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ON THE PROPERTIES OF PROPER (M, S) OPTIMAL BLOCK DESIGNS

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Properties of designs which are (M, S) optimal within various classes of proper block designs are studied. The classes of designs considered are not restricted to connected designs. Connectedness is shown to be a property generally possessed by designs which are (M, S) optimal within these more general classes of designs. In addition, we show that the complement of any proper binary (M, S) optimal design is (M, S) optimal within an appropriate class of complementary designs and that the dual of any proper equireplicated (M, S) optimal design is (M, S) optimal within an appropriate class of dual designs.

1. Introduction and summary. Let $\mathfrak{D}$ denote the collection of all proper block designs having v treatments arranged in b blocks of size k such that treatment i is replicated r_i times for $i = 1, \cdots, v$. This paper is an investigation into the properties of designs which are (M, S) optimal in various classes $\mathfrak{D}$. The (M, S) optimality criterion was introduced by Eccleston and Hedayat (1974) as a generalization of the S-optimality criterion suggested by Shah (1960). This criterion selects from the subclass of designs in $\mathfrak{D}$ whose information matrices have maximal trace those designs for which the trace of the square of the information matrix is minimal.

A property which is generally desirable in any block design is that of connectedness. Such a property is also desirable in a design which is optimal within $\mathfrak{D}$. The notion of connectedness has not in general been related to the (M, S) optimality criteria. Eccleston and Hedayat (1974) provide some results concerning (M, S) optimality and various types of connectedness, but their results are only applicable within classes of proper connected designs. In Section 4, we show that connectedness is a property generally associated with designs which are determined to be (M, S) optimal within $\mathfrak{D}$. In particular, we show that when connected designs exist in $\mathfrak{D}$, there will exist connected (M, S) optimal designs. A commonly occurring sufficient condition is also given which guarantees that (M, S) optimal designs in $\mathfrak{D}$ be connected. Section 5 deals with the invariance of (M, S) optimality under complementation, i.e., the complement of an (M, S) optimal proper binary design in $\mathfrak{D}$ is shown to be (M, S) optimal in an appropriate class of complementary designs. Section 6 is used to study the relationship between (M, S) optimality and duality for designs in $\mathfrak{D}$.

2. Preliminaries. Throughout the sequel we will let $\mathfrak{D}$ denote the class of all proper block designs having v treatments arranged in b blocks each containing k

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experimental units such that treatment i is replicated r_i times for $i = 1, \cdots, v$. Each design in $\mathfrak{D}$ can be identified with a $v \times b$ incidence matrix N whose entries n_{ij} are nonnegative integers indicating the number of times treatment i occurs in block j. Thus $\mathfrak{D}$ can be thought of as a class of incidence matrices whose ith row sum is r_i and whose column sums are all k. A design is said to be equireplicated if $r_i = r$ for $i = 1, \cdots, v$.

The statistical analysis of interest in this paper is the intrablock analysis with the usual fixed effects two-way classification model. The matrix of coefficients of the reduced normal equations for obtaining intrablock estimates of the treatment effects for any design $N \in \mathfrak{D}$ is given by

$$C = R - k^{-1}NN' \tag{2.1}$$

where $R = \operatorname{diag}(r_1, \cdots, r_v)$. The matrix C defined by (2.1) is called the *information matrix* or *C-matrix* of the design.

If N' denotes the transpose of N, then NN' is called the association matrix and $N'N$ the block characteristic matrix of the design. We denote the entries of the $v \times v$ association matrix by λ_{ij} and the entries of the $b \times b$ block characteristic matrix by μ_{ij}. When the entries of N assume only the values zero or one, the design is said to be binary, otherwise the design is called nonbinary. It is straightforward to verify that for fixed i, the entries of the association matrix of any binary design $N \in \mathfrak{D}$ satisfy the relationship

$$\Sigma_{j \neq i}\lambda_{ij} = r_i(k - 1). \tag{2.2}$$

A property which is usually desirable in any block type experiment is the ability to estimate all possible treatment differences unbiasedly. Any block design having this property is said to be connected.

REMARK 2.3. It will be convenient later to have access to two characterizations of connectedness which were given by Eccleston and Hedayat (1974). For reference purposes, these two characterizations are stated below.

