}\mu_k$, so that $P_U \mu_k = \theta_k e_p$, $P_{U\perp}\mu_k = \delta_k$. Hence this representation for μ_k is not restrictive and is unique. Therefore, the transformation from μ_k to (θ_k, δ_k) is just a reparameterization of the problem. Let

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}, \theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}, \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}. \tag{3.2}$$

Then $\mu = \theta * e_p + \delta, \theta \in R^n, \delta \in (U\perp)^n$.

In order to define the parameter space, let T be a t-dimensional subspace of R^n, $t < n$, and let V be a v-dimensional subspace of R^{np} such that $V \subset (U\perp)^n$ and $v < n(p-1)$ (the dimension of $(U\perp)^n$). In this paper it is assumed that the parameter space is given by

$$\theta \in T, \delta \in V, \sigma^2 > 0, -\frac{1}{p-1} < \rho < 1. \tag{3.3}$$

(The last inequality is equivalent to assuming that $\Sigma > 0$.)

We consider three different hypothesis testing problems for this model. For all three problems the alternative set is the parameter space defined in (3.3).

A. Let $S \subset T$ be an s-dimensional subspace, $s < t$. In the first problem, we test that $\theta \in S$.
B. Let $W \subset V$ be a w-dimensional subspace, $w < v$. The second problem is to test that $\delta \in W$.
C. The third problem is to test that $\rho = 0$.

In C we are just testing that the repeated measurements on a particular individual are in fact independent. We now give some examples of types A and B.

EXAMPLE 1. Suppose that $\mu_k = (\gamma + \alpha_1, \cdots, \gamma + \alpha_p)'$, $\Sigma\alpha_i = 0$ (μ_k is independent of k). This model corresponds to a one-way analysis of variance where each individual receives each treatment. We want to test that $\alpha_i = 0$. In the notation of (3.1), $\theta_k = \gamma$, $\delta_k = (\alpha_1, \cdots, \alpha_p)'$. This hypothesis, therefore, is of type B.

EXAMPLE 2. Suppose that $\mu_k = (\gamma + \alpha_1 + \beta_1, \cdots, \gamma + \alpha_1 + \beta_c, \cdots, \gamma + \alpha_r + \beta_c)'$, where $\Sigma\alpha_i = 0$, $\Sigma\beta_j = 0$. This model corresponds to a two-way anova with no interaction where each individual receives each pair of treatments. We want to test that $\alpha_i = 0$, and we also want to test that $\beta_j = 0$. In the notation of (3.1), $\theta_k = \gamma$, $\delta_k = (\alpha_1 + \beta_1, \cdots, \alpha_r + \beta_c)$. Therefore, both these hypotheses are of type B.

A similar model would be possible allowing for interactions between the α and β effects. In fact, any balanced replicated analysis of variance model can be used to generate a model of this type by assuming that the kth replication in each class is made on the kth individual, or, equivalently, that every individual receives each treatment combination. All hypotheses of interest for this sort of model would be of type B.

The following model gives an example of a situation in which the μ_k are different and we are interested in hypotheses of type A.

EXAMPLE 3. In this example we use a doubly indexed collection of random vectors, $Y_{jk}, j = 1, \cdots, c, k = 1, \cdots, m$. We assume that the Y_{jk} are independently normally distributed with covariance matrix given by (3.1). Suppose $\mu_{jk} = EY_{jk} = (\gamma + \alpha_1 + \beta_j + \alpha\beta_{1j}, \cdots, \gamma + \alpha_p + \beta_j + \alpha\beta_{pj})'$, where $\Sigma\alpha_i = 0, \Sigma\beta_j = 0, \Sigma_i\alpha\beta_{ij} = 0, \Sigma_j\alpha\beta_{ij} = 0$. This model corresponds to a two-way analysis of variance in which each individual receives one β treatment level and all the α treatment levels. In this model $\theta_{jk} = \gamma + \beta_j$, and $\delta_{jk} = (\alpha_1 + \alpha\beta_{1j}, \cdots, \alpha_p + \alpha\beta_{pj})'$. (It is clear that $\mu_{jk} = \theta_{jk}e_p + \delta_{jk}$, and that $e_p'\delta_{jk} = \Sigma_i(\alpha_i + \beta_{ij}) = 0$.) Therefore, the hypothesis that $\beta_j = 0$ is of type A, while the hypothesis that $\alpha_i = 0$ is of type B, as is the hypothesis that $\alpha\beta_{ij} = 0$.

