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ASYMPTOTIC NORMALITY OF PERMUTATION STATISTICS DERIVED FROM WEIGHTED SUMS OF BIVARIATE FUNCTIONS

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Statistics of the form $H_n = \Sigma d_{ijn} h_n(X_i, X_j)$ are considered, where $X_1, X_2, \cdots,$ are independent and identically distributed random variables, the diagonal terms, d_{iin}, are equal to zero, and $h_n(x, y)$ is a symmetric real valued function. The asymptotic normality of such statistics is proven and the result then combined with work of Jogdeo on statistics that are weighted sums of bivariate functions of ranks to find sufficient conditions for asymptotic normality of permutation statistics derived from H_n.

1. Introduction. Suppose that $X_1, X_2, \cdots,$ are independent and identically distributed random variables. For each $n = 1, 2, \cdots,$ let $h_n(x, y)$ denote a symmetric real valued function such that $Eh_n(X_1, X_2)^2$ is finite, and let $\mathbf{D}_n$ denote an $n \times n$ symmetric nonzero matrix with elements $\{d_{ijn}\}$ such that $d_{iin} = 0$ for all $i = 1, \cdots, n$ and all n. Statistics of the form

$$H_n = \Sigma_{i \neq j} d_{ijn} h_n(X_i, X_j) \tag{1.1}$$

are considered. Such statistics have been widely studied with various restrictions on $\mathbf{D}_n$ or h_n. If $h_n(x, y) = f_n(x) f_n(y)$, then H_n is a quadratic form and results on limiting distributions have been reported by Whittle (1964), deWet and Venter (1973) and Sen, A. (1976), among others. If $d_{ijn} = 1$, then H_n is a U-statistic (Hoeffding, 1948). If d_{ijn} is restricted to 0 or 1, recent results have been reported by Brown and Kildea (1978).

The primary purpose of this work is to find sufficient conditions for the asymptotic normality as $n \to \infty$ of permutation statistics derived from (1.1). To accomplish this, the limiting distribution of H_n is studied in more general situations than mentioned above. The limiting distribution of H_n in this more general setting is of interest in its own right. For example, in biometry, $h_n(x, y) = |x - y|$ has been suggested by Mantel (1967) to define H_n as a raw index of disease contagion. Royaltey, Astrachan and Sokal (1975) also use $h_n = |x - y|$ to define a measure of spatial autocorrelation in geography. The existing literature does not yield asymptotic normality for H_n derived from $h_n = |x - y|$ except for special d_{ij} values such as $d_{ij} = 1$, $j = i + 1$, which gives H_n equal to a sum of two-dependent random variables. In addition, practitioners are well aware of the lack of general results (Mantel, 1967), and the limiting normality for more general h_n and $\mathbf{D}_n$ should encourage the definition of other statistics more appropriate in their fields.

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The use of permutation statistics derived from (1.1) has been suggested by Mantel and Valand (1970) in biometry, Cliff and Ord (1973) in geography, and by Hubert and Schultz (1977) in clustering studies. The major work on limiting distributions for such permutation statistics was done by Jogdeo (1968) but limiting normality was not shown in generality. Some additional work has been done by Abe (1969) in the graph theory setting (restricting d_{ijn} and h_n to values 0 and 1).

In Section 2, the asymptotic normality of H_n is proven (Theorem 2.1). In Section 3, the results of Theorem 2.1 are used to extend some of Jogdeo's work (Theorem 3.1) and to develop sufficient conditions for the asymptotic normality of permutation statistics derived from H_n (Theorem 3.2).

