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HADAMARD MATRICES AND THEIR APPLICATIONS

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An $n \times n$ matrix H with all its entries $+1$ and -1 is *Hadamard* if $HH' = nI$. It is well known that n must be 1, 2 or a multiple of 4 for such a matrix to exist, but is not known whether Hadamard matrices exist for every n which is a multiple of 4. The smallest order for which a Hadamard matrix has not been constructed is (as of 1977) 268. Research in the area of Hadamard matrices and their applications has steadily and rapidly grown, especially during the last three decades. These matrices can be transformed to produce incomplete block designs, t-designs, Youden designs, orthogonal F-square designs, optimal saturated resolution III designs, optimal weighing designs, maximal sets of pairwise independent random variables with uniform measure, error correcting and detecting codes, Walsh functions, and other mathematical and statistical objects. In this paper we survey the existence of Hadamard matrices and many of their applications.

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1. Introduction.

Very often the most difficult problems can be stated with deceptive simplicity. This is particularly true in mathematics, and many well-known problems are of this type. Typical are:

Fermat's last theorem, that if the integer n is three or greater, there exist no nonzero integral solutions x, y, z to the equation $x^n + y^n = z^n$;

Guthrie's four-color conjecture for planar maps, that the regions of any map on the plane can be colored with four or fewer colors so that no two adjacent regions are assigned the same color;

the van der Waerden conjecture, that the permanent of a doubly stochastic square matrix of side n is at least $n!n^{-n}$;

Goldbach's conjecture, that every even integer larger than 2 is a sum of two primes.

All of these problems have been known for a long time; the solution of the second has only recently been announced by Appel and Haken (1976), and the others are unsolved.

We are concerned here with a similar conjecture which has stimulated considerable interest among mathematicians and statisticians over recent years. This is the Hadamard matrix conjecture, sometimes referred to as Paley's conjecture, although it is implicit in some writings from before Paley's time. If n is a positive integer divisible by 4, is there a square matrix H of order n, having all its entries $+1$ or -1, such that $HH' = nI$?

The first four problems which we mentioned are peculiar in that it can be argued —and has been argued—that their real worth lies in the mathematical by-products which have resulted from failure to solve them. These attendant results have often been more useful than would be the solution of these problems themselves by elementary means. For example, attempts to prove Fermat's last theorem and the other number-theoretical problems have led to the theory of algebraic numbers, the concept of ideals in rings, as well as a number of beautiful insights into the nature of prime numbers.

However, the Hadamard matrix conjecture is different in nature. Although a number of associated ideas have been developed in the search for Hadamard matrices, the very existence of these matrices has extensive consequences in many fields of research, such as optimal design theory, information theory and graph theory. For example, a Hadamard matrix can be interpreted directly as a weighing design. The equivalence of Hadamard matrices and a class of balanced incomplete block designs (see Theorem 4.1, below) means that one can use them to construct a range of block designs, Youden "squares" and generalized Youden designs. They can be used in forming optimal fractional factorial designs, orthogonal arrays and orthogonal F-square designs. For this reason, the present paper presents a brief review of the importance of these matrices in various fields with special reference to areas of statistical interest. We also present a comprehensive bibliography. It should be realized that we do not attempt anything like a complete survey of the construction or the pure-mathematical properties of Hadamard matrices; the interested reader should consult Wallis, Street and Wallis (1972), and J. Wallis (1973c). More elementary introductions can be found in the relevant chapters of Ryser (1963), Hall (1967), and Street and Wallis (1977).

We adopt the following notations:

(i) In writing out matrices, $+$ and $-$ are used as abbreviations for $+1$ and -1.

(ii) I denotes an identity matrix, J a square or rectangular matrix with all entries $+1$, and $\mathbf{1}$ a column vector with all entries $+1$. In all cases the dimensions should be deduced from the context.

(iii) A' denotes the transpose of the matrix A.

2. Hadamard matrices. A square matrix H of order n whose entries are $+1$ or -1 is called a *Hadamard matrix of order n* provided that its rows are pairwise orthogonal, in other words

$$HH' = nI. \tag{2.1}$$

Equation (2.1) implies that H is nonsingular, and has an inverse $n^{-1}H'$; consequently

$$H'H = nI.$$

This tells us that the columns of a Hadamard matrix are also pairwise orthogonal. Furthermore, it can be interpreted as saying that a matrix is Hadamard if and only if its transpose is Hadamard.

As an example, a Hadamard matrix of order 4 is given by

$$H = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}. \tag{2.2}$$

Historically, these matrices were discussed as far back as 1867 by Sylvester, who found that they arose in connection with a problem on tesselations. However, they arose in a very natural way from the following considerations. Suppose $A = (a_{ij})$ is a square matrix of order n with real entries such that $|a_{ij}| \leqslant M$ for every i and j. Hadamard (1893) proved that the absolute value of the determinant of A cannot exceed $M^n n^{\frac{1}{2}n}$. Since $(\det H)^2 = \det H' \cdot \det H = \det(H'H) = \det(nI) = n^n$, therefore for $M = 1$, a Hadamard matrix of order n attains the maximum determinant value. Hadamard showed further that these matrices are the only ones which attain the bound. (This is why the name "Hadamard matrix" has been used.) For a brief survey of results on this "maximum determinant" problem, see Brenner and Cummings (1972). Statistical aspects of the maximum determinant problem are discussed by Hedayat (1978); see also Section 5.5.

There exist Hadamard matrices of orders 1 and 2, but it can be shown that every other Hadamard matrix has order $4t$ for some positive integer t. Hadamard matrices of infinitely many orders have been constructed, and it has been conjectured that one exists for every t, but no general proof is available, and the number of unsettled orders is infinite. However, no case is known of an order divisible by 4 which has no Hadamard matrix. The smallest order which is undecided is 268. The most recent and comprehensive listing of the orders for which Hadamard matrices are known is that of Seberry (1978).

Prior to Paley's time Sylvester (1867) noted Hadamard matrices of orders which are powers of 2, essentially using Theorem 3.1 below, and Scarpis (1898) proved that when p is a prime congruent to 3 (modulo 4) then there is a Hadamard matrix of order $p + 1$, while if p is a prime congruent to 1 (modulo 4), then there is a Hadamard matrix of order $2(p + 1)$. Paley (1933) discovered two constructions (Theorems 3.2 and 3.3 below) which generalize Scarpis' theorems. These results enable one to find Hadamard matrices of many small orders, and after Paley's paper there were only six orders not yet constructed in the range from 1 to 200, namely 92, 116, 156, 172, 184, and 188.

The next major constructions were discovered by John Williamson. In 1944 he published valuable generalizations of some of Paley's work, which did not, however, yield any new orders under 200. He also gave a new type of construction, which we outline below in Section 3.3. Williamson constructed a Hadamard matrix of order 172 by this method. Later, Baumert, Golomb and Hall (1962) used Williamson's method to construct a Hadamard matrix of order 92, and consequently one of order 184; and then Baumert (1966) constructed a matrix of order 116 similarly.

Williamson's method was generalized by Baumert and Hall (1965a) who found a matrix of order 156. Further generalizations were given by Cooper and Wallis (1972), J. Wallis (1973a), and Turyn (1974), and research in this area is continuing.

Table 1 presents a summary of the construction of small Hadamard matrices (orders up to 200).

Suppose H is a Hadamard matrix of order n containing a submatrix of order n_1 which is itself Hadamard. What do we know about n_1? Cohn (1965) showed that $n_1 \leqslant n/2$. For example, no Hadamard matrix of order 12 can contain a Hadamard matrix of order 8. Thus, if one hopes to construct a Hadamard matrix of order n by augmenting a Hadamard matrix of order n_1 with $n - n_1$ rows and columns with entries $+1$ or -1 then a necessary condition is $n - 2n_1 \geqslant 0$.

All of the above discussion ignores the question of how many different Hadamard matrices of a given order might exist. This is a very difficult question to answer. First, it is necessary to decide what is meant by "different" Hadamard matrices. It is usually agreed that two Hadamard matrices are essentially the same if one can be obtained from the other by a permutation of the rows, or of the columns, or by negating certain rows, or columns. Two Hadamard matrices are called *equivalent* (or *Hadamard equivalent*) if one can be obtained from the other by a sequence of these operations; or alternatively two Hadamard matrices H_1 and H_2 of the same order are equivalent if there exist signed permutation matrices P_1 and P_2 for which $P_1H_1 = H_2P_2$, where a signed permutation matrix is a matrix in which each row and each column has exactly one nonzero entry, and that entry is from the set $\{1, -1\}$. It is known that Hadamard matrices of orders up to 12 are uniquely determined up to equivalence; Hall (1961) showed that there are precisely five equivalence classes of matrices of order 16 and (1965) three of order 20. Other results have been discovered by Rutledge (1952), Stiffler and Baumert (1961), Baumert (1962), Wallis and Wallis (1969), Bussemaker and Seidel (1970), Newman (1971), W. D. Wallis (1971a), (1971b), (1972a), (1972b); Gordon (1974), Norman (1976), Longyear (1978), Cooper, Milas and Wallis (1978), who used integral equivalence of matrices to obtain lower bounds on the number of equivalence classes, including the following theorem:

THEOREM 2.1. *Given any positive integer N, there are infinitely many orders at which there are at least N equivalence classes of Hadamard matrices under Hadamard equivalence.*

The methods show that the smallest such order will be at most 32^n, where $n = \left[\dfrac{N+9}{10}\right]$ (square brackets denoting integral part); but this is clearly not the best possible result.

Another method of discussing equivalence is to consider the *weight* of a Hadamard matrix, the number of entries equal to $+1$, as introduced by Schmidt and Wang (1977). W. D. Wallis (1977) investigated the use of $W(H)$, the maximum of all the weights of Hadamard matrices equivalent to H, but his results tend to suggest that $W(H)$ will not be a useful tool in investigating Hadamard equivalence.

Bussemaker and Seidel (1970) and Cooper, Milas and Wallis (1978) have examined other equivalence relations; and various authors (such as Bhat (1972a), (1972b) and Singhi (1974), (1975)) have discussed isomorphism among the balanced incomplete block designs which we shall associate with Hadamard matrices in Theorem 4.1 below.

Given a Hadamard matrix, one can negate every row whose first element is -1, thus obtaining an equivalent matrix whose first column is all $+1$'s. Similarly, the first row can be converted to all $+1$'s. A matrix in this form will be called normalized. For example, given

$$\begin{bmatrix} + & - & + & + \\ - & + & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix} \tag{2.3}$$

one can first negate row 2, forming

$$\begin{bmatrix} + & - & + & + \\ + & - & - & - \\ + & + & - & + \\ + & + & + & - \end{bmatrix} \tag{2.4}$$

and then column 2, to obtain

$$\begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & - & + \\ + & - & + & - \end{bmatrix} \tag{2.5}$$

which is normalized. A matrix whose first column is all positive will be called *seminormalized*; (2.4) presents a seminormalized matrix which is in fact not normalized. It is clear that one can normalize or seminormalize a Hadamard matrix in many different ways. This observation is important because, as we shall see in Section 4.1, one may be able to construct nonisomorphic BIB designs from the same Hadamard matrix.

Table 2 presents representative normalized Hadamard matrices of all orders up to 32. Because of the special interest in order 188, a Hadamard matrix of that order is exhibited in Table 3.

There has been some interest in the existence of a square $(1, -1)$ matrix $\overline{H}_{n+1}$ of order $n + 1$ all of whose $n \times n$ diagonal submatrices are Hadamard. Pesotan and Rahgavarao (1975) have found that the determinant of an $\overline{H}_{n+1}$ is, at most,

$$n^{n/2}\left(1 + \frac{[nn^{\frac{1}{2}}]}{n}\right).$$

For $n + 1 = 5$ there exists an $\overline{H}_5$ with determinant 48; this is the maximum possible determinant in the class of all matrices of order 5 with entries $+1$ and -1 (see Section 4.1). This matrix is exhibited below.

$$\overline{H}_5 = \left[\begin{array}{cccc|c} + & + & + & + & + \\ + & - & + & - & + \\ + & + & - & - & + \\ + & - & - & + & - \\ \hline - & - & - & + & + \end{array}\right]$$

Further discussion of this problem can be found in Pesotan, Raghavarao and Raktoe (1977) and Raghavarao and Pesotan (1977).

The problem of whether an incomplete Hadamard matrix can be extended to a Hadamard matrix has been discussed by Shrikhande and Bhagwandas (1970), and by Vijayan (1976). Here by an incomplete Hadamard matrix we mean an $m \times n$ matrix B having all its entries $+1$ or -1, such that $m < n$ and $BB' = mI$.

3. Construction of Hadamard matrices

3.1. *Early constructions.* The easiest construction of Hadamard matrices is embodied in Theorem 3.1. The symbol $\otimes$ denotes direct product of matrices: if A is the matrix with typical entry a_{ij}, then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \cdots & \cdots & \ddots \end{bmatrix}.$$

THEOREM 3.1. *If H_1 is a Hadamard matrix of order m and H_2 is a Hadamard matrix of order n then $H_1 \otimes H_2$ is a Hadamard matrix of order mn.*

EXAMPLE 3.1. There is a Hadamard matrix of order 4, namely

$$\begin{bmatrix} + & + \\ + & - \end{bmatrix} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}.$$

(Note: this matrix is the matrix of (2.2) with its second and third rows interchanged.)

By repeated use of the Hadamard matrix of order 2 as H_1, we obtain

COROLLARY 3.1. *There is a Hadamard matrix of order 2^k for every positive integer k.*

THEOREM 3.2 (Paley (1933). *If p^α is a prime power and $p^\alpha + 1 \equiv 0 \pmod 4$, then there is a Hadamard matrix of order $p^\alpha + 1$.*

PROOF. Suppose the members of the field $GF(p^\alpha)$ are labelled $a_0, a_1, a_2, \cdots$, in some order. A matrix Q of order p^α is defined as follows. The (i, j) entry of Q equals $\chi(a_i - a_j)$, where χ is the quadratic character on $GF(p^\alpha)$, namely

$$\begin{aligned} \chi(0) &= 0, \\ \chi(b) &= 1 \quad \text{if } b \text{ is a nonzero quadratic element (perfect square) in } GF(p^\alpha), \\ \chi(b) &= -1 \quad \text{if } b \text{ is not quadratic.} \end{aligned}$$

Then writing

$$S = \begin{bmatrix} 0 & \mathbf{1}' \\ -\mathbf{1} & Q \end{bmatrix}, \qquad H = I + S,$$

H is a Hadamard matrix.