In the last example it is not necessary to assume that the same number of individuals receive each treatment. In fact, the general model given at the beginning of this section would include any model in which each individual was measured the same number of times and each treatment was of one of the following two types:

1. each individual receives only one level of the treatment (type A); or
2. the average treatment effect for each individual is 0 or only the deviations from the average effect on each individual are of interest (type B).

We now consider an example with a covariate which does not fall into the above framework.

EXAMPLE 4. Let $X_k \sim N_p(\mu_k, \Sigma)$, as before. Suppose that $\mu_k = (\gamma + \alpha_1 + \varepsilon u_{1k}, \cdots, \gamma + \alpha_p + \varepsilon u_{pk})'$. This would be a one-way analysis of variance with each individual receiving each treatment level and each measurement having a covariate. Then $\theta_i = \gamma + \varepsilon\bar{u}_{\cdot k}$, $\delta_i = (\alpha_1 + \varepsilon(u_{1k} - \bar{u}_{\cdot k}), \cdots, \alpha_p + \varepsilon(u_{pk} - \bar{u}_{\cdot k}))'$. We want to test the hypothesis that $\varepsilon = 0$. If $\bar{u}_{\cdot k} = 0$ for all k, this hypothesis is of type B. If $u_{ik} = \bar{u}_{\cdot k}$, or, in other words, if the covariate depends only on the individual, not on the treatment level, then the hypothesis is of type A. If neither of these conditions is satisfied, then the model is not of type A, type B or type C, since it involves both θ_i and δ_i.

4. The basic results. In this section we show how to transform the problems defined in the last section to problems that are easier to handle. Let C_p be a $(p - 1) \times p$ matrix such that

$$\Gamma_p = \begin{bmatrix} p^{-\frac{1}{2}}e_p' \\ C_p \end{bmatrix}$$

is orthogonal. Define

$$\begin{pmatrix} Y_{1k} \\ Y_{2k} \end{pmatrix} = \Gamma_p X_k, \quad Y_1 = \begin{pmatrix} Y_{11} \\ Y_{1n} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} Y_{21} \\ Y_{2n} \end{pmatrix}, \tag{4.1}$$

where Y_{1k} is one-dimensional and Y_{2k} is $(p - 1)$ dimensional. Since Γ_p is an invertible matrix that does not depend on any unknown parameters, observing the X_k is equivalent to observing Y_1 and Y_2. We now find the joint distribution of Y_1 and Y_2.

LEMMA 1. *Y_1 and Y_2 are independent.* $Y_1 \sim N_n\left(p^{\frac{1}{2}}\theta, \sigma^2(1 + (p-1)\rho)I_n\right)$, $Y_2 \sim N_{n(p-1)}((C_p * I_n)\delta, \sigma^2(1-\rho)I_{n(p-1)})$.

PROOF. It is easily verified that

$$Y^k = \begin{pmatrix} Y_{1k} \\ Y_{2k} \end{pmatrix} \sim N_p\left[\begin{bmatrix} p^{\frac{1}{2}}\theta_i \\ C_p\delta_i \end{bmatrix}, \begin{pmatrix} \sigma^2(1+(p-1)\rho) & 0 \\ 0 & \sigma^2(1-\rho)I_{(p-1)} \end{pmatrix}\right],$$

and that the Y^k are independent. The result follows directly. □

LEMMA 2. *$C_p * I_n$ is an invertible function from $(U\perp)^n$ to $R^{n(p-1)}$.*

PROOF. Note that the rows of C form an orthonormal basis for U. Therefore $C_p z$ is an invertible function from $U\perp$ to R^{p-1} and

$$(C_p * I_n)Z = \begin{pmatrix} C_p Z_1 \\ C_p Z_n \end{pmatrix}$$

is, therefore, an invertible function from $(U\perp)^n$ to $(R^{p-1})^n = R^{n(p-1)}$. □

Define

$$\beta_1 = p^{\frac{1}{2}}\theta, \quad \beta_2 = (C_p * I_n)\delta, \quad \tau_1^2 = \sigma^2(1 + (p+1)\rho), \quad \tau_2^2 = \sigma^2(1-\rho). \tag{4.2}$$

COROLLARY. *The transformation from $(\theta, \delta, \sigma^2, \rho)$ to $(\alpha_1, \alpha_2, \tau_1^2, \tau_2^2)$ is an invertible function.*

Therefore $\beta_1, \beta_2, \tau_1^2, \tau_2^2$ is just a reparameterization of the problem.

For any subspace $Q \subset (U\perp)^n$, define Q^* to be the image of Q under the transformation $C_p * I_n$. Q^* is a subspace and by Lemma 2 has the same dimension as Q. The following lemma follows directly from the definitions.