(i) A design N is disconnected if and only if after a suitable permutation of rows and columns, N can be written in the form $\operatorname{diag}(N_1, \cdots, N_a)$, $1 < a \leqslant v$ where each N_i is the incidence matrix of a connected subset of treatments.

(ii) A design N is connected if and only if after a suitable permutation of rows of N, $NN' = (\lambda_{ij})$ has the property that for each $j \geqslant 2$, there exists an i, $1 \leqslant i < j$, such that $\lambda_{ij} \geqslant 1$.

3. The (M, S) optimality criterion. A design $N \in \mathfrak{D}$ is said to be optimal within $\mathfrak{D}$ provided it is determined to be "best" by some well-defined optimality criterion. The (M, S) optimality criterion was given by Eccleston and Hedayat (1974) and is a two-stage optimization process. Let $\mathfrak{M}$ denote the subclass of designs $N \in \mathfrak{D}$ whose C-matrices have maximal trace (denoted by $\operatorname{tr} C$) among designs in $\mathfrak{D}$. A design $N \in \mathfrak{D}$ is said to be (M, S) optimal if $N \in \mathfrak{M}$ and if the square of its C-matrix has minimum trace among designs in $\mathfrak{M}$.

Observe that for $N \in \mathcal{D}$,

$$\operatorname{tr} C = \Sigma_i r_i - k^{-1}\Sigma_i\Sigma_j n_{ij}^2 \tag{3.1}$$

and that

$$\operatorname{tr} C^2 = \Sigma_i r_i^2 - 2k^{-1}\Sigma_i r_i \lambda_{ii} + k^{-2}\operatorname{tr}(NN')^2. \tag{3.2}$$

If $N \in \mathcal{D}$ is binary, $\lambda_{ii} = r_i$ and (3.2) takes the simpler form

$$\operatorname{tr} C^2 = \left(\Sigma_i r_i^2\right)(1 - 2k^{-1}) + k^{-2}\operatorname{tr}(NN')^2. \tag{3.3}$$

4. Connectedness. The primary purpose of this section is to show that connectedness is a property generally associated with (M, S) optimal designs in $\mathcal{D}$.

LEMMA 4.1. *If $N \in \mathcal{D}$ is a disconnected nonbinary design, then $N \notin \mathcal{M}$.*

PROOF. Suppose $N \in \mathcal{D}$ is a disconnected nonbinary design. Without loss of generality, assume $n_{11} \geqslant 2$. Observe now that there must exist p such that $\lambda_{1p} = 0$; otherwise, N would be connected by Remark 2.3(i). Let u be such that $n_{pu} \neq 0$. Construct a new design $\bar{N}$ having entries $\bar{n}_{11} = n_{11} - 1$, $\bar{n}_{1u} = 1$, $\bar{n}_{p1} = 1$, $\bar{n}_{pu} = n_{pu} - 1$ and $\bar{n}_{ij} = n_{ij}$ for all other i, j. Note that $\bar{N} \in \mathcal{D}$. Also note that if C and $\bar{C}$ denote the information matrices of N and $\bar{N}$, then

$$\operatorname{tr}\bar{C} - \operatorname{tr} C = k^{-1}(2n_{11} + 2n_{pu} - 4).$$

Since $n_{11} \geqslant 2$ and $n_{pu} \geqslant 1$, it follows that $\operatorname{tr} C$ is not maximal in $\mathcal{D}$.

COROLLARY 4.2. *If $k \geqslant v$, then any design in $\mathcal{M}$ is connected.*

PROOF. Let $N \in \mathcal{M}$. If N is nonbinary, then N is connected by Lemma 4.1. If N is binary, then $k = v$ and each treatment occurs once in each block. Connectedness follows.

From Corollary 4.2, we see that when $k \geqslant v$, the search for (M, S) optimal designs in $\mathcal{D}$ can be limited to connected designs.

We observe now that any connected design containing more than $v + b - 1$ experimental units possesses as least one treatment replication which can be removed such that the resulting design is still connected. This observation is of use in the proof of the following theorem and follows directly from the well-known fact that the rank of the design matrix of any connected design is $v + b - 1$.