EXAMPLE 3.2. To construct a Hadamard matrix of order 12, we observe that $12 = 11 + 1$. The quadratic elements of $GF(11)$ are 1, 3, 4, 5 and 9; using the natural ordering $a_0 = 0, a_1 = 1, \cdots, a_{10} = 10$, Q and H are

$$Q = \begin{bmatrix} 0 & - & + & - & - & - & + & + & + & - & + \\ + & 0 & - & + & - & - & - & + & + & + & - \\ - & + & 0 & - & + & - & - & - & + & + & + \\ + & - & + & 0 & - & + & - & - & - & + & + \\ + & + & - & + & 0 & - & + & - & - & - & + \\ + & + & + & - & + & 0 & - & + & - & - & - \\ - & + & + & + & - & + & 0 & - & + & - & - \\ - & - & + & + & + & - & + & 0 & - & + & - \\ - & - & - & + & + & + & - & + & 0 & - & + \\ + & - & - & - & + & + & + & - & + & 0 & - \\ - & + & - & - & - & + & + & + & - & + & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} + & + & + & + & + & + & + & + & + & + & + & + \\ - & + & - & + & - & - & - & + & + & + & - & + \\ - & + & + & - & + & - & - & - & + & + & + & - \\ - & - & + & + & - & + & - & - & - & + & + & + \\ - & + & - & + & + & - & + & - & - & - & + & + \\ - & + & + & - & + & + & - & + & - & - & - & + \\ - & + & + & + & - & + & + & - & + & - & - & - \\ - & - & + & + & + & - & + & + & - & + & - & - \\ - & - & - & + & + & + & - & + & + & - & + & - \\ - & - & - & - & + & + & + & - & + & + & - & + \\ - & - & + & - & - & - & + & + & + & - & + & + \end{bmatrix}.$$

THEOREM 3.3. *Suppose p^α is a prime power and $p^\alpha + 1 \equiv 2 \pmod 4$. Then there is a Hadamard matrix of order $2(p^\alpha + 1)$.*

PROOF. Using the quadratic character χ in $GF(p^\alpha)$, a matrix Q is constructed as in Theorem 3.2, and

$$S = \begin{bmatrix} 0 & \mathbf{1}' \\ \mathbf{1} & Q \end{bmatrix},$$

then

$$H = S \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} + I \otimes \begin{bmatrix} + & - \\ - & - \end{bmatrix}$$

is the required Hadamard matrix.

EXAMPLE 3.3. As $12 = 2(5 + 1)$, where 5 is prime and $5 + 1 \equiv 2 \pmod 4$, we can use Theorem 3.3 to construct a matrix of order 12. The quadratic elements in $GF(5)$ are 1 and 4, so

$$Q = \begin{bmatrix} 0 & + & - & - & + \\ + & 0 & + & - & - \\ - & + & 0 & + & - \\ - & - & + & 0 & + \\ + & - & - & + & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & + & + & + & + & + \\ + & 0 & + & - & - & + \\ + & + & 0 & + & - & - \\ + & - & + & 0 & + & - \\ + & - & - & + & 0 & + \\ + & + & - & - & + & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} + & - & + & + & + & + & + & + & + & + & + & + \\ - & - & + & - & + & - & + & - & + & - & + & - \\ + & + & + & - & + & + & - & - & - & - & + & + \\ + & - & - & - & + & - & - & + & - & + & + & - \\ + & + & + & + & + & - & + & + & - & - & - & - \\ + & - & + & - & - & - & + & - & - & + & - & + \\ + & + & - & - & + & + & + & - & + & + & - & - \\ + & - & - & + & + & - & - & - & + & - & - & + \\ + & + & - & - & - & - & + & + & + & - & + & + \\ + & - & - & + & - & + & + & - & - & - & + & - \\ + & + & + & + & - & - & - & - & + & + & + & - \\ + & - & + & - & - & + & - & + & + & - & - & - \end{bmatrix}.$$

Gruner (1939–40) showed that a Hadamard matrix of order $n = k^2$ can be constructed if $k - 1$ and $k + 1$ are both prime powers. The case $k = 18$ gave a Hadamard matrix of order 324; this order had not been constructed at that time.

Williamson (1944) generalized Theorems 3.2 and 3.3 to:

THEOREM 3.4. *If there exists a Hadamard matrix of order h, $h > 1$, and p^α is an odd prime power, then there is a Hadamard matrix of order $h(p^\alpha + 1)$.*

The proof may be found in Williamson (1944), Hall (1967), and Wallis, Street and Wallis (1972).

3.2. *Construction of special Hadamard matrices.* The matrix S of Theorem 3.2 is skew, and it has been found that matrices like this play a special role in the theory of Hadamard matrices. So we define a *skew-Hadamard* matrix to be a Hadamard matrix of the form

$$H = S + I$$

where S is skew. S will satisfy

$$SS' = (n - 1)I, \tag{3.0}$$

where n is the order of H. The matrix S of Theorem 3.3 also satisfies (3.0) when $n = p^\alpha + 1$; but in this case S is symmetric. A *symmetric* matrix with zero diagonal and ± 1 elsewhere is called a (*symmetric*) *conference matrix* if it satisfies (3.0). These matrices can be used in the construction of Theorem 3.3 and will yield a Hadamard matrix. Conference matrices were introduced by Belevitch (1950) and their relation to Hadamard matrices was studied by Goldberg (1966), Goethals and Seidel (1967), Belevitch (1968) and J. Wallis (1971a). Skew-Hadamard matrices and conference matrices are in a sense analogous objects, as is clear from the treatment in Delsarte, Goethals and Seidel (1971), J. Wallis (1971a), (1972b), and Turyn (1971). Skew-Hadamard matrices and conference matrices have been used in many constructions, most of which are detailed in Wallis, Street and Wallis (1972).

Because of the importance of skew-Hadamard matrices it is of interest to determine which matrices are equivalent to skew-Hadamard matrices. Longyear (1976) obtained three criteria for determining if a given Hadamard matrix is skew-Hadamard equivalent. One of her criteria states that a Hadamard matrix H is skew-Hadamard equivalent if and only if there exists a signed permutation matrix P such that $H + 2P$ is a Hadamard matrix. While this criterion is quite general it is not useful in computation. Her other two tests are most practical for computer use and for use by hand. As an application of her results, she has shown that of the five equivalence classes of Hadamard matrices of order 16, exactly two do and three do not contain skew-Hadamard representatives, and that of the three classes of order 20 one does and two do not.

There has been some special interest in the construction of symmetric Hadamard matrices. These are used in a number of constructions; in particular, a pair comprising a skew-Hadamard matrix M and a symmetric Hadamard matrix N such that the product MN is symmetric is called a pair of *amicable Hadamard matrices*; amicable Hadamard matrices arise in several constructions (see J. Wallis (1971b), (1973b), and Wallis, Street and Wallis (1972)). Symmetric Hadamard matrices with constant diagonal, such as

$$\begin{bmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}$$

have also been studied. Such matrices can only exist when the order is a perfect square; it seems reasonable to conjecture that they exist for all such orders, but infinitely many cases remain unconstructed. For further details see Bose and Shrikhande (1970), and W. D. Wallis (1969a), (1971c), (1972c). These special matrices are just one example of a special class of block designs, discussed in the above papers and also by Ahrens and Szekeres (1969), W. D. Wallis (1969b), (1970a), Bose and Shrikhande (1971) and Rudvalis (1971). One result to which we shall refer later is:

THEOREM 3.5. *If there exists a set of $n-2$ pairwise orthogonal Latin squares of order $2n$, then there is a symmetric Hadamard matrix with constant diagonal of order $4n^2$.*

A Hadamard matrix is called *regular* if the sum of the elements in any row equals a constant. It can be shown that a regular Hadamard matrix has order a perfect square, $4n^2$, say. The number of entries equal to $+1$ in a row will be constant, either $2n^2 - n$ or $2n^2 + n$. In the first case any two rows will have $n^2 - n$ positions wherein both have entry $+1$; the second case has the same property but the constant is $n^2 + n$. The matrices of the second type can be derived from the first by simple negation.

THEOREM 3.6 (Szekeres (1969)). *If there is a Hadamard matrix of order $4s$, then there is a regular Hadamard matrix of order $16s^2$.*

PROOF. Suppose $H = (h_{ij})$ is a normalized Hadamard matrix of order $4s$. We construct a matrix G, of size $16s^2 \times 16s^2$, by defining

$$\begin{aligned} g_{4sp+q,\,4si+j} &= h_{q,\,i+1}h_{j,p} && \text{if } p > i \\ g_{4sp+q,\,4si+j} &= -h_{q,\,i}h_{j,p+1} && \text{if } p < i \\ g_{4sp+q,\,4si+j} &= -1 && \text{if } p = i \end{aligned}$$

for $0 < p, i < 4s - 1$ and $1 < q, j < 4s$. Then G is a regular Hadamard matrix.

Several other techniques have been used to construct regular Hadamard matrices. In particular, Shrikhande and Singh (1962) prove that the Hadamard matrices of Theorem 3.5 are regular. Goethals and Seidel (1970) prove that if both $n - 1$ and $n + 1$ are odd prime powers, then there is a regular (and symmetric) Hadamard matrix of order n^2. This is particularly useful when $n \equiv 2 \pmod 4$, as for example when $n = 10$ a matrix of order 100 is obtained. Another construction for regular Hadamard matrices is given by Wallis and Whiteman (1972).

3.3. *Williamson's method.* Many exciting results have stemmed from the basic ideas put forward by Williamson (1944). Consider the array

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix}. \tag{3.1}$$

If A, B, C and D are replaced by square matrices of order n, H becomes a square matrix of order $4n$. One can attempt to choose A, B, C and D in such a way that H will be a Hadamard matrix.

If HH' is considered as a block matrix with $n \times n$ blocks, then the diagonal blocks each equal $AA' + BB' + CC' + DD'$. This must be $4nI$ for H to be a Hadamard matrix. The (1, 2) block is $BA' - AB' + DC' - CD'$. This will be zero if $AB' = BA'$ and $CD' = DC'$. Similar results hold for the other off-diagonal elements. So we have:

THEOREM 3.7. *If there exist square* (1, − 1) *matrices* A, B, C *and* D *of order* n *which satisfy*

$$AA' + BB' + CC' + DD' = 4nI \tag{3.2}$$

and, for every pair X, Y *of distinct matrices chosen from* A, B, C, D,

$$XY' = YX', \tag{3.3}$$

then they can be used in (3.1) *to construct a Hadamard matrix of order* $4n$.

EXAMPLE 3.4. Let

$$A = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}, \qquad B = C = D = \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}.$$

Then

$$AB' = BA' = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

and clearly every other possible combination satisfies $XY' = YX'$. Moreover

$$AA' = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}, \qquad BB' = CC' = DD' = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

so $AA' + BB' + CC' + DD' = 12I = 4nI$. So these matrices satisfy the theorem. The Hadamard matrix is

$$H = \begin{bmatrix} + & + & + & + & - & - & + & - & - & + & - & - \\ + & + & + & - & + & - & - & + & - & - & + & - \\ + & + & + & - & - & + & - & - & + & - & - & + \\ - & + & + & + & + & + & - & + & + & + & - & - \\ + & - & + & + & + & + & + & - & + & - & + & - \\ + & + & - & + & + & + & + & + & - & - & - & + \\ - & + & + & + & - & - & + & + & + & - & + & + \\ + & - & + & - & + & - & + & + & + & + & - & + \\ + & + & - & - & - & + & + & + & + & + & + & - \\ - & + & + & - & + & + & + & - & - & + & + & + \\ + & - & + & + & - & + & - & + & - & + & + & + \\ + & + & - & + & + & - & - & - & + & + & + & + \end{bmatrix}.$$

The basic difficulty lies in finding the matrices A, B, C, D. Williamson made certain simplifying assumptions: first, he assumed that A, B, C and D are *symmetric* matrices, so that the condition $XY' = YX'$ reduces to saying that A, B, C and D commute. Then he made the further assumption that in each of the matrices, the $(i + 1, j + 1)$ entry was equal to the (i, j) entry; for example:

$$(3.4) \qquad A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \cdots a_n \\ a_n & a_1 & a_2 & a_3 \cdots a_{n-1} \\ a_{n-1} & a_n & a_1 & a_2 \cdots a_{n-2} \\ a_{n-2} & a_{n-1} & a_n & a_1 \cdots a_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ a_2 & a_3 & a_4 & a_5 \cdots a_1 \end{bmatrix}$$

and the whole matrix is determined by its first row. (One assumes that row and column numbers are reduced modulo n when necessary.) Such a matrix is called *circulant*. Circulant matrices have already occurred above: when $\alpha = 1$, the matrices Q of Theorems 3.2 and 3.3 are circulant. Every such matrix is a polynomial in the matrix K,

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

(The matrix of (3.4) is $a_1 I + a_2 K + a_3 K^2 + \cdots + a_n K^{n-1}$.) Since polynomials in a given matrix commute, Williamson's A, B, C and D are commutative, and only the condition (3.2) remains to be satisfied.

While Williamson's assumptions are very restrictive, it has been found possible to satisfy them in all the small cases. Williamson (1944) solved the case $n = 43$ ($4n = 172$), Baumert, Golomb and Hall (1962) solved $n = 23$ ($4n = 92$), and Baumert (1966) solved $n = 29$ ($4n = 116$). More recent results, including the first construction of infinite classes of matrices, are in Turyn (1972), Whiteman (1973) and J. Wallis (1973d), (1974), (1975a). In Table 4 we list first rows for circulant matrices A, B, C and D which may be used to construct Hadamard matrices of the

Williamson type for order $4n$, odd $n \leqslant 25$. Many of these matrices are taken from Baumert and Hall (1965b) and Wallis, Street and Wallis (1972).

Various constructions for skew-Hadamard matrices have been made using the Williamson method. J. Wallis (1971c) used the array (3.1) to construct the first known skew-Hadamard matrix of order 92. See also Blatt and Szekeres (1969), Hunt (1972) and Hunt and Wallis (1972). Another approach was used by Goethals and Seidel (1970), who modified the array (3.1). Further developments of their construction, and other adaptations and results, can be found in Yang (1971), Whiteman (1972), Wallis and Whiteman (1972), J. Wallis (1975b) and Spence (1975b), (1977).

Another method of generalization is to change the size of the array (3.1). The original array has every variable exactly once per row and column. It has been shown by various authors in various ways that such an array exists at sizes 1, 2, 4 and 8, and no other (Folkman (1967), Spencer (1967), J. Wallis (1970), Storer (1971), Taussky (1971), Spence (1972)). Storer (1971) points out that the theorem is essentially contained in Hurwitz (1898).

The arrays of size 1 and 2 are trivial, and the array of size 8 can give rise to no new Hadamard matrices. So attention was directed toward arrays in which the same variable could occur more than once in the same row. We define a *Baumert-Hall array* H of order $4t$ to be a $4t \times 4t$ array whose elements are $\pm A$, $\pm B$, $\pm C$ and $\pm D$, constructed in such a way that if A, B, C and D are replaced by $(1, -1)$ matrices which commute, then HH' is the $4t \times 4t$ block matrix with diagonal elements $t(AA' + BB' + CC' + DD')$ and other elements zero.