LEMMA 3. a. *If Q is a subspace of T, then $\theta \in Q$* iff *$\beta_1 \in Q$.*
b. *If Q is a subspace of V, then $\delta \in Q$* iff *$\beta_2 \in Q^*$.*
c. $\sigma^2 > 0$, $-1/(p-1) < \rho < 1$ iff $\tau_1^2 > 0$, $\tau_2^2 > 0$.
d. $\rho = 0$ iff $\tau_1^2 = \tau_2^2$.
e. $\rho \geqslant 0$ iff $\tau_1^2 \geqslant \tau_2^2$.

The important fact about Lemmas 1 and 3 is that they factor the problem into two pieces. Y_1 and Y_2 are independent and the parameters of their distributions are unrelated. We are now ready to derive optimality results.

5. Optimal procedures.

5.1. *Tests concerning θ.* We now consider the problem of testing hypothesis A of Section 3. After transforming to (Y_1, Y_2) and $(\beta_1, \beta_2, \tau_1^2, \tau_2^2)$, (using Lemmas 1–3), this problem becomes the testing problem in which we observe Y_1 and Y_2 independent,

$$Y_1 \sim N_n(\beta_1, \tau_1^2 I_n), \; Y_2 \sim N_{n(p-1)}(\beta_2, \tau_2^2 I_{n(p-1)}) \tag{5.1}$$

and we are testing

$$H_0 : \beta_1 \in S, \tau_1^2 > 0, \beta_2 \in V^*, \tau_2^2 > 0$$
$$H_1 : \beta_1 \in T, \tau_1^2 > 0, \beta_2 \in V^*, \tau_2^2 > 0.$$

Call this problem P. P is then the product of the testing problems P_1 and P_2 where P_1 is the independent measures model in which we observe $Y_1 \sim N_n(\beta_1, \tau_1^2 I_n)$ and we are testing

$$H_0 : \beta_1 \in S, \tau_1^2 > 0,$$
$$H_1 : \beta_1 \in T, \tau_1^2 > 0,$$

and P_2 is the trivial problem in which we observe $Y_2 \sim N_{n(p-1)}(\beta_2, \tau_2^2 I_{n(p-1)})$ and we are testing

$$H_0 : \beta_2 \in V^*, \tau_2^2 > 0$$
$$H_1 : \beta_2 \in V^*, \tau_2^2 > 0.$$

Since P_2 is a trivial problem, a good procedure for P_1 will be good for P. Therefore, let F_1 be the usual F statistic and ϕ_1 the usual F critical function for testing P_1. That is

$$(5.2) \qquad F_1 = \frac{\|P_{T|S} Y_1\|^2}{\|P_{T\perp} Y_1\|^2} \frac{n-t}{t-s}, \quad \phi_1(F_1) = \begin{cases} 1 & \text{if } F_1 > F_1^\alpha \\ \gamma & \text{if } F_1 = F_1^\alpha \\ 0 & \text{if } F_1 < F_1^\alpha \end{cases}$$

where F_1^α is the upper α point of a central F distribution with $t - s$ and $n - t$ degrees of freedom.

THEOREM 1. a. $F_1 \sim F_{t-s, n-p}(\|P_{T|S}\beta_1\|^2/\tau_1^2)$.
b. ϕ_1 *is size* α, UMP *invariant, most stringent, Bayes, admissible and the likelihood ratio test for* P.

PROOF. F_1 has the given distribution for P_1 and hence for P. The results for ϕ_1 are known for the problem P_1 (see Lehmann (1959)) and by Theorem B of Arnold (1973) they are true for P. (It is clear the maximum likelihood estimators exist for P_2.) □

We note that the assumption that V, and hence V^*, be a subspace is overly restrictive. The only property of P_2 used in this theorem is the existence of MLE's for P_2. Therefore V could be an arbitrary set such that MLE's exist. It could not, for example, be the whole of $(U\perp)^n$.