2. Asymptotic normality of H_n. Define h_n and $\mathbf{D}_n$ as in Section 1 and recall that $Eh_n(X_1, X_2)^2 < \infty$ is assumed. The following notation will be needed:

$$\begin{aligned} Eh_n(X_i, X_j) &= \theta_n, \qquad i \neq j; \\ g_n(X_i, X_j) &= h_n(X_i, X_j) - \theta_n; \\ G_n &= \Sigma_{i \neq j} d_{ijn} g_n(X_i, X_j); \\ g_n^*(X_i) &= E[\, g_n(X_i, X_j) | X_i], \qquad i \neq j; \\ d_{i \cdot n} &= \Sigma_j d_{ijn}; \\ G_n^* &= 2\Sigma_i d_{i \cdot n} g_n^*(X_i). \end{aligned}$$

Finally, let

$$\begin{aligned} \sigma_n^2 &= \operatorname{Var} g_n^*(X_i) = \operatorname{Cov}[\, g_n(X_1, X_2), g_n(X_1, X_3)], \\ \sigma_{1n}^2 &= \operatorname{Var} g_n(X_1, X_2) = \operatorname{Var} h_n(X_1, X_2). \end{aligned}$$

Throughout the paper, assume that $\Sigma_i (d_{i \cdot n})^2 > 0$ and $\sigma_n^2 > 0$ so that G_n^* is never identically zero.

With the notation above, $E(G_n | X_i) = 2 d_{i \cdot n} g_n^*(X_i)$ and G_n^* is the projection of G_n onto the linear space composed of $L_n = \Sigma f_i(X_i)$ where $Ef_i(X_i)^2 < \infty$.

LEMMA 2.1. *If*
(AE) $[\sigma_{1n}^2 \Sigma_{i \neq j} d_{ijn}^2] / [\sigma_n^2 \Sigma_k d_{k \cdot n}^2] \to 0$
then $G_n (\operatorname{Var} G_n)^{-\frac{1}{2}} - G_n^* (\operatorname{Var} G_n^*)^{-\frac{1}{2}} \to 0$ *(in probability).*

PROOF. Dropping the subscript n, direct computation yields

$$0 \leqslant \operatorname{Var} G - \operatorname{Var} G^* = 2\Sigma_{i \neq j} d_{ij}^2 (\sigma_1^2 - 2\sigma^2). \tag{2.1}$$

Bound (2.1) implies that

$$E\left[\, G (\operatorname{Var} G^*)^{-\frac{1}{2}} - G^* (\operatorname{Var} G^*)^{-\frac{1}{2}} \right]^2 \leqslant 2\sigma_1^2 \Sigma_{i \neq j} d_{ij}^2 / \left[4\sigma^2 \Sigma_k d_{k.}^2 \right]$$

which tends to zero by (AE). Furthermore, bound (2.1) and (AE) imply that $\operatorname{Var} G / \operatorname{Var} G^*$ tends to 1 which completes the proof.

Next, conditions yielding the asymptotic normality of the projection, G_n^*, are needed. Define $Y_n = g_n^*(X_1)$, $A_{kn} = [\Sigma_i d_{i \cdot n}^2]^{\frac{1}{2}} / |d_{k \cdot n}|$, and $A_n = \min_{1 \leqslant k \leqslant n} A_{kn}$.

LEMMA 2.2.
(N1) *If for each* $\varepsilon > 0$, $\sigma_n^{-2}\int_{\{|y_n| \geq \sigma_n \varepsilon A_n\}} y_n^2\, dP \to 0$,
then $G_n^*(\text{Var } G_n^*)^{-\frac{1}{2}} \to Z$ *(in distribution), where Z is normal with mean zero and variance* 1.

PROOF. Let $Y_{kn} = g_n^*(X_k)$. Then the normal convergence criterion (Loeve, page 295) requires that for each $\varepsilon > 0$,

$$\left[\sigma_n^2 \Sigma d_{j\cdot n}^2\right]^{-1} \Sigma\left(d_{k\cdot n}^2\right) \int_{\{|y_{kn}| \geq \sigma_n \varepsilon A_{kn}\}} y_{kn}^2\, dP \tag{2.2}$$

tend to zero. But $A_n = \min A_{kn}$ and Y_{kn} equal in distribution to Y_n imply that (2.2) is less than or equal to

$$\sigma_n^{-2}\int_{\{|y_n| \geq \sigma_n \varepsilon A_n\}} y_n^2\, dP \tag{2.3}$$

which tends to zero by (N1).