THEOREM 4.3. *Let $\mathcal{D}$ be such that $bk \geqslant v + b - 1$. If $N \in \mathcal{M}$ is a disconnected design, then there exists a connected design $\bar{N} \in \mathcal{M}$ with $\operatorname{tr} \bar{C}^2 \leqslant \operatorname{tr} C^2$.*

PROOF. Suppose $N \in \mathcal{M}$ is disconnected. By Lemma 4.1 and Remark 2.3(i), we can assume that N is binary and that NN' has the form

$$\operatorname{diag}(N_1N_1', \cdots, N_aN_a'), \qquad 1 < a \leqslant v,$$

where each N_i is the incidence matrix of a connected subset of treatments. We can also assume that the rows of each N_i have been permuted so that the corresponding λ_{ij} have the property mentioned in Remark 2.3(ii). We will first let $a = 2$ and let N_1 and N_2 denote $s \times t$ and $(v - s) \times (b - t)$ matrices.

Since $bk \geqslant v + b - 1$, we must have either $tk > s + t - 1$ or $(b - t)k > (v - s) + (b - t) - 1$. Without loss of generality, assume $tk > s + t - 1$. Using the observation made following Corollary 4.2, we know that there exists a replication of some treatment p, $1 \leqslant p \leqslant s$, occurring in a block u, $1 \leqslant u \leqslant t$, which can be removed from N_1 such that N_1 still has the property described in Remark 2.3(ii). Let w be such that $n_{s+1,w} = 1$ and form a new design $\bar{N}$ having entries $\bar{n}_{pu} = 0$, $\bar{n}_{pw} = 1$, $\bar{n}_{s+1,u} = 1$, $\bar{n}_{s+1,w} = 0$, and $\bar{n}_{ij} = n_{ij}$ for all other i, j. Note that $\bar{N} \in \mathfrak{D}$. Note also that $\overline{NN'} = (\bar{\lambda}_{ij})$ where $\bar{\lambda}_{pl} = \lambda_{pl} - 1$ and $\bar{\lambda}_{s+1,l} = 1$ for all $l \neq s + 1$ having $\bar{n}_{lu} = 1$, $\bar{\lambda}_{pl} = 1$ and $\bar{\lambda}_{s+1,l} = \lambda_{s+1,l} - 1$ for all $l \neq p$ having $\bar{n}_{lw} = 1$, and $\bar{\lambda}_{ij} = \lambda_{ij}$ for all other i, j.

Since the rows of N_1 and N_2 satisfy the condition given in Remark 2.3(ii) and since the entries of $\overline{NN'}$ are as given in the previous paragraph, we have by 2.3(ii) that $\bar{N}$ is connected. Also, if C and $\bar{C}$ denote the information matrices of N and $\bar{N}$ respectively, then

$$\operatorname{tr} C^2 - \operatorname{tr} \bar{C}^2 = 2k^{-2}\left[\Sigma_{l \neq p} n_{lu}\lambda_{pl} + \Sigma_{l \neq s+1} n_{lw}\lambda_{s+1,l} - 2(k-1)\right].$$

Since $\lambda_{pl} \geqslant 1$ for all treatments $l \neq p$ having $n_{lu} = 1$ and $\lambda_{s+1,l} \geqslant 1$ for all treatments $l \neq s + 1$ having $n_{lw} = 1$, it follows that $\operatorname{tr} C^2 \geqslant \operatorname{tr} \bar{C}^2$.

Now if $N \in \mathfrak{D}$ is disconnected and $a > 2$, there will always exist two N_i satisfying the same conditions as N_1 and N_2 above, and by repeating the above argument consecutively to pairs of connected subsets of treatments satisfying those conditions, we will eventually arrive at a connected design whose information matrix squared has trace at least as small as that of the original disconnected design.

THEOREM 4.4. *There exists a connected (M, S) optimal design in $\mathfrak{D}$ if and only if $bk \geqslant v + b - 1$.*

PROOF. Sufficiency follows from Theorem 4.3 and necessity follows from the fact that the number of experimental units occurring in any connected design must be at least $v + b - 1$.

While Theorem 4.4 guarantees the existence of connected (M, S) optimal designs in $\mathfrak{D}$ when $bk \geqslant v + b - 1$, it does not guarantee that an (M, S) optimal design must be connected. In fact, the following example shows that an (M, S) optimal design need not be connected.