THEOREM 3.8. *If there exist square matrices satisfying the conditions of Theorem* 3.7 *and if there exists a Baumert-Hall array of size* $4t$, *then there is a Hadamard matrix of size* $4tn$.

EXAMPLE 3.5. The array

$$\begin{bmatrix}
A & A & A & B & -B & C & -C & -D & B & C & -D & -D \\
A & -A & B & -A & -B & -D & D & -C & -B & -D & -C & -C \\
A & -B & -A & A & -D & D & -B & B & -C & -D & C & -C \\
B & A & -A & -A & D & D & D & C & C & -B & -B & -C \\
B & -D & D & D & A & A & A & C & -C & B & -C & B \\
B & -C & -D & D & A & -A & C & -A & -D & C & B & -B \\
D & -D & B & -B & A & -C & -A & A & B & C & D & -D \\
-C & -C & -C & -D & C & A & -A & -A & -D & B & -B & -B \\
D & -D & -B & -B & -B & C & C & -D & A & A & A & D \\
-D & -C & C & C & C & B & B & -D & A & -A & D & -A \\
C & -B & -C & C & D & -B & -D & -B & A & -D & -A & A \\
-C & -D & -D & -C & -C & -B & B & B & D & A & -A & -A
\end{bmatrix}$$

is a Baumert-Hall array of size 12. If we replace A, B, C, D by the circulant symmetric matrices generated by the following first rows:

A:	+	+	−	−	+	−	+	+	−	+	−	−	+
B:	+	−	−	−	+	+	+	+	+	+	−	−	−
C:	+	+	+	−	+	+	−	−	+	+	−	+	+
D:	+	+	−	+	−	+	+	+	+	−	+	−	+

we obtain a Hadamard matrix of order 156.

The array of Example 3.5 was found by Baumert and Hall (1965a) and gave the first proof of the existence of a Hadamard matrix of order 156. In 1970, Welch found an array of order 20. Subsequently other arrays have been found by J. Wallis (1973a), Cooper and Wallis (1972), and Turyn (1974); see also Lakein and Wallis (1975). Using a computer, Turyn (1975) constructed a Baumert-Hall array of order 188; the construction depends on Theorem 6 of Turyn (1974). This gave the first known construction of a Hadamard matrix of order 188.

These ideas were generalized by Geramita, Geramita and Wallis (1976). They define an *orthogonal design* of order n and type $(s_1, s_2, \cdots, s_l)$, $s_i > 0$, on the commuting variables $x_1, x_2, \cdots, x_l$, to be an $n \times n$ array A with entries from $\{0, \pm x_1, \pm x_2, \cdots, \pm x_l\}$ whose rows are formally orthogonal, such that each row has precisely s_i entries either x_i or $-x_i$. It can be shown that if A is an orthogonal design row-wise, it is also orthogonal column-wise. Therefore, if A is an orthogonal design, then

$$AA' = A'A = \Sigma_{i=1}^{l}(s_i x_i^2) I_n.$$

A Baumert-Hall array is an orthogonal design of order $4t$ with $s_1 = s_2 = s_3 = s_4 = t$. Orthogonal designs have application in the construction of Hadamard matrices, and also in the construction of weighing designs (see Section 4.4 below). Plotkin (1972) makes the very strong conjecture that every Hadamard matrix of order $8n$ can be obtained from specializing some orthogonal design of order $8n$ and type (n, n, n, n, n, n, n, n), i.e., every Hadamard matrix of order $8n$ may be obtained from an orthogonal design of the order and type above by setting the variables all equal to 1. He shows that the existence of a Hadamard matrix of order n implies the existence of three types of orthogonal designs.

Orthogonal designs are surveyed by Geramita and Wallis (1974), which appeared while Geramita, Geramita and Wallis (1976) was still in press. Subsequent developments are found in Geramita and Verner (1976) and Cooper and Wallis (1976a), (1976b).

3.4. *Other constructions.* Many constructions of Hadamard matrices involve difference sets. If G is an abelian group with v elements, a (v, k, λ) difference set D in G is a set of k elements $g_1, g_2, \cdots, g_k$ of G such that the differences $\pm(g_i - g_j)$, where $1 \leqslant i \leqslant j \leqslant k$, comprise every nonzero element of G, precisely λ times each.

THEOREM 3.9. *If there is a* $(4t - 1, 2t - 1, t - 1)$ *difference set, then there is a Hadamard matrix of order* $4t$.

PROOF. Suppose D is the difference set, and G is the abelian group of order $4t-1$ containing D. Construct a square matrix A of order $4t-1$ as follows. First, label the elements of G as $g_1, g_2, \cdots, g_{4t-1}$ in some order. Then put $a_{ij} = 1$ if $g_j - g_i$ is in D, and $a_{ij} = -1$ otherwise. Finally, let

$$H = \begin{bmatrix} 1 & \mathbf{1}' \\ \mathbf{1} & A \end{bmatrix}.$$

REMARK. The constructions of Paley essentially involve difference set methods. Difference sets are discussed at length by Ryser (1963), Hall (1967), Storer (1967), and Baumert (1971). They have been generalized to supplementary difference sets (see J. Wallis (1972a)). Difference sets and supplementary difference sets have been used by many authors in the construction of Hadamard matrices; see, for example, Brauer (1953), Turyn (1965), Johnsen (1966), Szekeres (1971), Wallis and Whiteman (1972), Mann and McFarland (1973), Spence (1975), Dillon (1976), among others. Difference set methods have been useful in the construction of Williamson-type matrices.

Dade and Goldberg (1959) used permutation groups to construct Hadamard matrices. Unfortunately, Hall (1969) showed that as a consequence of the Feit-Thompson theorem, their methods do not give matrices of any orders not already found by Paley.

Bush (1971a), (1971b), (1971c) used finite projective planes to construct Hadamard matrices; however, as was pointed out by W. D. Wallis (1972c) and by Bush (1973), the actual prerequisite is the existence of a sufficiently large set of orthogonal Latin squares; see Theorem 3.5.

An important recent technique is that of J. Wallis (1976), who has used orthogonal designs and other methods to prove that given any positive integer q, there exists an integer r, dependent on q such that a Hadamard matrix exists for every order $2^t q$ where $t > r$. The Hadamard conjecture is that $r = 2$; the best known result is:

THEOREM 3.10 (J. Wallis (1976)). *Suppose q and t are natural numbers, and $t \geqslant [2 \log_2(q-3)]$. Then there is a Hadamard matrix of order $2^t q$.*

This means that for each q there is only a finite number of orders $2^t q$ for which a Hadamard matrix is not known.

Further techniques for the construction of Hadamard matrices can be found in Spence (1967), J. Wallis (1969a), (1969b), (1972c), and Whiteman (1971), (1976). Various attempts have been made to construct Hadamard matrices by computer; for the application of "backtrack" programming to this end the reader is referred to Hall and Knuth (1965).

4. Applications of Hadamard matrices in design theory. Research on the application of Hadamard matrices has been steadily growing over recent decades, and this is why we have written this outline of their uses. In particular, the application of Hadamard matrices to the theory and construction of experimental designs has been of considerable importance. It was realized very early that Hadamard matrices are equivalent to certain block designs; but other links to block designs have only been realized more recently. Similarly, the application of Hadamard matrices as weighing designs is natural and obvious, but the applications to other types of design such as group divisible designs and optimal resolution 3 designs have only been discovered recently. In this section we demonstrate how Hadamard matrices can be used to construct all these designs, and also 3-designs, Youden designs, factorial designs and orthogonal arrays; and we indicate the relationship to generalized Youden designs and to orthogonal F-designs.

4.1. *Hadamard matrices and balanced incomplete block designs.* Let T be a set of v treatments. A *t-design* with parameters v, b, r, k, λ_t on T is a block design consisting of b blocks each of size k such that (i) no treatment appears more than once in a block, (ii) every treatment appears in r blocks, (iii) every subset of size t appears in exactly λ_t blocks of the design. Several interesting applications of t-designs are given by Hedayat and John (1974). A 2-design is better known as a balanced incomplete block design (BIB design), and some authors refer to a 3-design as a doubly balanced incomplete block design. When $v = b$ the 2-design is called *symmetric*. It is useful to notice that a t-design is also an e-design for every $e < t$, so parameters $\lambda_1, \lambda_2, \cdots, \lambda_{t-1}$ are also defined for every t-design; $\lambda_1 = r$, and λ_2 is usually denoted simply by λ.

With each t-design we can associate a $v \times b$ matrix N, called the incidence matrix of the design, by letting $n_{ij} = 1$ if the ith treatment appears in the jth block and zero otherwise. It is clear that N uniquely determines the design. Two designs having the same parameters, with incidence matrices N_1 and N_2, are said to be isomorphic if there exist permutation matrices P and Q such that $N_1 = PN_2Q$.

If H is a normalized Hadamard matrix of order $4t$ then the matrix obtained by deleting the first row and the first column of H will be called the *core* of H; if A is the core then

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ 1 & & & & & \\ \vdots & & & & & \\ & & A & & & \\ 1 & & & & & \end{bmatrix}.$$

THEOREM 4.1. *The existence of a Hadamard matrix of order* $4t$ *is equivalent to the existence of symmetric* BIB *designs with the parameters*

(i) $v = b = 4t - 1, r = k = 2t - 1, \lambda = t - 1$;
(ii) $v = b = 4t - 1, r = k = 2t, \lambda = t$.

PROOF. Assume H to be normalized Hadamard matrix of order $4t$ and let A be the core of H. Then $N_1 = \frac{1}{2}(J + A)$, $N_2 = \frac{1}{2}(J - A)$ are incidence matrices of BIB designs with the parameters (i) and (ii) respectively. It is clear that the above processes are reversible.

EXAMPLE 4.1. If we start with the normalized Hadamard matrix

$$H = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & + & + & - & + & - & - & - \\ + & - & + & + & - & + & - & - \\ + & - & - & + & + & - & + & - \\ + & - & - & - & + & + & - & + \\ + & + & - & - & - & + & + & - \\ + & - & + & - & - & - & + & + \\ + & + & - & + & - & - & - & + \end{bmatrix}$$

we obtain the incidence matrices

$$N_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Numbering the rows of N_1 as 0, 1, 2, 3, 4, 5, 6, we see that N_1 corresponds to the BIB designs with blocks

046, 015, 126, 023, 134, 245, 356,

while N_2 is the incidence matrix of the complementary design, $J - N_1$.

It is known that from the family of BIB designs with $v = b$ one can construct other BIB designs. Therefore one can relate more BIB designs to Hadamard matrices. For example,

COROLLARY 4.1. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of* BIB *designs with the parameters*

(i) $v = 2t - 1, b = 4t - 2, r = 2t - 2, k = t - 1, \lambda = t - 2$;
(ii) $v = 2t, b = 4t - 2, r = 2t - 1, k = t, \lambda = t - 1$;
(iii) $v = 2t - 1, b = 4t - 2, r = 2t, k = t, \lambda = t$.

PROOF. Let N_1 be as in the construction of Theorem 4.1. Rearrange (if necessary) rows of N_1 and bring it to the form

$$\begin{bmatrix} \mathbf{1} & N_3 \\ \mathbf{0} & N_4 \end{bmatrix}.$$

Then N_3 and N_4 are the incidence matrices for designs with the parameters in (i) and (ii) above. It can be easily shown that $N_5 = \frac{1}{2}(J - N_3)$ is the incidence matrix for a design with the parameters in (iii).

It is to be noted that from a given Hadamard matrix one may be able to produce several nonisomorphic BIB designs with the parameters specified in Theorem 4.1 and Corollary 4.1. For example, Bhat (1972b) has shown that each of the three nonisomorphic Hadamard matrices of order 20 gives rise to two nonisomorphic BIB (19, 19, 9, 9, 4) designs, and thus has constructed six nonisomorphic designs with these parameters. Recently, Singhi (1974) has shown that these are the only nonisomorphic solutions of BIB (19, 19, 9, 9, 4) designs.

Bhat and Shrikhande (1970) have proved that if for any t a solution exists for a BIB design with $v = b = 4t - 1$, $r = k = 2t - 1$, $\lambda = t - 1$, then the number of nonisomorphic solutions to a BIB design with $v = b = t2^{n+2} - 1$, $r = k = t2^{n+1} - 1$, $\lambda = t2^n - 1$ tends to infinity as n tends to infinity. Singhi (1975) has obtained some related results and has developed a technique for generating a large number of nonisomorphic BIB designs with $v = b = 4t - 1$, $r = k = 2t - 1$, $\lambda = t - 1$. For example, he has shown that there exist at least 57 nonisomorphic BIB designs with $v = b = 31$, $r = k = 15$, $\lambda = 7$.

THEOREM 4.2. *The existence of a Hadamard matrix of order* $4s$ *implies the existence of a* 3-*design with the parameters* $v = 4s$, $b = 8s - 2$, $r = 4s - 1$, $k = 2s$, $\lambda_3 = s - 1$.

PROOF. Let A be the core of a normalized Hadamard matrix of order $4s$. Then

$$N = \begin{bmatrix} \frac{1}{2}(J + A) & \frac{1}{2}(J - A) \\ \mathbf{1}' & \mathbf{0}' \end{bmatrix}$$

is the incidence matrix of the required 3-design.

A BIB design is called *quasi-symmetric* if every block intersects exactly one block in y treatments and intersects all other blocks in x treatments, for some constants x and y. The design formed by taking two complete replications of a symmetric BIB design is quasi-symmetric; Stanton and Kalbfleisch (1968) show that any other quasi-symmetric design must have the same parameters as the design in Theorem 4.2, and in fact, W. D. Wallis (1970b) shows that the 3-designs of Theorem 4.2 are the only quasi-symmetric designs with these parameters.

It should be mentioned that there do exist BIB designs with these parameters which are not 3-designs; Stanton and Mullin (1969) give examples in the case $s = 2$.

EXAMPLE 4.2. Consider the Hadamard matrix of order 8 which was used in Example 4.1. The corresponding 3-design of Theorem 4.2 has incidence matrix

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Recall that a *regular* Hadamard matrix is one in which the sum of the elements in any row equals a constant.

THEOREM 4.3. *The existence of a regular Hadamard matrix H of order $4n^2$ is equivalent to the existence of symmetric* BIB *designs with parameters*

(i) $v = b = 4n^2, r = k = 2n^2 - n, \lambda = n^2 - n,$
(ii) $v = b = 4n^2, r = k = 2n^2 + n, \lambda = n^2 + n.$

PROOF. It is easy to see that $\frac{1}{2}(J - H)$ and $\frac{1}{2}(J + H)$ are the incidence matrices of the required designs; and the construction is clearly reversible.

