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BOUNDS ON EXPECTATIONS OF LINEAR SYSTEMATIC STATISTICS BASED ON DEPENDENT SAMPLES

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David summarized distribution-free bounds for $E(X_{k:n})$, the expected value of the kth order statistic, and for the expected value of certain linear combinations of the order statistics, when sampling n i.i.d. observations from a population with expectation μ and variance σ^2. Here the problem of finding distribution-free bounds for the expectations of linear systematic statistics is considered in the case in which the observations X_i, $i = 1, 2, \cdots, n$, satisfy only $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$. The observations may be dependent and have different distributions. Bounds are obtained for the expectations of the kth order statistic, the trimmed mean, the range, and quasi-ranges, the spacings and Downton's estimator of σ. The sharpness of these bounds is considered. In contrast with the i.i.d. case all the bounds obtained are shown to be sharp.

1. Introduction. Let $X_1, X_2, \cdots, X_n$ be random variables with the corresponding order statistics denoted by $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$. Bounds on the expected values of the order statistics when the X_i are i.i.d., with expectation μ and variance σ^2, are well known. A convenient reference for these results is David (1970, pages 46ff.). The earliest result due to Gumbel (1954) and Hartley and David (1954) concerns the maximum, i.e.,

$$E(X_{n:n}) \leqslant \mu + \sigma(n-1)(2n-1)^{-\frac{1}{2}}. \tag{1}$$

Here we obtain bounds for the expectations of order statistics and of linear systematic statistics in the case of possibly dependent random variables with possibly different marginal distributions. Results such as those discussed in this paper can be expected to be useful in determining conservative significance levels for statistics which are of the form $\max(T_1, T_2, \cdots, T_n)$ or range $(T_1, T_2, \cdots, T_n)$, where the T_i's have common mean and variance but are possibly dependent.

2. The basic inequality. Denote n random variables and their order statistics as in the first section, with $E(X_i) = \mu_i$, $\sigma^2(X_i) = \sigma_i^2$, where the X_i are not necessarily independent. Define $\mu_{i:n} = E(X_{i:n})$ and $\bar{\mu} = n^{-1}\Sigma\,\mu_{i:n} = n^{-1}\Sigma\,\mu_i$. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be any real numbers with $\bar{\lambda} = n^{-1}\Sigma\lambda_i$. As

$$\begin{aligned}\Sigma(\mu_{i:n} - \bar{\mu})^2 &= \Sigma\,\mu_{i:n}^2 - n\bar{\mu}^2 \leqslant \Sigma E(X_{i:n}^2) - n\bar{\mu}^2\\ &= \Sigma E(X_i^2) - n\bar{\mu}^2 = \Sigma\sigma_i^2 + \Sigma\,\mu_i^2 - n\bar{\mu}^2\\ &= \Sigma\{\sigma_i^2 + (\mu_i - \bar{\mu})^2\},\end{aligned}$$

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the Cauchy-Schwarz inequality yields

$$(2)\qquad \begin{aligned} |\Sigma\lambda_i(\mu_{i:n} - \bar{\mu})| &= |\Sigma(\lambda_i - \bar{\lambda})(\mu_{i:n} - \bar{\mu})| \\ &\leqslant \left[\Sigma(\lambda_i - \bar{\lambda})^2\Sigma(\mu_{i:n} - \bar{\mu})^2\right]^{\frac{1}{2}} \\ &\leqslant \left[\Sigma(\lambda_i - \bar{\lambda})^2\Sigma\{\sigma_i^2 + (\mu_i - \mu)^2\}\right]^{\frac{1}{2}}. \end{aligned}$$

It is interesting to note that bounds for $\Sigma\lambda_i\mu_{i:n}$ can often be obtained by choosing numbers $\lambda_1', \cdots, \lambda_n'$ and $\lambda_1'', \cdots, \lambda_n''$ such that

$$\Sigma\lambda_i'\mu_{i:n} \leqslant \Sigma\lambda_i\mu_{i:n} \leqslant \Sigma\lambda_i''\mu_{i:n}$$

and using (2) to obtain a lower bound for $\Sigma\lambda_i'\mu_{i:n}$ and an upper bound for $\Sigma\lambda_i''\mu_{i:n}$. This device will be used in examples (a) and (b) in Section 3. It is easy to show by an argument involving isotonic regression that the choices of λ_i' and λ_i'' made in these examples are the best possible, and that in example (c) no improvement of (8) is possible by using this device.

3. Applications. Throughout this section we assume the $X_1, X_2, \cdots, X_n$ to be jointly distributed with common expectation μ and variance σ^2. In many instances the X_i will be identically distributed but, since this extra assumption does not lead to better bounds, we allow the marginal distributions to be possibly different.