In applications it is useful to have other conditions which imply (N1). Define (i)–(iii) below:

(i) $\liminf_{n\to\infty} \sigma_n^2 > 0$, and $\max_{1 \leq i \leq n} d_{i\cdot n}^2 / \Sigma d_{k\cdot n}^2 \to 0$.
(ii) The sequence $\{Y_n^2\}$ is uniformly integrable.
(iii) For some $\delta > 0$, $E|h_n(X_1, X_2)|^{2+\delta}$ is uniformly bounded in n.

Then condition (N1) of Lemma 2.2 is implied by either
(N2) (i) and (ii), or
(N3) (i) and (iii).

The main theorem follows immediately from the lemmas above.

THEOREM 2.1. *Conditions* (AE) *and* (N1) *imply that*

$$(H_n - EH_n)(\text{Var } H_n)^{-\frac{1}{2}} \to Z \text{ (in distribution)}$$

where Z is normal with mean 0 *and variance* 1.

Note that Var G_n^* may replace Var H_n in Theorem 2.1. The theorem above does not prove asymptotic normality for all statistics of form (1.1) which are asymptotically normal. Rather, the theorem yields asymptotic normality for those H_n whose projections are asymptotically equivalent to H_n and also asymptotically normal.

Special case 1. Suppose that $h_n = h$ for all n. Then $\sigma_n^2 = \sigma^2 > 0$ and conditions (AE) and (N1) become

$(\text{AE})_0$ $\Sigma_{i \neq j} d_{ijn}^2 / \Sigma_k d_{k\cdot n}^2 \to 0$,
$(\text{N1})_0$ $\max_{1 \leq i \leq n} d_{i\cdot n}^2 / \Sigma_k d_{k\cdot n}^2 \to 0$,

which are easily checked. If $h(x, y) = xy$, then $EX_1^2 < \infty$ and $\sigma^2 = [EX_1]^2\text{Var}(X_1) > 0$ are required along $(\text{AE})_o$ and $(\text{N1})_0$ on the $\{d_{ijn}\}$ to conclude that H_n is asymptotically normal from Theorem 2.1. These conditions are not comparable with Whittle (1967) who requires 4 + moments on the X_i's along with square summability of $\{d_j\}$ where $d_{ij} = d_{i-j}$. Also, $EX_1 \neq 0$ is needed to obtain $\sigma^2 > 0$ for this choice of h. That is, if $EX_1 = 0$, then $G_n^* = 0$ and Theorem 2.1 does not apply. Such degeneracy will occur whenever $h(x, y) = f(x)f(y)$ and $Ef(X_1) = 0$.

Special case 2. Suppose that $h_n = h$ and $d_{ijn} = 0$ or 1 for all i, j, n. If the number of 1's in each row and column is at least K_{*n} and at most K_n^*, respectively, then $(\text{AE})_0$ and $(\text{N1})_0$ are implied by $K_n^* K_{*n}^{-2} \to 0$. If $K_n^* = K_{*n} = K_n$, then this is equivalent to $K_n \to \infty$. Brown and Kildea (1978) obtained asymptotic normality using moments in this special case without the restriction of $K_n \to \infty$. Thus, Theorem 2.1 essentially gives a subclass of (1.1) with $d_{ijn} = 0$ or 1 which are amenable to projection methods or asymptotically equivalent to their projections. Also, if $d_{ij} = 1, j = i + 1$, then H_n is a sum of two-dependent random variables and is asymptotically normal. However, the (AE) condition is not satisfied since $\mathbf{D}_n$ contains too many zeros.

Alternate centering of H_n. An alternate centering for H_n not involving θ_n is obtainable in most situations by using the deviations $\delta_{ijn} = d_{ijn} - \bar{d}_n$, with $\bar{d}_n = 2\Sigma_{i<j} d_{ijn}/n(n-1)$ in place of the original d_{ijn}. However, if the row sums are all equal, then $\delta_{i\cdot n} = 0$ for all i which gives $G_n^* = 0$ and this centering will not work.