The various constructions for regular Hadamard matrices, outlined in Section 3.2, can be used to construct block designs via Theorem 4.3. For example, from Theorem 3.6 we have:

THEOREM 4.4. *If there is a Hadamard matrix of order $4s$, then there exist symmetric* BIB *designs with parameters*

(i) $v = b = 16s^2, r = k = 8s^2 - 2s, \lambda = 4s^2 - 2s,$
(ii) $v = b = 16s^2, r = k = 8s^2 + 2s, \lambda = 4s^2 + 2s.$

4.2 *Hadamard matrices and group divisible designs.* Various generalizations of BIB designs have been considered. We discuss one case—group divisible designs—which form a subclass of partially balanced incomplete block designs (PBIB) with two associate classes.

A group divisible design is a block design based on $v = mn$ treatments, consisting of b blocks of size k, $k < v$, in which each treatment appears r times, whose blocks and treatments can be rearranged so that the $v \times b$ incidence matrix N of the design satisfies

$$NN' = \begin{bmatrix} G & D & \cdot & \cdot & \cdot & D \\ D & G & \cdot & \cdot & \cdot & D \\ \cdot & \cdot & \cdot & & & \\ & & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ D & D & \cdot & \cdot & \cdot & G \end{bmatrix}$$

where G is the $n \times n$ matrix $(r - \lambda_1)I + \lambda_1 J$ and D is the $n \times n$ matrix $\lambda_2 J$, for some nonnegative integers λ_1 and λ_2. Two treatments either occur together in λ_1 blocks, in which case they are called first associates, or else they occur together in λ_2 blocks, and are called second associates. (Observe that if $\lambda_1 = \lambda_2$ then we have a BIB design.) Group divisible designs are commonly classified into three subfamilies:

singular, where $r - \lambda_1 = 0$;
semiregular, where $r - \lambda_1 > 0$ but $rk - v\lambda_2 = 0$;
regular, where $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$.

Hadamard matrices can be utilized in the construction of group divisible designs, as is seen in the following three theorems which are based on some results of Bush (1977b).

THEOREM 4.5. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of group divisible designs with parameters*

(i) $v = 12t$, $b = 16t - 4$, $r = 8t - 2$, $k = 6t$; $m = 3$, $n = 4t$, $\lambda_1 = 4t - 2$, $\lambda_2 = 4t - 1$

(ii) $v = 16t - 4$, $b = 12t$, $r = 6t$, $k = 8t - 2$; $m = 4t - 1$, $n = 4$, $\lambda_1 = 2t$, $\lambda_2 = 3t$.

PROOF. Let B be a normalized Hadamard matrix of order $4t$ whose first column is deleted. Then $N_1 = (J + K)/2$ and $N_2 = N_1'$ are the incidence matrices for designs with parameters (i) and (ii) respectively, where

$$K = \begin{bmatrix} B & B & -B & -B \\ B & -B & B & -B \\ B & -B & -B & B \end{bmatrix}.$$

Two treatments in design (i) are first associated if they are within the first $4t$, the second $4t$ or the last $4t$ row indices of N_1. Otherwise they are second associates. In design (ii) the first associates are those groups of treatments (columnwise in N_1) with indices i, $i + 4t - 1$, $i + 8t - 2$, $i + 12t - 3$, for various values of i. The theorem follows from the fact that the block matrices in N_1 are the incidence matrices of BIB designs with parameters listed in Theorem 4.1.

THEOREM 4.6. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of group divisible designs with parameters*

(i) $v = 16t - 4, b = 16t - 4, r = 8t - 3, k = 8t - 3; m = 4, n = 4t - 1, \lambda_1 = 4t - 3, \lambda_2 = 4t - 2$

(ii) $v = 16t - 4$, $b = 16t - 4$, $r = 8t - 1$, $k = 8t - 1$; $m = 4$, $n = 4t - 1$, $\lambda_1 = 4t - 1, \lambda_2 = 4t$.

PROOF. Let A be the core of a Hadamard matrix of order $4t$. Then $N_1 = (J + L)/2$ and $N_2 = (J - L)/2$ form the incidence matrices for designs with parameters (i) and (ii) respectively, where

$$L = \begin{bmatrix} -A & A & A & A \\ A & -A & A & A \\ A & A & -A & A \\ A & A & A & -A \end{bmatrix}.$$

The association schemes of these designs are defined similarly to those of design (i) in Theorem 4.5.

THEOREM 4.7. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of a group divisible design with parameters*

$$v = 4t, b = 8t - 4, r = 4t - 2, k = 2t;$$
$$m = 2, n = 2t, \lambda_1 = 2t - 2, \lambda_2 = 2t - 1.$$

PROOF. Let M be the incidence matrix of a design with parameters (ii) as in Corollary 4.1 based on a Hadamard matrix of order $4t$. Then

$$N = \begin{bmatrix} M & M \\ M & J - M \end{bmatrix}$$

is the incidence matrix of the required design. The association schemes are defined according to the partitioned form of N.

Bush (1977a) shows that nonisomorphic solutions to a family of semiregular group divisible designs can be given if there exists a Hadamard matrix of order $4t$, $t > 1$. His result is:

THEOREM 4.8. *If a Hadamard matrix of order* $4t$, $t > 1$, *exists, then there will be always multiple nonisomorphic solutions to the semiregular group divisible design with parameters*

$$v = 4t - 2, b = 4t, r = 2t, k = 2t - 1; m = 2t - 1, n = 2, \lambda_1 = 0, \lambda_2 = t.$$

PROOF. Construct the incidence matrix of a design with parameters (i) as in Theorem 4.1 based on a Hadamard matrix of order $4t$. Select arbitrarily any $2t - 1$ rows. Denote these rows as $1, 2, \cdots, 2t - 1$. With this enumeration, supply an additional block containing $1, 2, \cdots, 2t - 1$. Then in the ith row of our numbered subsets of rows put in i whenever a one is encountered in the incidence matrix, but put in $2t - 1 + i$ whenever the entry is zero. The result will be a design with the required parameters. By an elegant argument Bush (1977a) shows that at least two nonisomorphic solutions can be obtained by different selections of rows even though the same Hadamard matrix is used.

4.3. *Hadamard matrices and Youden designs.* Let T be a set of v treatments. A $k \times v$ Youden design on T is a $k \times v$ array filled out with the elements of T with the properties that every row of the array is a permutation of the set T (i.e., a randomized complete block in the usual terminology of statistical design), and the array is a BIB design with respect to the columns.

It is well known that the existence of a $k \times v$ Youden design is equivalent to the existence of a symmetric BIB design based on v treatments in blocks of size k; an easy algorithm for converting symmetric BIB designs to Youden designs is given in Hartley and Smith (1948). Thus it follows from Theorems 4.1 and 4.4 that there are Youden designs of sizes $(2t - 1) \times (4t - 1)$, $(2t) \times (4t - 1)$ and $(8t^2 - 2t) \times (16t^2)$ whenever there is a Hadamard matrix of order $4t$. Similarly, the existence of a regular Hadamard matrix of order $4n^2$ implies the existence of $(2n^2 - n) \times (4n^2)$ and $(2n^2 + n) \times 4n^2$ Youden designs.

EXAMPLE 4.3. The following two Youden designs are constructed based on N_1 and N_2 in Example 4.1.

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 6 & 0 & 1 & 2 & 3 & 4 & 5 \end{matrix}, \qquad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{matrix}.$$

Kiefer (1975a), (1975b) has generalized the concept of Latin squares and Youden designs to generalized Youden designs as follows: A $b_1 \times b_2$ array is said to be a generalized Youden design based on v treatments if the array is a generalized balanced block design row-wise and column-wise. Here by a generalized balanced block design consisting of b blocks of size k each and v treatments (k may exceed v) we mean a block design in which a treatment may appear more than once per block; if treatment i appears n_{ij} times in block j, then there are constants r and λ such that

$$\Sigma_{j=1}^{b} n_{ij} = r, \qquad \Sigma_{j=1}^{b} n_{ij} n_{hj} = \lambda$$

for all h and i, $h \neq i$, and that for all i and j

$$|n_{ij} - k/v| < 1.$$

Such a generalized Youden design will be denoted by GYD (v, b_1, b_2). A GYD is said to be regular if $b_1 \equiv 0(\text{mod } v)$ or $b_2 \equiv 0(\text{mod } v)$; otherwise it is called nonregular. Youden designs and Latin squares are examples of regular generalized Youden designs. Kiefer (1975a) has proved that, under the usual linear additive model for two-way elimination of heterogeneity, a GYD (if it exists) is A-, D-, and E-optimal in the regular case. A nonregular GYD is A- and E-optimal; a nonregular GYD is D-optimal unless $v = 4$.

The basic ingredients for the construction of generalized Youden designs are Latin squares and BIB designs. Indeed, the BIB designs associated with Hadamard matrices can be directly utilized for the construction of these designs. Kiefer (1975b), among other results, has shown that the existence of a Hadamard matrix of order $4t$ implies the existence of the following series of generalized Youden designs:

$$v = 4t, b_i = 2t(4t - 1)(2f_i - 1), i = 1, 2, f_1 > 0, f_2 \geqslant 2.$$

For the construction of this series of designs and other series based on Hadamard matrices we refer the reader to Kiefer (1975b).

4.4. *Hadamard matrices and fractional factorial designs.* Hadamard matrices are intimately connected to factorial experiments in which each factor is at two levels. Plackett and Burman (1946) utilized Hadamard matrices for the construction of optimum multifactorial experiments. Other statisticians have used Hadamard matrices for a variety of experiments under a variety of optimality criteria. Applications of Hadamard matrices to the area of optimal regression theory have been noticed recently by workers in the area of optimal design theory. A complete treatment of the subject can be found in Hedayat (1977). We now survey the key results:

Let t be an integer. A fractional factorial design is said to be of resolution $2t + 1$ if it satisfies the condition that under the usual model all effects of order t or less are estimable whenever all effects of order higher than t are assumed to be zero. A fractional factorial design is said to be of resolution $2t$ if all effects of order $t - 1$ or less are estimable whenever all effects of order $t + 1$ or higher are assumed to be zero. A fractional factorial design is said to be saturated if the number of observations (runs) is equal to the number of parameters in the model to be estimated. A fractional factorial design is said to be orthogonal if the covariance between any two estimable effects is zero.

THEOREM 4.9. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of an orthogonal saturated D-optimal resolution* 3 *fractional factorial design for* $4t - 1$ *factors each of two levels.*

PROOF. Let H be a Hadamard matrix of order $4t$. Seminormalize H and delete the first column. Denote the remaining $4t \times (4t - 1)$ matrix by $\bar{H}$. Then $F = \frac{1}{2}(\bar{H} + J)$ is a desired fraction if we consider the rows of F as the treatment combinations from a $4t - 1$ factorial each at two levels denoted by 0 and 1.

THEOREM 4.10. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of a resolution* 4 *design with* $8t$ *runs, an experiment on* $4t$ *factors each at two levels.*

PROOF. Let H be a seminormalized Hadamard matrix of order $4t$. Let $4t$ treatment combinations be the rows of $(H + J)/2$ and additional $4t$ observations be $(J - H)/2$.

REMARK. It is known that the number of observations in the design constructed by Theorem 4.10 is minimum.

THEOREM 4.11. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of an orthogonal saturated resolution* 3 *fractional factorial design based on* $8t - 4$ *factors each at two levels and one factor at four levels.*

PROOF. Let $\bar{H}$ be a normalized Hadamard matrix of order $4t$ whose first column is deleted. Write $\bar{H}$ as $\bar{H} = [C_1 : C_2]$ where C_1 is the first column of $\bar{H}$. Then

$$D = \begin{bmatrix} C_1 & C_2 & C_2 \\ 3C_1 & C_2 & -C_2 \end{bmatrix}$$

is the required design where the columns of D correspond to factors and the rows correspond to treatment combinations.

It is to be noted that one can collapse the four-level factor to three levels and obtain a fractional factorial design equivalent to the design of Margolin (1968). Further information regarding such designs are given in Dey and Ramakrishna (1977).

THEOREM 4.12. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of a D-optimal design with* $4t$ *runs,* $(x_{i1}, x_{i2}, \cdots, x_{i4t-1})$, $i = 1, 2, \cdots, 4t$, *to fit the first order model in* $4t - 1$ *variables.*

$$E(Y) = \theta_0 + \theta_1 x_1 + \cdots + \theta_{4t-1} x_{4t-1}$$

assuming that the design point coordinates x_{ij} *all lie in the range* $[-1, 1]$.

PROOF. Let H be a Hadamard matrix of order $4t$. Seminormalize H and delete the first column. The rows of the remaining matrix when considered as the points of the design have the stated property.

REMARK. One may consider Theorem 4.9 as a corollary to Theorem 4.12.

Because of its symmetry and uniformity properties, the vertex-set V_d of a d-dimensional cuboctahedron (the difference body of a regular d-simplex) is useful as an experimental design for the exploration of a response over a spherical region. Doehlert and Klee (1972) have studied the problem of coordinating the containing space so as to minimize the number of levels at which the controllable variables are required to appear. The authors have used Hadamard matrices and certain other combinatorial configurations for reducing the number of levels for each controllable variable. Further work on these problems has been done by McCarthy, Stanton and Vanstone (1976). In this context it should be pointed out that the problem of constructing a Hadamard matrix of order $n + 1$ is equivalent to the problem of constructing an n-dimensional regular simplex whose vertices are a subset of the vertices of an n-cube (see, for example, Dedo (1968)).

4.5. *Hadamard matrices and optimal weighing designs.* Suppose it is required to determine the weights of p objects using a chemical balance (two-pan balance) and standard weights; n weighings may be made. In each weighing there are two decisions to be made:

(i) should a given object be included in the weighing or not;

(ii) in which pan should it be weighed? Assume the objects to be numbered $1, 2, \cdots, p$; and define x_{ij} by

$x_{ij} = +1$ if object j is to be placed in the left-hand pan during weighing i;

$x_{ij} = -1$ if object j is to be placed in the right-hand pan during the weighing i;

$x_{ij} = 0$ if object j is to be omitted from weighing i.

Then the $n \times p$ matrix $X = (x_{ij})$ completely characterizes the weighing experiment.

Let us write $w_1, w_2, \cdots, w_p$ for the true weights of the p objects, and $y_1, y_2, \cdots, y_n$ for the results of the n weighings (so that the readings indicate that the weight of the left-hand pan exceeds that of the right-hand pan by y_i in the weighing of i); denote the column vectors of w's and y's by $\mathbf{W}$ and $\mathbf{Y}$ respectively. Then the readings can be represented by the linear model

$$\mathbf{Y} = X\mathbf{W} + \mathbf{e},$$

where $\mathbf{e}$ is the column vector of $e_1, e_2, \cdots, e_n$, and e_i is the error between observed and expected readings. We assume that $\mathbf{e}$ is a random vector distributed with mean zero and covariance matrix $\sigma^2 I$. This is a reasonable assumption in the case where the objects to be weighed have small mass compared to the mass of the balance. Under these assumptions the best linear unbiased estimator of W is $\hat{W} = (X'X)^{-1}X'\mathbf{Y}$ with covariance of $\hat{W} = \sigma^2(X'X)^{-1}$. It has been shown by Hotelling (1944) that for any weighing design the variance of $\hat{w}_i$ cannot be less than σ^2/n.