EXAMPLE 4.5. Suppose $b = 6$, $k = 2$, $v = 6$, and $r_i = 2$ for all i. Then the design N given below is such that the off-diagonal elements of NN' differ by one. Thus by Proposition 4.7 of Jacroux and Seely (1977), N is an (M, S) optimal design.

$$N = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Also, by 2.3(i), N is clearly disconnected. However, by interchanging the replications of treatments three and four occurring in blocks three and four respectively, a design is obtained which is connected and (M, S) optimal.

A commonly occurring sufficient condition is now given which guarantees that an (M, S) optimal design within $\mathfrak{D}$ must be connected.

THEOREM 4.6. *Let $\mathfrak{D}$ be such that $r_p = \max\{r_i | 1 \leqslant i \leqslant v\}$ and $r_q = \min\{r_i | 1 \leqslant i \leqslant v\}$. If $r_p(k-1) + r_q(k-1) \geqslant v-1$, then an (M, S) optimal design in $\mathfrak{D}$ must be connected.*

PROOF. Suppose $N \in \mathfrak{M}$ is disconnected. As in the proof of Theorem 4.3, we may assume that N is binary, that NN' has the form

$$\operatorname{diag}(N_1N_1', \cdots, N_aN_a'), \qquad 1 < a \leqslant v,$$

and that the rows of each N_i have been permuted so that the corresponding λ_{ij} satisfy condition 2.3(ii). We will also assume to begin with that $a = 2$, that N_1 and N_2 denote $s \times t$ and $(v-s) \times (b-t)$ matrices, and that treatment p occurs in N_1.

Since $r_p(k-1) + r_q(k-1) \geqslant v-1$, we may conclude that either $r_p(k-1) > s - 1$ or $r_{s+1}(k-1) > v - s - 1$. Without loss of generality, assume that $r_p(k-1) > s-1$. Note that $\lambda_{ij} = 0$ for all $1 \leqslant i \leqslant s$, and $j > s$. By 2.2, $\Sigma_{j \neq p} \lambda_{pj} = r_p(k-1)$. Now since the λ_{pj} are nonnegative integers and $r_p(k-1) > s-1$, we may conclude that $\lambda_{pm} \geqslant 2$ for some $m \neq p$, $1 \leqslant m \leqslant s$. Let u denote a block in which treatments p and m occur together, and let w be such that $n_{s+1,w} = 1$. Note that if we remove the replication of treatment p occurring in block u from the design, the rows of N_1 will still satisfy 2.3(ii), hence the treatments in N_1 will still be connected. Form a new design $\bar{N}$ having entries $\bar{n}_{pu} = 0$, $\bar{n}_{s+1,u} = 1$, $\bar{n}_{pw} = 1$, $\bar{n}_{s+1,w} = 0$, and $\bar{n}_{ij} = n_{ij}$ for all other i, j. Note that $\bar{N} \in \mathfrak{D}$. As in the proof of Theorem 4.3, $\bar{N}$ is connected, and if $\overline{NN'} = (\bar{\lambda}_{ij})$,

$$\operatorname{tr} C^2 - \operatorname{tr} \bar{C}^2 = 2k^{-2}\left[\Sigma_{l \neq p} n_{lu} \lambda_{pl} + \Sigma_{l \neq s+1} n_{lw} \lambda_{s+1,l} - 2(k-1)\right] > 0.$$

Now if $N \in \mathfrak{D}$ is disconnected and $a > 2$, there will always exist two N_i satisfying the same conditions as N_1 and N_2 above, and the result follows by applying the above argument consecutively to pairs of connected sets of treatments satisfying these conditions.

REMARK 4.7. Let $\mathfrak{D}$ be a class of proper designs having parameters b, k, v, and r_i where $\alpha b \leqslant r_i \leqslant (\alpha+1)b$ for all i and α is some positive integer. Jacroux and Seely (1977) have shown that the problem of finding an (M, S) optimal design in $\mathfrak{D}$ can be reduced to finding an (M, S) optimal design in the class $\mathfrak{D}'$ of proper binary designs having parameters $b' = b$, $k' = k - \alpha v$, $v' = v$, and $r_i' = r_i - \alpha b$ for all i. By the results of this section, we can limit the search for an optimal design in $\mathfrak{D}'$ to connected designs when $b'k' \geqslant b' + v' - 1$. Eccleston and Hedayat (1974) provide results which, when appropriate, limit the search even further to specific types of connected designs in $\mathfrak{D}'$.