5.2. *Hypotheses about* δ. In this section we consider testing hypotheses of type B. After transforming the problem and applying Lemmas 1–3, this problem becomes the problem in which we observe Y_1 and Y_2 independent, having the distributions given in (5.1), and we are testing

$$H_0 : \beta_1 \in T, \tau_1^2 > 0, \beta_2 \in W^*, \tau_2^2 > 0$$
$$H_1 : \beta_1 \in T, \tau_1^2 > 0, \beta_2 \in V^*, \tau_2^2 > 0.$$

Call this problem Q. As before, Q is the product of a trivial problem Q_1 and the independent measures model Q_2 in which we observe Y_2 having the given distribution and we are testing

$$H_0 : \beta_2 \in W^*, \tau_2^2 > 0$$
$$H_1 : \beta_2 \in V^*, \tau_2^2 > 0.$$

Since Q_1 is trivial, a good procedure for Q_2 is good for Q. Therefore, define

(5.3) $$F_2 = \frac{\|P_{V^*|W^*}Y_2\|^2}{\|P_{V^*\perp}Y_2\|^2}\frac{n(p-1)-v}{v-w}, \quad \phi_2(F_2) = \begin{cases} 1 & \text{if } F_2 > F_2^\alpha \\ \gamma & \text{if } F_2 = F_2^\alpha \\ 0 & \text{if } F_2 < F_2^\alpha \end{cases}$$

where F_2^α is the upper α point of a central F distribution with $v - w$ and $n(p-1) - v$ degrees of freedom. The following theorem follows in exactly the same way as Theorem 1.

THEOREM 2. a. $F_2 \sim F_{v-w,\, n(p-1)-v}(\|P_{V^*|W^*}\beta_2\|^2/\tau_2^2)$.
b. ϕ_2 *is size* α, UMP *invariant, most stringent, Bayes, admissible and the likelihood ratio test.*

As before, the assumption that T be a subspace is more restrictive than it need be.

5.3. *Testing that* $\rho = 0$. Now, consider the problem of testing that $\rho = 0$. After transforming to Y_1 and Y_2, this problem becomes the problem in which we observe Y_1 and Y_2 independent, having the distributions given in (5.1). We are testing

$$H_0 : \beta_1 \in T, \beta_2 \in V, \tau_1^2 = \tau_2^2$$
$$H_1 : \beta_1 \in T, \beta_2 \in V, \tau_1^2 > 0, \tau_2^2 > 0.$$

This problem is not a product of problems. However, it is already a problem about which much is known. It is the problem in which we have two independent measures models and are testing for the equality of variances. Let

(5.4)

$$F_3 = \frac{\|P_{T\perp}Y_1\|^2}{\|P_{V\perp}Y_2\|^2}\frac{n(p-1)-v}{n-t}, \quad \phi_3^{a,b}(F_3) = \begin{cases} 1 & \text{if } F > a \text{ or } F < b \\ \gamma & \text{if } F = a \text{ or } F = b \\ 0 & \text{if } a > F > b. \end{cases}$$

The following theorem summarizes some results about F_3 and $\phi_3^{a,b}$. (It is proved in the same fashion as for the two-sample problem.)

THEOREM 3. a. $(\tau_2^2/\tau_1^2)F_3 \sim F_{n-t,\,n(p-1)-v}(0)$.
b. *There exist a and b such that* $\phi_3^{a,b}$ *is* UMP *unbiased.*
c. *There exist a and b such that* $\phi_3^{a,b}$ *is the likelihood ratio test.*

6. Calculating the statistics. In order to calculate the statistics, F_1, F_2 and F_3, one could choose a particular C_p, and hence a particular Γ_p, and use it to compute the Y_i. He could then compute the various projections, and hence the F statistics. In this section we derive some elementary results that make such calculations unnecessary. We show how it is possible to get almost all the appropriate formulae from the formulae for the similar independent measures model. We assume that these formulae are known. Define

$$M_1 = \frac{\|P_{T|S}Y_1\|^2}{t-s}, \quad M_2 = \frac{\|P_{V^*|W^*}Y_2\|^2}{v-w}, \quad M_3 = \frac{\|P_{T\perp}Y_1\|^2}{n-t},$$

$$M_4 = \frac{\|P_{V^*\perp}Y_2\|^2}{n(p-1)-v}.$$

These quantities are the various mean squares, and

$$F_1 = \frac{M_1}{M_3}, \quad F_2 = \frac{M_2}{M_4}, \quad F_3 = \frac{M_3}{M_4}.$$

Now consider the independent measures model, i.e., the model in which it is assumed that $\rho = 0$, or equivalently that $\tau_1^2 = \tau_2^2$. We consider this model in the transformed problem. Define

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \tau^2 = \tau_1^2 = \tau_2^2.$$

If $\rho = 0$, then $Y \sim N_{np}(\beta, \tau^2 I_{np})$. Testing the hypothesis that

$$H_0 : \beta_1 \in S, \beta_2 \in V, \tau^2 > 0,$$
$$H_1 : \beta_1 \in T, \beta_2 \in V, \tau^2 > 0,$$

is equivalent to testing that

$$H_0 : \beta \in S \times V, \tau^2 > 0,$$
$$H_1 : \beta \in T \times V, \tau^2 > 0.$$