Therefore, we shall call a weighing design X *optimal* if it estimates each of the weights with this minimum variance, σ^2/n. By Proposition 1′ of Kiefer (1975a), an optimal weighing design in our sense is actually optimal with respect to a very general class of criteria. It can be shown that X is optimal if and only if $X'X = nI$. This means that a chemical balance weighing design X is optimal if it is an $n \times p$ matrix of ± 1 whose columns are orthogonal. Thus we have:

THEOREM 4.13. *Any $p(\leqslant n)$ columns of a Hadamard matrix of order n constitute an $n \times p$ optimal chemical balance weighing design.*

An optimal weighing design is clearly A-, D-, and E-optimal in the following sense. A weighing design X is said to be A-optimal if the trace of $(X'X)^{-1}$ is minimum, D-optimal if the determinant of $(X'X)^{-1}$ is minimum, and E-optimal if the maximum eigenvalue of $(X'X)^{-1}$ is minimum among the class of all $n \times p$ weighing designs and for the model of response specified above. If the balance has not been corrected for the bias we may assume the bias to be one of the p objects, say the first one.

Hadamard matrices can also be used for weighing objects using spring balance (one-pan balance). The spring balance problem is different from the chemical balance problem in that the elements x_{ij} of the matrix X are restricted to the values of 0 and 1. In this case no design exists with Var of $\hat{w}_i = \sigma^2/n$ because $X'X$ never becomes nI. However, a D-optimal spring balance design can be constructed for $p = n - 1$ objects using $n - 1$ measurements if a Hadamard matrix of order $4n$ exists. The procedure is easy. Let $X = N_2$, the incidence matrix of the BIB design in (ii) of Theorem 4.1. In this case the variance of each estimated weight is $4(n - 1)\sigma^2/n$. Thus we have:

THEOREM 4.14. *The existence of a Hadamard matrix of order n implies the existence of a saturated D-optimal spring balance design for $n - 1$ objects using $n - 1$ weighings.*

The weighing design

$$X = \begin{bmatrix} 1 & \mathbf{1}' \\ \mathbf{1} & J - N_2 \end{bmatrix}$$

can be used as a biased spring balance design, where the bias corresponds to the first object. With such a design the variance of estimated weights of objects $2, 3, \cdots, n$ will be $4\sigma^2/n$. This is the minimum possible variance as has been shown by Moriguti (1954). Therefore we have:

THEOREM 4.15. *The existence of a Hadamard matrix of order n implies the existence of a saturated A-optimal biased spring balance design for $n - 1$ objects using n weighings.*

In Theorems 4.14 and 4.15 the term *saturated* implies that in the usual analysis of variance no degrees of freedom will be left for the errors. For a more detailed study of optimal weighing designs, the reader should consult Mood (1946), Raghavarao (1971) and Banerjee (1975).

It is sometimes not possible to weigh all the objects simultaneously in one weighing. Accordingly one asks about weighing designs in which exactly k objects are weighed each time. In the case of a design with as many weighings as objects, so that the matrix is square, one asks for a matrix X of order p with exactly k entries ± 1 per row and the rest zero, such that $X'X$ is diagonal. In the case of an unbiased balance it is best to have exactly k entries ± 1 per column, so one seeks a matrix X with entries 0, -1 and $+1$, which satisfies

$$X'X = kI.$$

It has been conjectured, see J. Wallis (1972b), that such a matrix exists whenever $0 \leqslant k \leqslant p$. Considerable work has been done on this conjecture recently; for a survey of the results up to 1974 and a bibliography, see Geramita and Wallis (1974). Hadamard matrices are important in the constructions.

Federer, Hedayat, Lowe and Raghavarao (1976) found an interesting application of spring balance weighing designs in crop experiments. Forage crop researchers determine the proportion of legume, weed, and grass contents of hay by sampling and hand separation or by visual estimates of the relative proportions. The sampling and hand separation method is costly and time consuming and both procedures are subject to biases. A method employing spring balances weighing design theory is presented by the above authors as an alternative to the presently used methods.

In closing this section it should be mentioned that the designs used in the weighing problems are applicable to any problem of measurements, in which the measure of a combination is a known linear function of the separate measures with numerically equal coefficient and a homoscedastic error term.

4.6. *Hadamard matrices and orthogonal arrays.* An *orthogonal array* of size N, with k constraints, s levels, strength t and index λ is an $N \times k$ array whose entries are drawn from a set S of s objects, with the following property. Whenever a subarray is formed by deleting all but t of the columns, the N rows of the remaining $N \times t$ array contain each of the s^t possible row vectors of length t over S precisely λ times in its rows.

To avoid triviality, it is assumed that $s \geqslant 2$. The array is referred to as an *OA* (N, k, s, t, λ). These parameters are clearly redundant, as N must equal λs^t.

EXAMPLE 4.4. Here is an *OA* (9, 4, 3, 2, 1):

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{bmatrix}.$$

THEOREM 4.16. *The existence of a Hadamard matrix of order* $4t$ *is equivalent to the existence of*

(i) *an* $OA(4t, 4t - 1, 2, 2, t)$;
(ii) *an* $OA(8t, 4t, 2, 3, t)$.

PROOF. (i) Suppose H is a seminormalized Hadamard matrix of order $4t$. Delete the first column of H. The remaining matrix is the desired array. Conversely, given an OA $(4t, 4t - 1, 2, 2, t)$, relabel its elements as $+1$ and -1. Then the above procedure can be reversed.

(ii) If H is a Hadamard matrix of order $4t$ then

$$A = \begin{bmatrix} H \\ -H \end{bmatrix}$$

is an array with the desired parameters.

EXAMPLE 4.5. Consider the Hadamard matrix

$$\begin{bmatrix} - & - & + & + \\ + & - & - & + \\ + & + & + & + \\ + & - & + & - \end{bmatrix}.$$

(i) If the first row is negated so as to seminormalize the matrix and the first column is deleted, one obtains the $OA(4, 3, 2, 2, 1)$

$$\begin{bmatrix} + & - & - \\ - & - & + \\ + & + & + \\ - & + & - \end{bmatrix}.$$

(ii) The $OA(8, 4, 2, 3, 1)$ obtained from the original matrix is

$$\begin{bmatrix} - & - & + & + \\ + & - & - & + \\ + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ - & + & + & - \\ - & - & - & - \\ - & + & - & + \end{bmatrix}.$$

4.7. *Hadamard matrices and orthogonal F-square designs.* An *F-square design* of order n and frequency vector $(\lambda_1, \cdots, \lambda_m)$ on a set $\{a_1, \cdots, a_m\}$ is an $n \times n$ array such that a_i appears λ_i times in each row and column of the array. Such a design will be denoted by $F(n; \lambda_1, \cdots, \lambda_m)$. Note that $\lambda_1 + \cdots + \lambda_m = n$, and that an $F(n; 1, 1, \cdots, 1)$ is a Latin square of order n. If F_1 and F_2 are two F-square designs of order n on $\{a_1, \cdots, a_r\}$ and $\{b_1, \cdots, b_s\}$ with frequency vectors $(\lambda_1, \cdots, \lambda_r)$ and $(\mu_1, \cdots, \mu_s)$ respectively, then we say F_1 is orthogonal to F_2 if, upon superposition of F_2 on F_1, a_i is superposed upon b_j exactly $\lambda_i \mu_j$ times. Let $\mathcal{F}$ be a set of t pairwise orthogonal F-square designs each based on a set consisting of k elements. Then from Hedayat, Raghavarao and Seiden (1975) we know that $t \leqslant (n - 1)^2/(k - 1)$. $\mathcal{F}$ is said to be a complete set if $t = (n - 1)^2$

$/(k-1)$. For more information on the theory and applications of these designs see Hedayat and Seiden (1970), Hedayat, Raghavarao and Seiden (1975), Federer (1977) and Kirton and Seberry (1978).

THEOREM 4.17. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of a complete set of* $(4t-1)^2$ *orthogonal* $F(4t;\, 2t, 2t)$ *designs.*

PROOF. Let H be a normalized Hadamard matrix of order $4t$. Let C_i be the ith column of H. For each pair C_i and C_j, $i, j = 2, \cdots, 4t$, construct the array F_{ij} by putting $+$ in the (s, t) entry of F_i, if the product of the sth entry of C_i and the tth entry of C_j is $+$ and putting $-$ otherwise. Then it is not difficult to verify that the resulting $(4t-1)^2$ arrays form a complete set of orthogonal $F(4t;\, 2t, 2t)$ designs.

EXAMPLE 4.6. Let

$$H = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}.$$

Then we have:

$$F_{22} = \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}, \quad F_{23} = \begin{matrix} + & + & - & - \\ - & - & + & + \\ + & + & - & - \\ - & - & + & + \end{matrix}, \quad F_{24} = \begin{matrix} + & - & - & + \\ - & + & + & - \\ + & - & - & + \\ - & + & + & - \end{matrix},$$

$$F_{32} = \begin{matrix} + & - & + & - \\ + & - & + & - \\ - & + & - & + \\ - & + & - & + \end{matrix}, \quad F_{33} = \begin{matrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{matrix}, \quad F_{34} = \begin{matrix} + & - & - & + \\ + & - & - & + \\ - & + & + & - \\ - & + & + & - \end{matrix},$$

$$F_{42} = \begin{matrix} + & - & + & - \\ - & + & - & + \\ - & + & - & + \\ + & - & + & - \end{matrix}, \quad F_{43} = \begin{matrix} + & + & - & - \\ - & - & + & + \\ - & - & + & + \\ + & + & - & - \end{matrix}, \quad F_{44} = \begin{matrix} + & - & - & + \\ - & + & + & - \\ - & + & + & - \\ + & - & - & + \end{matrix}.$$

5. Other applications of Hadamard matrices. It is only recently that the applicability of Hadamard matrices outside of design theory has been recognized. In this section we outline the applications in sets of pairwise independent random variables, binary codes, information processing and maximum determinant problems.

5.1. *Hadamard matrices and a maximal set of pairwise independent random variables.* Suppose $R = \{X_1, \cdots, X_v\}$ is a set of v mutually independent random variables on a set of n points. Then it is well known that

$$v \leqslant \log_2 n. \tag{5.1}$$

If we relax the condition of mutual independence and assume only that the elements of R are pairwise independent, then clearly the constraint (5.1) can be relaxed to

$$v \leqslant n - 1. \tag{5.2}$$

If $v = n - 1$ then we know something about the elements of R; the X_i can take precisely two distinct values with positive probability. For each $n > 3$ one can always construct a maximal set of pairwise independent random variables if we put no additional restriction on X_i. Suppose we insist that each X_i should assign the measure n^{-1} to each point (uniform measures); then the problem of constructing a maximal set of such pairwise independent random variables is not completely solved because of the following theorem which is due to Lancaster (1965).

THEOREM 5.1. *The existence of a maximal set of pairwise independent random variables on n points (such that each of its members assigns the measure n^{-1} to each point) is equivalent to the existence of a Hadamard matrix of order n.*

PROOF. Let H be a seminormalized Hadamard matrix of order n. Delete the first column of H and denote the remaining matrix by $\bar{H}$. Now associate each of the given n points with one of the rows of $\bar{H}$, and associate the $n - 1$ random variables with $n - 1$ columns of $\bar{H}$, so that the jth random variable takes the value x_{ij} at point i, where x_{ij} is the entry in the (i, j) position in $\bar{H}$. Clearly these $n - 1$ random variables are pairwise independent because of the orthogonality of columns of $\bar{H}$ and the uniform measure which each random variable assigns to the n points associated with the rows of $\bar{H}$. Observe that each random variable takes two distinct values, 1 and -1. It is clear that this process is reversible.

EXAMPLE 5.1. Using the Hadamard matrix

$$H = \begin{bmatrix} + & + & - & + \\ + & - & - & - \\ + & + & + & - \\ + & - & + & + \end{bmatrix},$$

one obtains

$$\bar{H} = \begin{bmatrix} + & - & + \\ - & - & - \\ + & + & - \\ - & + & + \end{bmatrix}.$$

The three variables X_1, X_2 and X_3, defined by $\bar{H}$, are as follows:

X_1 takes value 1 on points 1 and 3, value -1 on points 2 and 4;
X_2 takes value 1 on points 3 and 4, value -1 on points 1 and 2;
X_3 takes value 1 on points 1 and 4, value -1 on points 2 and 3.

The variables are pairwise orthogonal.

REMARK. Theorem 5.1 shows that a maximal set of pairwise independent random variables with uniform measures on n points can exist only if $n \equiv 0 \pmod 4$.

5.2. *Hadamard matrices and binary codes.* We commence this section with a brief introduction to coding theory. For further information, the reader is referred to the recent books of Blake and Mullin (1975), MacWilliams and Sloane (1977), and the survey by Sloane (1972).

In transmitting information one represents the pieces of information by sequences of symbols. The process of representation is called *coding*, and the symbols are called *code symbols*. A sequence of code symbols is called a *code word*. This terminology is sufficiently broad to cover the case of ordinary conversation in ordinary human languages, but we are especially concerned with the types of codes which are needed when electronic transmission is used. In this case it is usual to require that the set of code symbols be small. In particular a *binary code* (2 symbols) is very applicable because of its natural correspondence to the two states in which a switch can be: open or closed. Code words are of constant length; this obviates the need for a special symbol to represent the break between words.

We refer to the medium through which communication will take place as a (communication) *channel*. Often the channel will contain random signals which can corrupt the message: we say the channel is *noisy*. For example, in radio communication over a long distance, the channel is often noisy, and this is very true of communication with satellites and spacecraft. For this reason it is desirable to use a code in which the various words are sufficiently different so as to make it unlikely that one word will be corrupted into another.

A *block code* is one in which all code words have the same length n. Let F be a set with q distinct symbols which is called the *alphabet*. In practice q is generally 2 and $F = \{0, 1\}$. In most of the theory one takes $q = p^r$ (p prime) and $F = GF(q)$, the finite field of order q. Using the q symbols of F one forms all n-tuples, i.e., F^n, and calls these n-tuples *words* and n the *word length*. Note that F^n is a vector space over F if F is a finite field. A block code of q symbols is simply a subset of F^n. If $F = GF(q)$ then a block code $\mathcal{C}$ over F is called a linear code or (n, k)-code over F if $\mathcal{C}$ is a k-dimensional linear subspace of F^n. Given any two code words, the *Hamming distance* between two code words is defined as the number of components in which the words disagree. A distance d code is one in which the minimum of all the Hamming distances between the words is at least d. The error-detecting and error-correcting capacity of a code are directly related to d as we shall see shortly.