5. Complementation. Suppose $N \in \mathcal{D}$ is a binary design. The complementary design of N is defined to be $\hat{N} = J - N$ where J is a $v \times b$ matrix of ones. Note that $\hat{N}$ has parameters $\hat{b} = b$, $\hat{k} = v - k$, $\hat{v} = v$, and $\hat{r}_i = b - r_i$ for each i. Our intent in this section is to show that (M, S) optimality is invariant under complementation.

Let $\mathcal{D}$ have parameters $r_i < b$ for $i = 1, \cdots, v$, and $k < v$ and let $\hat{\mathcal{D}}$ denote the class of all designs having parameters the same as $\hat{N}$ of the previous paragraph. If $\mathcal{M}$ and $\hat{\mathcal{M}}$ are defined in the obvious manner, then it is easily seen by Proposition 3.5 of Jacroux and Seely (1977) that $\mathcal{M}$ and $\hat{\mathcal{M}}$ consist of the binary designs in $\mathcal{D}$ and $\hat{\mathcal{D}}$ respectively. Note that for each binary design in $\mathcal{M}$ there is a corresponding complementary design in $\hat{\mathcal{M}}$, thus $\hat{\mathcal{M}} = \{J - N : N \in M\}$ where J is a $v \times b$ matrix of ones.

THEOREM 5.1. *Let $\mathcal{D}$ and $\hat{\mathcal{D}}$ be as defined above and let J be a $v \times b$ matrix of ones. Then N is (M, S) optimal in $\mathcal{D}$ if and only if $\hat{N} = J - N$ is (M, S) optimal in $\hat{\mathcal{D}}$.*

PROOF. Since b, k, v, and r_i are fixed for all designs in $\mathcal{D}$, by examining (3.3) it is clear that finding an (M, S) optimal design in $\mathcal{D}$ is equivalent to finding a design in $\mathcal{M}$ with minimum $\operatorname{tr}(NN')^2$. But $\operatorname{tr}(NN')^2 = \operatorname{tr}(N'N)^2$, hence finding an (M, S) optimal design in $\mathcal{D}$ can also be accomplished by finding a design in $\mathcal{M}$ with minimum $\operatorname{tr}(N'N)^2$. Similarly, finding an (M, S) optimal design in $\hat{\mathcal{D}}$ is equivalent to finding a design in $\hat{\mathcal{M}}$ with minimum $\operatorname{tr}(\hat{N}\hat{N}')^2$ or minimum $\operatorname{tr}(\hat{N}'\hat{N})^2$. Now if $\hat{N} \in \hat{\mathcal{M}}$ is expressed as $J - N$ where $N \in \mathcal{M}$, it is straightforward to verify that

$$\operatorname{tr}(\hat{N}'\hat{N})^2 = b^2(v - 2k)^2 + 2(v - 2k)(\Sigma_i r_i^2) + \operatorname{tr}(N'N)^2.$$

This last expression shows that minimizing $\operatorname{tr}(\hat{N}'\hat{N})^2$ over $\hat{\mathcal{M}}$ is equivalent to minimizing $\operatorname{tr}(N'N)^2$ over $\mathcal{M}$.

Jacroux and Seely (1977) have established some sufficient conditions for designs to be (M, S) optimal in $\mathcal{D}$. It may happen that there exists a design $N \in \mathcal{D}$ which does not satisfy any of these sufficient conditions but whose complement $\hat{N}$ may satisfy one of the conditions in the class $\hat{\mathcal{D}}$. Thus it may be possible to use Theorem 5.1 as a means to establish the (M, S) optimality of a design in $\mathcal{D}$.