3. Permutation statistics. In this section the results of Jogdeo (1968) are combined with Theorem 2.1 to complete some of Jogdeo's work (Theorem 3.1) and to conclude the asymptotic normality of a large class of permutation statistics derived from (1.1).

Let $V_1, V_2, \cdots,$ be independent and identically distributed random variables with a continuous distribution function, and let R_{in} denote the rank of V_i in the sample $V_1, \cdots, V_n$. Suppose that X-values, $x_1, \cdots, x_n$, are observed. The permutation statistic derived from (1.1) can be expressed as

$$S_n = \Sigma_{i \neq j} d_{ijn} h_n(x_{R_{in}}, x_{R_{jn}}), \tag{3.1}$$

or

$$\Sigma_{i \neq j} d_{R_{in} R_{jn} n} h_n(x_i, x_j). \tag{3.2}$$

In his 1968 paper, Jogdeo considers statistics of the form

$$T_n = \Sigma_{i \neq j} d_{ijn} a_{R_{in} R_{jn}}, \tag{3.3}$$

where for each n, $\mathbf{D}_n$ and $\mathbf{A}_n$ are nonzero symmetric $n \times n$ matrices with elements $\{d_{ijn}\}$ and $\{a_{ijn}\}$, respectively. Note that $T_n = S_n$ with the identification of a_{ijn} and $h_n(x_i, x_j)$. Jogdeo showed that under certain conditions, there exist symmetric functions $\{a_n(u, v)\}$ on $[0, 1]^2$ such that T_n is asymptotically equivalent to

$$W_n = \Sigma_{i \neq j} d_{ijn} a_n(U_i, U_j), \tag{3.4}$$

where $U_1, \cdots, U_n$ are independent uniform on [0, 1] and $U_i = U_n(R_{in})$ with $U_n(k)$ the kth order statistic in $(U_1, \cdots, U_n)$. However, asymptotic normality of W_n is not proven in general. Since (3.4) is exactly the type of statistic shown asymptotically normal in Theorem 2.1, Jogdeo's conditions for asymptotic equivalence of (3.3) and (3.4) can be combined with the asymptotic normality conditions of Theorem 2.1 to deduce the asymptotic normality of (3.1), (3.2) and (3.3). Since the diagonal terms are not involved in any of the sums, $a_{iin} = 0$ and $h_n(x, x) = 0$ are assumed for the remainder of this section.

The particular construction of functions $a_n(u, v)$ to be used here is

$$a_n(u, v) = \Sigma_{i,j} a_{ijn} I\left[\frac{i-1}{n} < u \leqslant \frac{i}{n}, \frac{j-1}{n} < v \leqslant \frac{j}{n}\right], \tag{3.5}$$

where $I[\cdot]$ is the set indicator function. Let $\bar{a}_{..n} = n^{-2}\Sigma_{i,j} a_{ijn}$ and $\bar{a}_{.jn} = n^{-1}\Sigma_i a_{ijn}$. To state the Jogdeo conditions for the asymptotic equivalence of (3.3) and (3.4), several definitions are needed.

(3.6) A set of n^2 numbers $\{a_{ij}\}$ is Δ-*monotone* if

$$a_{i+1,j+1} - a_{i,j+1} - a_{i+1,j} + a_{i,j} \geqslant 0 \text{ for all } i, j \quad \text{or}$$
$$\leqslant 0 \text{ for all } i, j.$$

(3.7) A set of n^2 numbers $\{a_{ij}\}$ is *piecewise* Δ-*monotone* if $a_{ij} = \Sigma_{m=1}^{k} a_{ij}^{(m)}$ where each set $\{a_{ij}^{(m)}\}$ is Δ-monotone and k does not depend on n.

Listed below are conditions needed to derive the asymptotic normality of T_n (Theorem 3.1). The first set of conditions are on the coefficients, $\{d_{ijn}\}$.