EXAMPLE 5.2. Consider the code with four words

$$\begin{aligned} A &= 111110 \\ B &= 111001 \\ C &= 000110 \\ D &= 000001 \end{aligned} \quad . \tag{5.3}$$

This is a block code of word length 6. The distances between A and D, and B and C, are 6; all other distances are 3. So it is a distance 3 code.

Suppose a word is sent in the code (5.3) and is received with one or two errors. Since the distance is 3, the word will not equal any code word. Thus one can detect as many as two errors in a code word. More generally, a distance d code is a $(d - 1)$-error detecting code.

Suppose the word 111111 is received. If the probability of error is sufficiently small that the chance of two errors is negligible, then the word which was transmitted must have been 111110, or A; if any one error is made then one can similarly work out the original code word. We say (5.3) is 1-error correcting; an e-error correcting code is one in which the assumption that no more than e errors are made per word enables one to decode any received message. A distance d code is $(\frac{1}{2})(d-1)$-error correcting.

These two results are best-possible. For example, the code (5.3) is not capable of detecting certain 3-error words (for example, A could be transmitted as B), nor is it 2-error detecting (either A or B could be transmitted as 111111). A similar observation applies to every distance d code which is not distance $d+1$. So codes with large distance are desirable. An $(n, M, d; q)$ code means a set of M code words of length n with q symbols and Hamming distance d. An $(n, M, d; q)$ code is *optimal* if M is as large as possible for given n, d and q. Plotkin (1960) obtained the following bounds for binary codes:

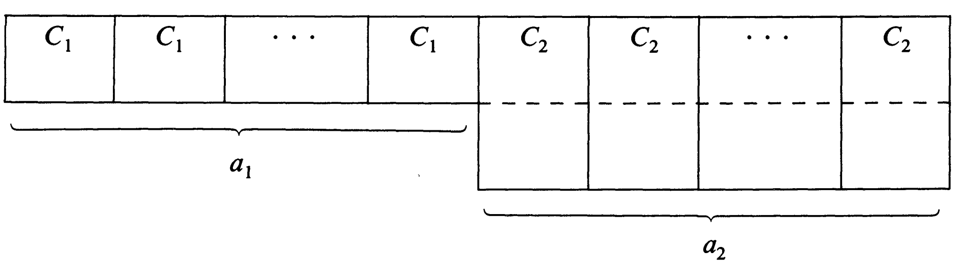
$$M \leqslant 2\left[\frac{d}{2d-n}\right] \quad \text{if } d \text{ is even and } d \leqslant n < 2d, \tag{5.4}$$

$$M \leqslant 2n \quad \text{if } d \text{ is even and } n = 2d, \tag{5.5}$$

$$M \leqslant 2\left[\frac{d+1}{2d+1-n}\right] \quad \text{if } d \text{ is odd and } d \leqslant n < 2d+1, \tag{5.6}$$

$$M \leqslant 2n+2 \quad \text{if } d \text{ is odd and } n = 2d+1. \tag{5.7}$$

Levenshtein (1964) proved that the Plotkin bounds are *tight*, in the sense that there exist binary codes which meet these bounds, provided that enough Hadamard matrices exist. It is well known (see, for example, MacWilliams and Sloane (1977)) that the existence of codes which meet bounds (5.4) and (5.5) implies the existence of codes which meet bounds (5.6) and (5.7). Therefore, in constructing codes which meet the Plotkin bounds we may assume that d is even. First, we need the following composition technique. Suppose we have an $(n_1, M_1, d_1; 2)$ code C_1 and an $(n_2, M_2, d_2; 2)$ code C_2, $M_2 \geqslant M_1$. We can construct an $(n, M_1, d; 2)$ code C with $n = a_1n_1 + a_2n_2$, and $d \geqslant a_1d_1 + a_2d_2$ for any values of a_1 and a_2 from C_1 and C_2 as follows. We paste a_1 copies of C_1 side by side, followed by a_2 copies of C_2 to obtain



Now omit the last $M_2 - M_1$ rows of C_2 to obtain the desired code. We denote this code by $a_1C_1 \oplus a_2C_2$.

We are now ready to construct binary codes which meet the Plotkin bounds. From Bose and Shrikhande (1959) and Levenshtein (1964) we can conclude the following results:

THEOREM 5.2. *The existence of a Hadamard matrix of order* $4t$ *implies the existence of the following optimal codes*: $(4t, 8t, 2t; 2)$, $(4t-1, 4t, 2t; 2)$, $(4t-1, 8t, 2t-1; 2)$ *and* $(4t-2, 2t, 2t; 2)$.

PROOF. Let H be a normalized Hadamard matrix of order $4t$. Then the $8t$ rows of the two matrices $W_{4t}^{(1)} = \frac{1}{2}(J+H)$ and $W_{4t}^{(2)} = \frac{1}{2}(J-H)$ form a $(4t, 8t, 2t; 2)$ code. To construct other codes form the matrix K by deleting the first column of H, and the matrix L by deleting all rows of K which start with $+1$ and deleting the first column from the resulting matrix. Then the $4t$ rows of $W_{4t}^{(3)} = \frac{1}{2}(J+K)$, the $8t$ rows of $W_{4t}^{(3)}$ and $W_{4t}^{(4)} = \frac{1}{2}(J-K)$, the $2t$ rows of $W_{4t}^{(5)} = \frac{1}{2}(J+L)$ form the remaining codes respectively.

EXAMPLE 5.3. From the Hadamard matrix H of order $4t = 12$,

$$H = \begin{bmatrix} + & + & + & + & + & + & + & + & + & + & + & + \\ + & - & + & - & + & + & + & - & - & - & + & - \\ + & - & - & + & - & + & + & + & - & - & - & + \\ + & + & - & - & + & - & + & + & + & - & - & - \\ + & - & + & - & - & + & - & + & + & + & - & - \\ + & - & - & + & - & - & + & - & + & + & + & - \\ + & - & - & - & + & - & - & + & - & + & + & + \\ + & + & - & - & - & + & - & - & + & - & + & + \\ + & + & + & - & - & - & + & - & - & + & - & + \\ + & + & + & + & - & - & - & + & - & - & + & - \\ + & - & + & + & + & - & - & - & + & - & - & + \\ + & + & - & + & + & + & - & - & - & + & - & - \end{bmatrix}$$

we can construct the following codes:

$$W_{12}^{(1)} = \begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

$$W_{12}^{(2)} = \begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}$$

(12, 24, 6; 2) code

$$W_{12}^{(3)} = \begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}$$

$$W_{12}^{(4)} = \begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}$$

(11, 24, 5; 2) code

$$W_{12}^{(3)} = \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

(11, 12, 6; 2) code

$$W_{12}^{(5)} = \begin{matrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{matrix}$$

(10, 6, 6; 2) code

We now analyze the bounds (5.4) and (5.5) for even values of d. The $(4t, 8t, 2t; 2)$ code constructed in the proof of Theorem 5.2 meets the bound (5.5) provided that a Hadamard matrix of order $4t$ exists. To construct codes via Hadamard matrices which meet the bound (5.4) we use the composition technique and the codes constructed by Theorem 5.2. Let the positive integers d and n be given such that d is even and $d \leqslant n < 2d$. We would like to construct an optimal $(n, M, d; 2)$ code such that $M = 2[d/(2d - n)]$. Define

$$r = \left[\frac{d}{2d-n}\right], \qquad a_1 = d(2r+1) - n(r+1), \qquad a_2 = rn - d(2r-1).$$

Then, using the notation introduced in the proof of Theorem 5.3, consider the code C, where:

$$C = \frac{a_1}{2} W_{4r}^{(5)} \oplus \frac{a_2}{2} W_{4r+4}^{(5)}, \qquad \text{if } n \text{ is even,}$$

$$C = a_1 W_{2r}^{(3)} \oplus \frac{a_2}{2} W_{4r+4}^{(5)}, \qquad \text{if } n \text{ is odd and } r \text{ is even,}$$

$$C = \frac{a_1}{2} W_{4r}^{(5)} \oplus a_2 W_{2r+2}^{(3)}, \qquad \text{if } n \text{ is odd and } r \text{ is odd.}$$

Then this code is optimal and meets the bound (5.4). Thus the existence of Hadamard matrices of order $2r$ (if r even), $2r + 2$ (if odd), $4r$, $4r + 4$ are sufficient for the existence of an optimal code which meets the bound (5.4).

EXAMPLE 5.4. Let $n = 27$ and $d = 16$. Here $r = 3$, $a_1 = 4$ and $a_2 = 1$. To construct an optimal (27, 6, 16; 2) code we need $W_{12}^{(5)}$ and $W_8^{(3)}$. We have already exhibited a $W_{12}^{(5)}$ in Example 5.3. Below is a $W_8^{(3)}$ constructed by the method of Theorem 5.3.

$$W_8^{(3)} = \begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}.$$

Thus the required code is $C = 2W_{12}^{(5)} \oplus W_8^{(3)} =$

$$\begin{array}{cccccccccc|cccccccccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{|ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{array}$$

Other codes can also be constructed via Hadamard matrices. One such series can be obtained by allowing the code words to be all the vectors in the linear span of $W_{4t}^{(3)}$ over $GF(2)$. If H_{4t} is a Paley type (Theorem 3.2), then it is known that the resulting code has dimension $2t$ and its minimum distance d is such that $d^2 - d + 1 \geqslant 4t$. For example, if we use the Hadamard matrix H_{24} of Theorem 3.2, then we obtain a (23, 4096, 7; 2) code which is essentially the well-known Golay perfect code. If H_{4t} is not a Paley type then unfortunately nothing is known about the dimension and the minimum distance of the code. Another series of codes can be obtained by taking the vectors in the linear span of $V = [I_{4t} : W_t^{(3)}]$ over $GF(2)$. But again, nothing is known about these codes.

The practicality of the codes derived from Hadamard matrices was demonstrated by the use of a (32, 64, 16; 2) code in the Mariner '69 telemetry system; see Posner (1968).

5.3. *Hadamard matrices and Barker sequences.* Suppose $x_1, x_2, \cdots, x_n$ is any sequence of complex numbers. Then the (aperiodic) correlation sequence

$c_1, c_2, \cdots, c_{n-1}$ is defined by

$$c_j = \Sigma_{i=1}^{n-j} x_i \bar{x}_{i+j}$$

where $\bar{x}$ denotes the complex conjugate of x and $i + j$ is reduced modulo n when necessary. (See Turyn (1968) for more details.)

EXAMPLE 5.5. The sequence $1, i, -1, -i$ gives rise to correlation sequence $-3i, -2, i$, as

$$c_1 = \Sigma_{i=1}^{3} x_i \bar{x}_{i+1} = -i - i - i = -3i,$$
$$c_2 = \Sigma_{i=1}^{2} x_i \bar{x}_{i+2} = -1 - 1 = -2,$$
$$c_3 = \Sigma_{i=1}^{1} x_i \bar{x}_{i+3} = 1.$$

The sequence i, i, i, i, i gives rise to correlation sequence 4, 3, 2, 1.

A sequence of elements $+1$ and -1, whose correlation sequence has small terms, can be used in digital communications theory to simulate white noise. In particular a sequence whose every c_j is ± 1 or 0 is called a *Barker sequence* and is the best possible sequence for this purpose.

EXAMPLE 5.6. The sequence

$$+, +, +, -$$

has correlation sequence

$$+, 0, -$$

so it is a Barker sequence of length 4. The sequence

$$+ \quad + \quad + \quad - \quad - \quad + \quad -$$

has correlation sequence

$$0, -1, 0, -1, 0, -1$$

so it is a Barker sequence of length 7.

Barker sequences of lengths 2, 3, 4, 5, 7, 11 and 13 are known. Turyn and Storer (1961) showed that any Barker sequence of length s longer than 13 can exist if and only if there is a circulant Hadamard matrix of order s, and in fact the Barker sequence could serve as the first row of the matrix.

It is clear, from the definition of circulant matrices in Section 3.3, that all circulant matrices are regular. As we noted in Section 3.2, the order of a regular Hadamard matrix must be a perfect square, so any later Barker sequences must have square order.

It has been proven that no further Barker sequence exists of length $s < 12{,}100$; so no circulant Hadamard matrix exists with order $< 12{,}100$, except for the order 4. Whether any exist at all is a matter for conjecture. Stanton and Mullin (1976) have proven the nonexistence of a related type of circulant weighing matrix, so a solution to the problem of the existence of Barker sequences may not be impossible.

5.4. *Hadamard matrices and Hadamard transform spectrometry.* Hadamard matrices have recently been found useful in spectrometry and pattern recognition, in the construction of masks. This is in fact a special application of the weighing designs which we mentioned in Section 4.4.

A spectrometer normally consists of a light separator, a small entrance slit through which a narrow beam of light is admitted, and the detecting and processing hardware. The narrowness of the slit means that the ratio of noise to signal can be high and so errors in observation can be significant.

In Hadamard transform spectrometry the separated light is sent on to a mask. Various parts of the mask will be clear, allowing the light to pass through; reflective (sending light to a secondary detector); or opaque. Let us represent clear, reflective and opaque by $+1$, -1 and 0 respectively. Then the configuration of the mask is represented by a sequence of elements $+1$, -1 and 0.

Suppose k measurements are to be made, and suppose it is convenient to measure the intensity of light at n points of the spectrum. Then the experiment will involve k masks, which can be thought of as $n \times k$ matrix of entries 1, 0 and -1. The efficiency of the experiment is the same as the efficiency of the matrix as a weighing design. The best systems of masks are thus derived from Hadamard matrices.

A discussion of the experimental-design features of Hadamard transform spectrometry, including the cases where n cannot be divisible by 4 or where the matrix can have no entries -1 (because only one detector is used) is given by Sloane and Harwit (1976). Discussion of the problems which arose can be found in Tai, Harwit and Sloane (1975). Decker (1973) gives a nontechnical introduction to the state of Hadamard transform spectrometry hardware as of the end of 1972. A complete survey will appear in Harwit and Sloane (1978).

5.5. *Hadamard matrices and maximum determinant problems.* Suppose $A = (a_{ij})$ is a real matrix of order n. Let

$$\begin{aligned} h(n) &= \max \det(A) \text{ subject to } a_{ij} = -1, 1, \\ f(n) &= \quad '' \quad '' \quad '' \quad '' \quad a_{ij} = 0, 1, \\ g(n) &= \quad '' \quad '' \quad '' \quad '' \quad a_{ij} = -1, 0, 1, \\ k(n) &= \quad '' \quad '' \quad '' \quad '' \quad 0 \leq a_{ij} \leq 1, \\ l(n) &= \quad '' \quad '' \quad '' \quad '' \quad -1 \leq a_{ij} \leq 1. \end{aligned}$$

Then it is easy to establish that:

$$h(n) = g(n) = k(n) = l(n) = 2^{n-1} f(n-1).$$

Therefore, the above five maximum determinant problems are equivalent. Moreover, we know that $|h(n)| \leq n^{n/2}$, with equality if and only if A is a Hadamard matrix of order n. Thus $|h(n)| < n^{n/2}$ if $n \not\equiv 0 (\text{mod } 4)$. The theory of Hadamard matrices deals with the above problems in $n \equiv 0 \pmod 4$). For other values of n not much is known in the literature. The interested reader on this and related topics is referred to Gilman (1931), Bellman (1943), Williamson (1946), Schmidt (1970), (1973), Payne (1973), Hedayat (1977), and Brenner and Cummings (1972). The last mentioned paper also contains a bibliography of papers on the case $n \not\equiv 0 \pmod 4$).