(a) *Bounds on the expectation of the* kth *order statistic.* Application of (2) to the left and right-hand sides in

$$(3)\qquad \Sigma_{i=1}^k\mu_{i:n}/k \leqslant \mu_{k:n} \leqslant \Sigma_{i=k}^n\mu_{i:n}/(n-k+1)$$

yields

$$(4)\qquad \mu - \sigma\{(n-k)/k\}^{\frac{1}{2}} \leqslant \mu_{k:n} \leqslant \mu + \sigma\{(k-1)/(n-k+1)\}^{\frac{1}{2}}.$$

Note that the bound in (1) becomes $\mu_{n:n} \leqslant \mu + \sigma(n-1)^{\frac{1}{2}}$, when independence cannot be assumed. Similarly, we have for the expected value of the trimmed mean:

$$(5)\qquad \begin{aligned} \mu - \sigma\{k_2/(n-k_2)\}^{\frac{1}{2}} &\leqslant \Sigma_{i=1}^{n-k_2}\mu_{i:n}/(n-k_2) \\ &\leqslant \Sigma_{i=k_1+1}^{n-k_2}\mu_{i:n}/(n-k_1-k_2) \\ &\leqslant \Sigma_{i=k_1+1}^{n}\mu_{i:n}/(n-k_1) \\ &\leqslant \mu + \sigma\{k_1/(n-k_1)\}^{\frac{1}{2}}. \end{aligned}$$

(b) *Bounds on the expected difference of two order statistics.* Application of (2) to the right-hand side in

$$\mu_{k_2:n} - \mu_{k_1:n} \leqslant \Sigma_{i=k_2}^n\mu_{i:n}/(n-k_2+1) - \Sigma_{i=1}^{k_1}\mu_{i:n}/k_1$$

yields for $1 \leqslant k_1 < k_2 \leqslant n$,

$$(6)\qquad \mu_{k_2:n} - \mu_{k_1:n} \leqslant \sigma\{n(n-k_2+1+k_1)/((n-k_2+1)k_1)\}^{\frac{1}{2}}.$$

Special cases of this inequality are for:

(i) The kth quasi-range:

(6a) $$\mu_{n-k+1:n} - \mu_{k:n} \leqslant \sigma(2n/k)^{\frac{1}{2}} \qquad \text{for } 2k \leqslant n;$$

(ii) The kth spacing:

(6b) $$\mu_{k+1:n} - \mu_{k:n} \leqslant \sigma n/\{k(n-k)\}^{\frac{1}{2}}.$$

(c) *A bound for the expectation of Downton's unbiased estimate of a normal standard deviation.* Downton (1966) suggested a scalar multiple of Gini's mean difference

(7) $$T_n = \frac{2\pi^{\frac{1}{2}}}{n(n-1)} \Sigma_{i=1}^{n} \left[i - \left(\frac{n+1}{2} \right) \right] X_{i:n}$$

as an unbiased estimate of σ in normal samples. We may obtain an upper bound on $E(T_n)$ even when the X_i's are not independent, not normal and, in fact, not even identically distributed. We only assume $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$. If we apply (2) it follows that

(8) $$E(T_n) \leqslant \sigma(\pi(n+1)/3(n-1))^{\frac{1}{2}}.$$
$$\doteq (1.023)\sigma((n+1)/(n-1))^{\frac{1}{2}}.$$

Hence T_n will in general not seriously overestimate σ, even when applied to situations markedly different from those for which it was designed.

4. Sharpness. In general, the inequalities described in this paper are sharp. Examples for which the bounds are achieved may be readily constructed. For example, suppose that for some $k \neq 1$, $n - k + 1$ of the elements of a finite population have common value $k - 1$, while the remaining $k - 1$ have common value $-(n - k + 1)$. Let $X_1, X_2, \cdots, X_n$ be the outcomes of n drawings without replacement. In this case $\mu = 0$, $\sigma^2 = (k-1)(n-k+1)$, and $\mu_{k:n} = (k-1)$ achieving the upper bound in (4).

The only exceptions are the two trivial bounds $\mu_{1:n} \leqslant \mu$ and $\mu_{n:n} \geqslant \mu$ included in (4) and (5). Hawkins (1971) obtained results analogous to (4) for samples from finite populations, including a slightly better lower bound (respectively upper) for $\mu_{n:n}$ (respectively, $\mu_{1:n}$) using a classical result of Pearson and Chandra Sekar (1936). Hawkins' bounds may be extended to yield, under the assumptions of Section 3:

(9a) $$\mu_{1:n} \leqslant \mu - \sigma/(n-1)^{\frac{1}{2}}$$

and

(9b) $$\mu_{n:n} \geqslant \mu + \sigma/(n-1)^{\frac{1}{2}}.$$

These bounds are now sharp (i.e., achievable).

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