(D1) For all n, $\Sigma_k d_{k\cdot n}^2 > 0$.

(D2) As $n \to \infty$, $\max_i d_{i\cdot n}^2 / \Sigma_k d_{k\cdot n}^2 \to 0$.

(D3) As $n \to \infty$, $\Sigma_{i \neq j} d_{ijn}^2 / \Sigma_k d_{k\cdot n}^2 \to 0$.

(D4) Both $\Sigma_{i \neq j} d_{ijn}^2$ and $\Sigma_k d_{k\cdot n}^2$ are uniformly bounded in n. Next, the conditions needed on $\{a_{ijn}\}$ where $a_n(u, v)$ is defined by (3.5) are given.

(A1) There exists $\delta > 0$ such that $n^{-2}\Sigma_{i \neq j}|a_{ijn}|^{2+\delta}$ is uniformly bounded in n.

(A2) $\liminf_{n\to\infty} n^{-1}\Sigma_j(\bar{a}_{.jn} - \bar{a}_{..n})^2 > 0$.

(A3) The $\{a_{ijn}\}$ are piecewise Δ-monotone.

(A4) As $n \to \infty$, $n^{-1}\max_{i,j}(a_{ijn} - \bar{a}_{..n})^4 \to 0$.

Conditions (D1)–(D3) appeared in Section 2, and (D4) essentially forces the preliminary normalization of the constants. If the original d_{ijn} do not satisfy (D4), then use $d_{ijn}^* = d_{ijn}/\Sigma|d_{krn}|$. Also, note that with $a_n(u, v)$ defined by (3.5), $n^{-2}\Sigma|a_{ijn}|^{2+\delta} = E|a_n(U_1, U_2)|^{2+\delta}$ and $\sigma_n^2 = \mathrm{Cov}[a_n(U_1, U_2), a_n(U_1, U_3)] = n^{-1}\Sigma_j(\bar{a}_{.jn} - \bar{a}_{..n})^2$.

LEMMA 3.1. *If* (D1), (D2), (D3), (A1), (A2) *are satisfied, then*

$$(W_n - EW_n)(\mathrm{Var}\ W_n)^{-\frac{1}{2}} \to Z \text{ (in distribution)}.$$

PROOF. The result follows directly from Theorem 2.1. Condition (N3) is implied by (A1), (A2) and (D3). Condition (AE) follows from (D2).

LEMMA 3.2. *If* (D4), (A3), (A4) *are satisfied, then*

$$W_n - T_n + B_n \to 0 \text{ (in probability)},$$

where

$$B_n = \Sigma d_{ijn} n^{-2} \Sigma_{k,r}[a_n(U_k, U_r) - a_{krn}].$$

PROOF. See Jogdeo, 1968, Theorem 4.1.

THEOREM 3.1. *If conditions* (D1)–(D4) *and* (A1)–(A4) *are satisfied then* $(T_n - ET_n)(\text{Var } T_n)^{-\frac{1}{2}} \to Z$ *(in distribution).*

PROOF. Note that (A1) and (D4) imply that $B_n \to 0$ in probability. Thus, $(T_n - EW_n)(\text{Var } W_n)^{-\frac{1}{2}} \to Z$ (in distribution). The proof is completed by noting that $1 \leq \text{Var } W_n(\text{Var } T_n)^{-1} \leq 1 + O_p(n^{-1})$ and $EW_n - ET_n = O_p(n^{-1})$ so that ET_n and Var T_n can replace EW_n and Var W_n.

Theorem 3.1 can be used directly to prove asymptotic normality of S_n with $a_{ijn} = h_n(x_i, x_j)$. However, several of the conditions become quite tractable when $h_n = h$. Note that $a_n(u, v)$ is not necessarily $h_n(u, v)$.