Here we list the first fourteen values of $h(n)$:

n	$h(n)$	n	$h(n)$
1	1	8	4,096
2	2	9	14,336
3	4	10	73,728
4	16	11	327,680
5	48	12	2,985,984
6	160	13	14,929,920
7	576	14	77,635,584

TABLE 1

Methods of construction of Hadamard matrices, orders up to 200. In each case one method of construction is shown:

1 –A	48–B	100–C	152–B
2 –A	52–C	104–B	156–E
4 –A	56–AB	108–B	160–AB
8 –A	60–B	112–AB	164–B
12–B	64–A	116–D	168–B
16–A	68–B	120–B	172–D
20–B	72–B	124–C	176–AB
24–B	76–C	128–A	180–B
28–B	80–B	132–B	184–AD
32–A	84–B	136–AB	188–F
36–C	88–AB	140–B	192–B
40–AB	92–D	144–AB	196–C
44–B	96–AB	148–C	200–B

A: Constructed in Corollary 3.1.
B: Constructed in Theorem 3.2.
C: Constructed in Theorem 3.3.
D: Constructed by Williamson's method (see also Table 4).
E: Constructed by Baumert and Hall.
F: Constructed by Turyn using Baumert-Hall arrays (see also Table 3).
AB: A product of orders, one constructable by method A and the other by method B; see Theorem 3.1. Similarly for AD.

TABLE 2
Examples of small Hadamard matrices

n=1

+

n=2

+	+
+	-

n=4

+	+	+	+
+	+	-	-
+	-	+	-
+	-	-	+

n=8

+	+	+	+	+	+	+	+
+	+	+	-	+	-	-	-
+	-	+	+	-	+	-	-
+	-	-	+	+	-	+	-
+	-	-	-	+	+	-	+
+	+	-	-	-	+	+	-
+	-	+	-	-	-	+	+
+	+	-	+	-	-	-	+

n=12

+	+	+	+	+	+	+	+	+	+	+	+
+	+	-	+	+	+	-	-	-	+	-	-
+	-	+	-	+	+	+	-	-	-	+	-
+	-	-	+	-	+	+	+	-	-	-	+
+	+	-	-	+	-	+	+	+	-	-	-
+	-	+	-	-	+	-	+	+	+	-	-
+	-	-	+	-	-	+	-	+	+	+	-
+	-	-	-	+	-	-	+	-	+	+	+
+	+	-	-	-	+	-	-	+	-	+	+
+	+	+	-	-	-	+	-	-	+	-	+
+	+	+	+	-	-	-	+	-	-	+	-
+	+	+	+	+	-	-	-	+	-	-	+

n=16

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	-	+	-	-	-	+	+	+	-	+	-	-	-
+	-	+	+	-	+	-	-	+	-	+	+	-	+	-	-
+	-	-	+	+	-	+	-	+	-	-	+	+	-	+	-
+	-	-	-	+	+	-	+	+	-	-	-	+	+	-	+
+	+	-	-	-	+	+	-	+	+	-	-	-	+	+	-
+	-	+	-	-	-	+	+	+	-	+	-	-	-	+	+
+	+	-	+	-	-	-	+	+	+	-	+	-	-	-	+
+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-
+	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+
+	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+
+	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-
+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+
+	+	-	-	-	-	+	+	-	-	+	+	+	+	-	-
+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-
+	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+

n=20

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	-	-	+	+	+	+	-	+	-	+	-	-	-	-	+	+	-	-
+	-	+	-	-	+	+	+	+	-	+	-	+	-	-	-	-	+	+	-
+	-	-	+	-	-	+	+	+	+	-	+	-	+	-	-	-	-	+	+
+	+	-	-	+	-	-	+	+	+	+	-	+	-	+	-	-	-	-	+
+	+	+	-	-	+	-	-	+	+	+	+	-	+	-	+	-	-	-	-
+	-	+	+	-	-	+	-	-	+	+	+	+	-	+	-	+	-	-	-
+	-	-	+	+	-	-	+	-	-	+	+	+	+	-	+	-	+	-	-
+	-	-	-	+	+	-	-	+	-	-	+	+	+	+	-	+	-	+	-
+	-	-	-	-	+	+	-	-	+	-	-	+	+	+	+	-	+	-	+
+	+	-	-	-	-	+	+	-	-	+	-	-	+	+	+	+	-	+	-
+	-	+	-	-	-	-	+	+	-	-	+	-	-	+	+	+	+	-	+
+	+	-	+	-	-	-	-	+	+	-	-	+	-	-	+	+	+	+	-
+	-	+	-	+	-	-	-	-	+	+	-	-	+	-	-	+	+	+	+
+	+	-	+	-	+	-	-	-	-	+	+	-	-	+	-	-	+	+	+
+	+	+	-	+	-	+	-	-	-	-	+	+	-	-	+	-	-	+	+
+	+	+	+	-	+	-	+	-	-	-	-	+	+	-	-	+	-	-	+
+	+	+	+	+	-	+	-	+	-	-	-	-	+	+	-	-	+	-	-
+	-	+	+	+	+	-	+	-	+	-	-	-	-	+	+	-	-	+	-
+	-	-	+	+	+	+	-	+	-	+	-	-	-	-	+	+	-	-	+

Table 2 continued

n=24

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-
+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+
+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+
+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+
+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+
+	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-
+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+
+	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-
+	+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+
+	-	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+
+	-	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-
+	+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-
+	+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+
+	-	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+
+	-	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-
+	+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-
+	-	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+
+	+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-
+	-	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+
+	-	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-
+	-	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-
+	-	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-
+	-	+	+	+	+	-	+	-	+	+	-	-	+	+	-	-	+	-	+	-	-	-	-

n=28

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	-	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+
+	-	+	-	+	+	-	-	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	-	-	+	+
+	+	-	-	+	-	-	+	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	-	+	+	-
+	-	+	+	+	-	+	+	-	-	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	-	-
+	+	+	-	-	-	+	-	-	+	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	-	+
+	-	-	-	+	+	+	-	+	+	-	-	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+
+	+	-	+	+	-	-	-	+	-	-	+	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-
+	-	+	+	-	-	+	+	+	-	+	+	-	-	+	+	+	+	-	-	-	-	-	-	-	-	+	+
+	+	+	-	-	+	+	-	-	-	+	-	-	+	+	-	+	-	-	+	-	+	-	+	-	+	+	-
+	-	+	+	+	+	-	-	+	+	+	-	+	+	-	-	+	+	+	+	-	-	-	-	-	-	-	-
+	+	+	-	+	-	-	+	+	-	-	-	+	-	-	+	+	-	+	-	-	+	-	+	-	+	-	+
+	-	-	-	+	+	+	+	-	-	+	+	+	-	+	+	-	-	+	+	+	+	-	-	-	-	-	-
+	+	-	+	+	-	+	-	-	+	+	-	-	-	+	-	-	+	+	-	+	-	-	+	-	+	-	+
+	-	-	-	-	-	+	+	+	+	-	-	+	+	+	-	+	+	-	-	+	+	+	+	-	-	-	-
+	+	-	+	-	+	+	-	+	-	-	+	+	-	-	-	+	-	-	+	+	-	+	-	-	+	-	+
+	-	-	-	-	-	-	-	+	+	+	+	-	-	+	+	+	-	+	+	-	-	+	+	+	+	-	-
+	+	-	+	-	+	-	+	+	-	+	-	-	+	+	-	-	-	+	-	-	+	+	-	+	-	-	+
+	-	-	-	-	-	-	-	-	-	+	+	+	+	-	-	+	+	+	-	+	+	-	-	+	+	+	+
+	+	-	+	-	+	-	+	-	+	+	-	+	-	-	+	+	-	-	-	+	-	-	+	+	-	+	-
+	-	+	+	-	-	-	-	-	-	-	-	+	+	+	+	-	-	+	+	+	-	+	+	-	-	+	+
+	+	+	-	-	+	-	+	-	+	-	+	+	-	+	-	-	+	+	-	-	-	+	-	-	+	+	-
+	-	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	-	-	+	+	+	-	+	+	-	-
+	+	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	-	+	+	-	-	-	+	-	-	+
+	-	-	-	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	-	-	+	+	+	-	+	+
+	+	-	+	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	-	+	+	-	-	-	+	-
+	-	+	+	-	-	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	-	-	+	+	+	-
+	+	+	-	-	+	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	-	+	+	-	-	-

Table 2 continued

n=32

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-
+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-
+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+
+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-
+	-	+	-	-	+	-	+	+	-	+	-	-	+	-	+	+	-	+	-	-	+	-	+	+	-	+	-	-	+	-	+
+	+	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-	+	+
+	-	-	+	-	+	+	-	+	-	-	+	-	+	+	-	+	-	-	+	-	+	+	-	+	-	-	+	-	+	+	-
+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-
+	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+
+	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+	+	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+
+	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-
+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+
+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-
+	+	-	-	-	-	+	+	-	-	+	+	+	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+	+	+	-	-
+	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+
+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+
+	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-
+	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+
+	-	+	-	-	+	-	+	+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	-	+	-	+	+	-	+	-
+	+	-	-	-	-	+	+	+	+	-	-	-	-	+	+	-	-	+	+	+	+	-	-	-	-	+	+	+	+	-	-
+	-	-	+	-	+	+	-	+	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+	-	+	+	-	+	-	-	+
+	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	+	+	+	+	+	+	+	+
+	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	+	-	+	-	+	-	+	-
+	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	+	+	-	-	+	+	-	-
+	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+	-	+	-	-	+	+	-	-	+
+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+	+	+	+	+	-	-	-	-
+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-	-	+	-	+	+	-	+	-	+	-	+	-	-	+	-	+
+	+	-	-	-	-	+	+	-	-	+	+	+	+	-	-	-	-	+	+	+	+	-	-	+	+	-	-	-	-	+	+
+	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+	-	+	+	-	+	-	-	+	+	-	-	+	-	+	+	-

TABLE 3

An example of a Hadamard matrix of order 188; *the matrix is*:

$$\begin{matrix} A & B & C & D \\ -B & A & -E & F \\ -C & E & A & G \\ -D & -F & -G & A \end{matrix}$$

where A, B, C, D, E, F, G are given in that order on the following pages

```
+-+----+--+----++---++++---+-+----++++++--+---+
++-+----+--+----++---++++---+-+----++++++--+---
-++-+----+--+----++---++++---+-+----++++++--+--
--++-+----+--+----++---++++---+-+----++++++--+-
---++-+----+--+----++---++++---+-+----++++++--+
+---++-+----+--+----++---++++---+-+----++++++--
-+---++-+----+--+----++---++++---+-+----++++++-
--+---++-+----+--+----++---++++---+-+----++++++
+--+---++-+----+--+----++---++++---+-+----+++++
++--+---++-+----+--+----++---++++---+-+----++++
+++--+---++-+----+--+----++---++++---+-+----+++
++++--+---++-+----+--+----++---++++---+-+----++
+++++--+---++-+----+--+----++---++++---+-+----+
++++++--+---++-+----+--+----++---++++---+-+----
-++++++--+---++-+----+--+----++---++++---+-+---
--++++++--+---++-+----+--+----++---++++---+-+--
---++++++--+---++-+----+--+----++---++++---+-+-
----++++++--+---++-+----+--+----++---++++---+-+
+----++++++--+---++-+----+--+----++---++++---+-
-+----++++++--+---++-+----+--+----++---++++---+
+-+----++++++--+---++-+----+--+----++---++++---
-+-+----++++++--+---++-+----+--+----++---++++--
--+-+----++++++--+---++-+----+--+----++---++++-
---+-+----++++++--+---++-+----+--+----++---++++
+---+-+----++++++--+---++-+----+--+----++---+++
++---+-+----++++++--+---++-+----+--+----++---++
+++---+-+----++++++--+---++-+----+--+----++---+
++++---+-+----++++++--+---++-+----+--+----++---
-++++---+-+----++++++--+---++-+----+--+----++--
--++++---+-+----++++++--+---++-+----+--+----++-
---++++---+-+----++++++--+---++-+----+--+----++
+---++++---+-+----++++++--+---++-+----+--+----+
++---++++---+-+----++++++--+---++-+----+--+----
-++---++++---+-+----++++++--+---++-+----+--+---
--++---++++---+-+----++++++--+---++-+----+--+--
---++---++++---+-+----++++++--+---++-+----+--+-
----++---++++---+-+----++++++--+---++-+----+--+
+----++---++++---+-+----++++++--+---++-+----+--
-+----++---++++---+-+----++++++--+---++-+----+-
--+----++---++++---+-+----++++++--+---++-+----+
+--+----++---++++---+-+----++++++--+---++-+----
-+--+----++---++++---+-+----++++++--+---++-+---
--+--+----++---++++---+-+----++++++--+---++-+--
---+--+----++---++++---+-+----++++++--+---++-+-
----+--+----++---++++---+-+----++++++--+---++-+
+----+--+----++---++++---+-+----++++++--+---++-
-+----+--+----++---++++---+-+----++++++--+---++
```

Table 3 continued

[illegible]

Table 3 continued

[illegible]

Table 3 continued

+	−	−	+	−	−	+	−	+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−
−	−	+	−	−	+	−	+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+
−	+	−	−	+	−	+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−
+	−	−	+	−	+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−
−	−	+	−	+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+
−	+	−	+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−
+	−	+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−
−	+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+
+	+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−
+	+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+
+	−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+	+
−	−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+	+	+
−	−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+	+	+	−
−	−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+	+	+	−	−
−	−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+	+	+	−	−	−
−	+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+	+	+	−	−	−	−
+	−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+	+	+	−	−	−	−	−
−	−	−	+	−	−	−	−	−	+	+	−	+	−	−	+	−	−	−	+	−	+	−	+	+	−	+	+	+	−	+	−	−	+	−	−	+	−	+	+	+	−	−	−	−	−	+
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-	-	+	+	-	+	+	+	-	+	-	+	-	-	-	-	+	-	-	+	-	-	-	-	+	+	-	-	-	+	+	+	-	+	+	+	-	+	-	+	+	+	+	-	-	-	-
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+	-	+	+	+	-	+	-	+	-	-	-	-	+	-	-	+	-	-	-	-	+	+	-	-	-	+	+	+	-	+	+	+	-	+	-	+	+	+	+	-	-	-	-	-	-	+
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-	+	-	+	-	-	-	-	+	-	-	+	-	-	-	-	+	+	-	-	-	+	+	+	-	+	+	+	-	+	-	+	+	+	+	-	-	-	-	-	-	+	+	-	+	+	+
+	-	+	-	-	-	-	+	-	-	+	-	-	-	-	+	+	-	-	-	+	+	+	-	+	+	+	-	+	-	+	+	+	+	-	-	-	-	-	-	+	+	-	+	+	+	-