6. Duality. Let $N \in \mathcal{D}$. Then the dual design is defined to be $\tilde{N} = N'$. Note that $\tilde{N}$ belongs to the class of all designs $\tilde{\mathcal{D}}$ having parameters $\tilde{v} = b$, $\tilde{r}_i = k$ for $i = 1, \cdots, v$, $\tilde{b} = v$, and $\tilde{k}_j = r_j$ for $j = 1, \cdots, v$. If $N \in \mathcal{D}$, the matrix of coefficients for estimating treatment effects in the corresponding dual design $\tilde{N}$ can be expressed in terms of v, r_i, b, and k as

$$\tilde{C} = kI_b - N'R^{-1}N \tag{6.1}$$

where I_b is the $b \times b$ identity matrix and $R = \operatorname{diag}(r_1, \cdots, r_v)$. From 6.1, we immediately get

$$\operatorname{tr}\tilde{C} = bk - \Sigma_i r_i^{-1}\Sigma_j n_{ij}^2 \tag{6.2}$$

and

$$\operatorname{tr} \tilde{C}^2 = bk^2 - 2k \operatorname{tr}(N'R^{-1}N) + \operatorname{tr}(N'R^{-1}N)^2. \tag{6.3}$$

THEOREM 6.4. *Let* $\mathfrak{D}$ *and* $\tilde{D}$ *be as defined above where* $\mathfrak{D}$ *is any class of proper equireplicate designs*, i.e., $r_i = r$ *for all* i. *Then* N *is* (M, S) *optimal in* $\mathfrak{D}$ *if and only if* $\tilde{N} = N'$ *is* (M, S) *optimal in* $\tilde{\mathfrak{D}}$.

PROOF. By examining (3.1) and (6.2), it is seen that a design $N \in \mathfrak{D}$ has maximal trace of C if and only if the corresponding $\tilde{N} \in \tilde{\mathfrak{D}}$ has maximal trace of $\tilde{C}$. Thus $\tilde{\mathfrak{M}} = \{N' : N \in \mathfrak{M}\}$. Since v, r, b, k, and $\Sigma_i \lambda_{ii}$ are constant for all designs in $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$, we see from (3.2) and (6.3) that finding (M, S) optimal designs in $\mathfrak{D}$ and $\tilde{\mathfrak{D}}$ is equivalent to finding designs in $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$ with minimum $\operatorname{tr}(NN')^2$ and minimum $\operatorname{tr}(\tilde{N}\tilde{N}')^2$ respectively. But $\operatorname{tr}(NN')^2 = \operatorname{tr}(\tilde{N}\tilde{N}')^2$, hence minimizing $\operatorname{tr}(NN')^2$ over $\mathfrak{M}$ is equivalent to minimizing $\operatorname{tr}(\tilde{N}\tilde{N}')^2$ over $\tilde{\mathfrak{M}}$ and the result follows.

Unfortunately, in classes of designs with unequal numbers of replicates, a result similar to that of Theorem 6.4 does not hold as the following example illustrates.

EXAMPLE 6.5. Consider the class of designs $\mathfrak{D}$ having the set of parameters $v = 5$, $r_1 = 6$, $r_2 = 5$, $r_3 = r_4 = 4$, $r_5 = 2$, $b = 7$ and $k = 3$. Then it can be shown by enumerating the designs in $\mathfrak{D}$ that N_1 given below is (M, S) optimal in $\mathfrak{D}$ and has $\operatorname{tr} C_1^2 = 55.778$.

$$N_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Next consider the design $N_2 \in \mathfrak{D}$ given by

$$N_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and having $\operatorname{tr} C_2^2 = 56.2219$. Now if $\tilde{C}_1$ and $\tilde{C}_2$ are the information matrices $\tilde{N}_1$ and $\tilde{N}_2$ respectively, it may be verified that $\operatorname{tr} \tilde{C}_1^2 = 43.48339$ and $\operatorname{tr} \tilde{C}_2^2 = 43.44179$.

We should observe that the designs in $\mathfrak{D}$ can be thought of as two-way classification designs with treatments as the levels of one factor and blocks as the levels of another factor. We should also note that the reduced normal equations for estimating the levels of the factors are the same as the reduced normal equations for estimating treatment and block effects. Thus the definition of (M, S) optimality given in Section 3 can be modified so as to be applicable to the estimation of factor levels, i.e., the definition can be given in terms of the reduced normal equations for estimating the levels of each factor. By Theorem 6.4 and Example 6.5, we see that a design in $\mathfrak{D}$ which is (M, S) optimal for estimating the levels of one factor may not be optimal for estimating the levels of the other factor unless the levels of each factor are equally replicated.

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