Therefore, let a be the dimension of $S \times V$, b be the dimension of $T \times V$. This model is an independent measures model, and the usual F statistic is given by

$$M_1^* = \frac{\|P_{T\times V|S\times V}Y\|^2}{b-a}, M_3^* = \frac{\|P_{(T\times V)\perp}Y\|^2}{np-b}, F_1^* = \frac{M_1^*}{M_3^*}.$$

Similarly the problem of testing that

$$H_0 : \beta_1 \in T, \beta_2 \in W, \tau^2 > 0,$$
$$\beta_1 \in T, \beta_2 \in V, \tau^2 > 0,$$

can be written as

$$H_0 : \beta \in T \times W, \tau^2 > 0,$$
$$H_1 : \beta \in T \times V, \tau^2 > 0.$$

Let c be the dimension of $T \times W$,

$$M_1^* = \frac{\|P_{T \times W} Y\|^2}{b - c}, \; F_2^* = \frac{M_2^*}{M_3^*}.$$

Then F_2^* is the F statistic for this problem.

THEOREM 4. a. $M_1 = M_1^*$, $M_2 = M_2^*$.
b. $(n - t)M_3 + (n(p - 1) - v)M_4 = (np - t - v)M_3^*$.

PROOF. The dimension, a, of $S \times V$ is $s + v$ and the dimension, b, of $T \times V$ is $t + v$. Therefore, $b - a = t - s$.

$$\|P_{T \times V|S \times V} Y\|^2 = \|P_{T|Sx0} Y\|^2 = \|P_{T|S} Y_1\|^2.$$

Therefore, $M_1 = M_1^*$. Similarly $M_2 = M_2^*$.

$$(np - t - v)M_3^* = \|P_{(T \times V)\perp} Y\|^2 = \|P_{(T\perp) \times (V\perp)} Y\|^2 = \|P_{T\perp} Y_1\|^2 + \|P_{V\perp} Y_2\|^2. \quad \square$$

Suppose now that a person is using a repeated measures model. If he has the formulae (or a computer that does the calculations) for the mean squares for the corresponding design when the measurements are assumed independent, he need only compute one more quantity, M_3. The mean square for each of the treatment effects is the same for the repeated measures design as for the independent measures design. However, for the repeated measures design there are two different variance estimators depending on whether the problem is of type A or type B. He therefore computes M_3. He has M_3^* from the independent measures model. Therefore, he can compute M_4 from Theorem 4B. (If (SS_3, df_3), (SS_4, df_4), (SS_3^*, df_3^*) are the respective sum of squares and degrees of freedom for M_3, M_4, M_3^*, then $SS_3 + SS_4 = SS_3^*$, $df_3 + df_4 = df_3^*$). To test hypotheses of type A, he puts the mean square for the effect over M_3. To test hypotheses of type B, he puts the mean square for the effect over M_4. To test hypotheses of type C, he puts M_3 over M_4. Note also that the degrees of freedom for the effects are unchanged in the transition from the independent measures model to the repeated measure model, while the degrees of freedom for variance in the independent measures model is the sum of the two degrees of freedom for variance in the repeated measures models.

The formulae for M_1^*, M_2^* and M_3^* are usually given in terms of the original random vectors X_k. We now give a formula for M_3 in terms of the original X_k, so that it will not be necessary to transform to the Y_k to compute any of the statistics considered in this paper. Let $X_k = (x_{1k}, \cdots, x_{pk})'$, and $\bar{x}_k = \Sigma_i x_{ik}/p$, the average of the observations on the kth individual. Let $\bar{X}' = (\bar{x}_1, \cdots \bar{x}_n)$. Then

(6.1) $$Y_{1k} = e'X_k/p^{\frac{1}{2}} = p^{\frac{1}{2}}\bar{X}_k,\ Y_1 = p^{\frac{1}{2}}\bar{X}$$

(6.2) $$M_3 = \frac{\|P_{V\perp}Y_1\|^2}{n-v} = \frac{p\|P_{V\perp}\bar{X}\|^2}{n-v},$$

(6.3) $$F_1 = \frac{\|P_{V|W}\bar{X}\|^2(n-v)}{\|P_{V\perp}\bar{X}\|^2(v-w)}.$$

(Note that F_1 can be computed by replacing each individual's vector of observations with its average value and treating the averages as independent measures.)

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