THEOREM 3.2. *Suppose that* $h_n = h$ *for all* n, (D1)–(D4) *hold, and that for almost every* **x** *sequence,* $\{h(x_i, x_j),\ i, j = 1, \cdots, n\}$ *is piecewise* Δ*-monotone. If* $n^{-1} \max_{1 \leq (i,j) \leq n} h(X_i, X_j)^4 \to 0$ a.s., *and*

$(A1)_0$ *there exists* $\alpha > \frac{1}{2}$ *such that* $E|h(X_1, X_2)|^{2+\alpha} < \infty$,

$(A2)_0$ *there exists* $\beta > \frac{2}{3}$ *such that* $E|h(X_1, X_2)h(X_1, X_3)|^{1+\beta} < \infty$,

then for a.e. **x** *sequence,* $(S_n - ES_n)(\text{Var } S_n)^{-\frac{1}{2}} \to Z$ *(in distribution).*

PROOF. In light of conditions for Theorem 3.1, it suffices to show that $(A1)_0$ and $(A2)_0$ imply (A1) and (A2) for a.e. sequence. For (A1), using $a_{ijn} = h(X_i, X_j)$, choose δ such that $\alpha - \delta > \frac{1}{2}$. Then

$$n^{-2}\Sigma|a_{ijn}|^{2+\delta} = n^{-2}\Sigma|h(X_i, X_j)|^{2+\delta}.$$

But, by Theorem 3 of Sen (1960) giving the a.s. convergence properties of U-statistics, the left hand side above converges a.s. to $E|h(X_1, X_2)|^{2+\delta}$ if there exist α, δ with $\alpha - \delta > \frac{1}{2}$ such that $E|h(X_1, X_2)|^{2+\delta+(\alpha-\delta)} < \infty$. This holds by $(A1)_0$ and the choice of δ.

For (A2), $n^{-1}\Sigma_j(\bar{a}_{.jn} - \bar{a}_{..n})^2$ is equal to

$$n^{-3}\Sigma_j(\Sigma_i h(X_i, X_j))^2 - (n^{-2}\Sigma_{k,r} h(X_k, X_r))^2. \tag{3.8}$$

The right term tends a.s. to $(Eh(X_1, X_2))^2$ again using Theorem 3 of Sen (1960). The left term in (3.8) is equal to

$$n^{-3}\Sigma_j\Sigma_i\Sigma_k h(X_i, X_j)h(X_k, X_j). \tag{3.9}$$

Define $f(x, y, z) = h(x, y)h(x, z) + h(x, y)h(y, z) + h(x, z)h(y, z)$. Then f is symmetric in the three arguments and Theorem 3 of Sen (1960) implies that

$$n^{-3}\Sigma f(X_i, X_j, X_k) \to Ef(X_1, X_2, X_3) \text{ a.s.} \tag{3.10}$$

if $Ef(X_1, X_2, X_3)^{1+\varepsilon} < \infty$ for some $\varepsilon > \frac{2}{3}$. This is implied by $(A2)_0$. Since (3.9) is merely $\frac{1}{3}$ of (3.10), then (3.9) tends a.s. to $Eh(X_1, X_2)h(X_1, X_3)$. Thus, (3.8) tends a.s. to $\sigma^2 = \text{Cov}(h(X_1, X_2), h(X_1, X_3))$ which is assumed positive throughout the paper.

The piecewise Δ-monotone property is essentially a property of the function h, and many functions satisfy the even stronger property of Δ-monotonicity. For example, if $h(x, y) = xy$, then $h(x_i, x_j)$ is Δ-monotone. If $h(x, y) = f(|x - y|) \geqslant 0$ with $f(0) = 0$, then the Δ-monotonicity of $\{h(x_i, x_j)\}$ is implied by convexity of f.

Often the function h is converted to a dimensionless quantity by considering $h(x_i, x_j)/B_n(x_1, \cdots, x_n)$. If B_n tends a.s. to some B_0, then Theorem 3.2 can be used on the original h. Otherwise, as long as B_n is symmetric, Theorem 3.1 can be used directly on $\{h(x_i, x_j)B_n^{-1}\}$.

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