TABLE 4
Hadamard matrices of Williamson type

If A, B, C, and D are the respective matrices obtained by developing the shown rows cyclically modulo n, then

$$\begin{array}{cccc} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{array}$$

is a Hadamard matrix of order $4n$

n=3:
```
+ + -
+ - -
+ - -
+ + +
```

n=5:
```
+ + + - -
+ - + + -
+ - - - -
+ - - - -
```

n=7:
```
+ + + + - - -
+ + + - - - -
+ + - + + - +
+ - + + + + -
```

n=9:
```
+ + + + - + - - -
+ + - + - - + - +
+ - - - + + - - -
+ + + - + + - + +
```

n=11:
```
+ + - - + - + - + + -
+ + + - - + + - - + +
+ + - + - + + - + - +
+ + - - - - - - - - +
```

n=13:
```
+ + - - - - - + + + + + -
+ - - + + + - - + + + - -
+ + + - + - + + - + - + +
+ - + + + - + + - + + + -
```

n=15:
```
+ + + + - + + + - - - + - - -
+ - + + + - + - - + - + + + -
+ - - + + - - - - - - + + - -
+ - - - - + - + + - + - - - -
```

n=17:
```
+ + - - + + - + - + - + - - + + -
+ + - - + - - - - - - - - + - - +
+ + + - - - - + - - + - - - - + +
+ + - - - + - + - - + - + - - - +
```

n=19:
```
+ - + - - - + + + + - - - - + + + - +
+ - + + - + + + + - - + + + + - + + -
+ + + - - + - - - - - - - - + - - + +
+ - + - + - - + + - - + + - - + - + -
```

n=21:
```
+ - - - + - + - - - + - + + + - + - + + +
+ + + + + - - - + - + + - + - - - + + + +
+ + - - + - - - - + - - + - - - - + - - +
+ - - - + + - + - - + + - - + - + + - - -
```

n=23:
```
+ + - - - + - - - + - + + - + - - - + - - - +
+ - + + - + + - - + + + + + + - - + + - + + -
+ + + - - - + + - + - + + - + - + + - - - + +
+ + + - + + + - + - - - - - - + - + + + - + +
```

n=25:
```
+ - - + + - + - + - - - - - - - - + - + - + + - -
+ - + + - + + + - + + - - - - + + - + + + - + + -
+ + + - + + + - - - + - + + - + - - - + + + - + +
+ - + + + + - - - - + + - - + + - - - - + + + + -
```

REFERENCES

AHRENS, R. W. and SZEKERES, G. (1969). On a combinatorial generalization of 27 lines associated with a cubic surface. *J. Austral. Mathl Soc.* **10** 485–492.

APPEL, K. and HAKEN, W. (1976). Every planar map is four colorable. *Bull. Amer. Math. Soc.* **82** 711–712.

BANERJEE, K. S. (1975). *Weighing Designs for Chemistry, Medicine, Economics, Operations Research and Statistics.* Dekker, New York.

BAUMERT, L. D. (1962). Six inequivalent Hadamard matrices of order 2n, n $\geqslant$ 5. *Jet Propulsion Lab. Research Summary*, 36–12, **1**, 74–76. J. P. L., Pasadena, California.

BAUMERT, L. D. (1966). Hadamard matrices of orders 116 and 232. *Bull. Amer. Math. Soc.* **72** 237.

BAUMERT, L. D. (1971). *Cyclic Difference Sets. Lecture Notes in Mathematics* **182**. Springer-Verlag, Berlin.

BAUMERT, L. D., GOLOMB, S. W. and HALL, M., JR. (1962). Discovery of an Hadamard matrix of order 92. *Bull. Amer. Math. Soc.* **68** 237–238.

BAUMERT, L. D. and HALL, M., JR. (1965a). A new construction for Hadamard matrices. *Bull. Amer. Math. Soc.* **71** 169–170.

BAUMERT, L. D. and HALL, M., JR. (1965b). Hadamard matrices of the Williamson type. *Math. Comp.* **19** 442–447.

BELEVITCH, V. (1950). Theory of 2^n-terminal networks with application to conference telephony. *Electron. Commun.* **27** 231–244.

BELEVITCH, V. (1968). Conference networks and Hadamard matrices. *Ann. Soc. Sci. Bruxelles Ser.* I **82** 13–32.

BELLMAN, R. (1943). A note on determinants and Hadamard's inequality. *Amer. Math. Monthly* **50** 550–551.

BHAT, V. N. (1972a). Non-isomorphic solutions of some balanced incomplete block designs. II. *J. Combinatorial Theory (A)* **12** 217–224.

BHAT, V. N. (1972b). Non-isomorphic solutions of some balanced incomplete block designs. III. *J. Combinatorial Theory (A)* **12** 225–252.

BHAT, V. N. and SHRIKHANDE, S. S. (1970). Non-isomorphic solution of some balanced incomplete block designs. I. *J. Combinatorial Theory* **9** 174–191.

BLAKE, I. F. and MULLIN, R. C. (1975). *The Mathematical Theory of Coding.* Academic Press, New York.

BLATT, D. and SZEKERES, G. (1969). A skew Hadamard matrix of order 52. *Canad. J. Math.* **21** 1319–1322.

BOSE, R. C. and SHRIKHANDE, S. S. (1959). A note on a result in the theory of code construction. *Information and Control* **2** 183–194.

BOSE, R. C. and SHRIKHANDE, S. S. (1970). Graphs in which each pair of vertices is adjacent to the same number d of other vertices. *Studia. Sci. Math. Hung.* **5** 181–195.

BOSE, R. C. and SHRIKHANDE, S. S. (1971). Some further constructions for $G_2(d)$ graphs. *Studia. Sci. Math. Hung.* **6** 127–132.

BRAUER, A. (1953). On a new class of Hadamard determinants. *Math. Z.* **58** 219–225.

BRENNER, J. and CUMMINGS, L. (1972). The Hadamard maximum determinant problem. *Amer. Math. Monthly* **79** 626–630.

BUSH, K. A. (1971a). Unbalanced Hadamard matrices and finite projective planes of even order. *J. Combinatorial Theory (A)* **11** 38–44.

BUSH, K. A. (1971b). An inner orthogonality of Hadamard matrices. *J. Austral. Math. Soc.* **12** 242–248.

BUSH, K. A. (1971c). Forms of Hadamard matrices induced by finite projective planes. In *Atti. Convegno Geometria Combinatoria Sue Applicazoni* (A. Barlotti, ed.), 131–136. University of Perugia, Perugia.

BUSH, K. A. (1973). Construction of symmetric Hadamard matrices. In *A Survey of Combinatorial Theory* (J. N. Srivastava, ed.), 81–83. North-Holland, Amsterdam.

BUSH, K. A. (1977a). An infinite class of semi-regular designs. *Ars Combinatoria* **3** 293–296.

BUSH, K. A. (1977b). Personal communication to A. Hedayat.

BUSSEMAKER, F. C. and SEIDEL, J. J. (1970). Symmetric Hadamard matrices of order 36. *Ann. New York Acad. Sci.* **175** 66–79.

COHN, J. H. E. (1965). Hadamard matrices and some generalizations. *Amer. Math. Monthly* **72** 515–518.

Cooper, J., Milas, J. and Wallis, W. D. (1978). Hadamard equivalence. *Proc. International Conference on Combinatorial Theory*, Canberra. Springer-Verlag. To appear.

Cooper, J. and Wallis, J. (1972). A construction for Hadamard arrays. *Bull. Austral. Math. Soc.* **7** 269–277.

Cooper, J. and Wallis, J. (1976a). A note on orthogonal designs in order eighty. *Ars Combinatoria* **1** 267–274.

Cooper, J. and Wallis, J. (1976b). Using cyclotomy to construct orthogonal designs. *J. Austral. Math. Soc. (A)* **22** 119–127.

Dade, E. C. and Goldberg, K. (1959). The construction of Hadamard matrices. *Michigan Math. J.* **6** 247–250.

Decker, J. A., Jr. (1973). Hadamard transforms spectrometry. *Industrial Res.* **15** (2) 60–64.

Dedo, M. (1968). Trattazione geometrica di un problems di Hadamard. *Period Math.* (4) **46** 107–114.

Delsarte, P., Goethals, J. M. and Seidel, J. J. (1971). Orthogonal matrices with zero diagonal II. *Canad. J. Math.* **23** 816–832.

Dey, A. and Ramakrishna (1977). A note on orthogonal maineffect plans. *Technometrics* **19** 511–512.

Dillon, J. F. (1976). On Hadamard difference sets. *Ars Combinatoria* **1** 275–279.

Doehlert, D. H. and Klee, V. L. (1972). Experimental designs through level reduction of the *d*-dimensional cuboctahedron. *Discrete Math.* **2** 309–334.

Ehlich, H. (1965). Neue Hadamard-Matrizen. *Arch. Math.* **16** 34–36.

Federer, W. T. (1977). On the existence and construction of a complete set of orthogonal $F(4t;\ 2t,\ 2t)$ – square design. *Ann. Statist.* **5** 561–564.

Federer, W. T., Hedayat, A., Lowe, C. C. and Raghavarao, D. (1976). An application of statistical design theory to crop estimation with special reference to legumes and mixtures of cultivars. *Agronomy Journal* **68** 914–919.

Folkman, J. (1967). A non-existence theorem for Hadamard matrices in many variables. Rand Document. D-16326-PR, Rand Corporation.

Geramita, A. V., Geramita, J. M. and Wallis, J. S. (1976). Orthogonal designs. *Lin Multilin Alg.* **3** 281–306.

Geramita, A. V. and Verner, J. H. (1976). Orthogonal designs with zero diagonal. *Canad. J. Math.* **28** 215–224.

Geramita, A. V. and Wallis, J. S. (1974). A survey of orthogonal designs. In *Proceedings of the Fourth Manitoba Conference on Numerical Mathematics* (B. L. Hartnell and H. C. Williams, eds.) 121–168, Utilitas Mathematica, Winnipeg, Canada.

Gilman, R. E. (1931). On Hadamard determinant theory and orthogonal determinants. *Bull. Amer. Math. Soc.* **37** 30–37.

Goethals, J. M. and Seidel, J. J. (1967). Orthogonal matrices with zero diagonal. *Canad. J. Math.* **19** 1001–1010.

Goethals, J. M. and Seidel, J. J. (1970). A skew Hadamard matrix of order 36. *J. Austral. Math. Soc.* **11** 343–344.

Goldberg, K. (1966). Hadamard matrices of order cube plus one. *Proc. Amer. Math. Soc.* **17** 744–746.

Gordon, B. (1974). A note on inequivalent Hadamard matrices. *J. Reine Angew. Math.* **268/269** 427–433. (Corrigendum: *J. Reine Angew Math.* **274/275** 93.)

Gruner, W. (1939–1940). Einlagerung des regulären n-Simplex in den n-dimensionalen Würfel. *Comment. Math. Helv.* **12** 149–152.

Hadamard, J. (1893). Résolution d'une question relative aux déterminants. *Bull. Sci. Math.* **17** (part 1) 240–246.

Hall, M., Jr. (1961). Hadamard matrices of order 16. *Jet Propulsion Lab. Research Summary* 36–10, **1** 21–26. J. P. L., Pasadena, California.

Hall, M., Jr. (1965). Hadamard matrices of order 20. *Jet Propulsion Lab. Technical Report* 32–761. J. P. L., Pasadena, California.

Hall, M., Jr. (1967). *Combinatorial Theory.* Blaisdell (Ginn), Waltham, Mass.

Hall, M., Jr. (1969). Automorphism of Hadamard matrices. *SIAM J. Appl. Math.* **17** 1094–1101.

Hall, M., Jr. and Knuth, D. E. (1965). Combinatorial analysis and computers. *Amer. Math. Monthly* **72** 21–28.

Hartley, H. O. and Smith, C. A. B. (1948). The construction of Youden squares. *J. Roy. Statist. Soc. (B)* **10** 262–263.

HARWIT, M. and SLOANE, N. J. A. (1978). *Hadamard Transform Methods in Optics and Spectroscopy.* Unpublished manuscript.

HEDAYAT, A. (1978). D-optimal and near D-optimal exact designs. Unpublished manuscript.

HEDAYAT, A. and JOHN, P. W. M. (1974). Resistant and susceptible BIB designs. *Ann. Statist.* **2** 148–158.

HEDAYAT, A., RAGHAVARAO, D. and SEIDEN, E. (1975). Further contributions to the theory of F-squares design. *Ann. Statist.* **3** 712–716.

HEDAYAT, A. and SEIDEN, E. (1970). F-square and orthogonal F-squares design: A generalization of Latin square and orthogonal Latin squares design. *Ann. Math. Statist.* **41** 2035–2044.

HOTELLING, H. (1944). Some improvements in weighing and other experimental techniques. *Ann. Math. Statist.* **15** 297–306.

HUNT, D. C. (1972). Skew-Hadamard matrices of order less than or equal to 100. In *Proc. First Australian Conference on Combinatorial Mathematics* (J. Wallis and W. D. Wallis, eds.), 23–27. TUNRA, Newcastle, Australia.

HUNT, D. C. and WALLIS, J. (1972). Cyclotomy, Hadamard array and supplementary difference sets. In *Proc. Second Manitoba Conference on Numerical Mathematics* (R. S. D. Thomas and H. C. Williams, eds.), 351–358. Utilitas Mathematics, Winnipeg, Canada.

HURWITZ, A. (1898). Uber die Komposition der quadratischen Formen von beliebig vielen Variabeln. *Nachr. Konigl. Ges. Wiss. Gottingen, Math-Phys. Klasse* 309–316.

JOHNSON, E. C. (1966). Skew-Hadamard Abelian group difference sets. *J. Algebra* **4** 388–402.

KIEFER, J. (1975a). Construction and optimality of generalized Youden designs. In *Statistical Design and Linear Models* (J. N. Srivastava, ed.), 333–353. North-Holland, Amsterdam.

KIEFER, J. (1975b). Balanced block designs and generalized Youden designs, I. Construction (patchwork). *Ann. Statist.* **3** 109–118.

KIRTON, H. C. and SEBERRY, J. (1978). Generation of a frequency square orthogonal to a 10 × 10 Latin square. *Proc. International Conference on Combinatorial Theory*, Canberra. Springer-Verlag, Berlin. To appear.

LAKEIN, R. B. and WALLIS, J. S. (1975). On the matrices used to construct Baumert-Hall arrays. In *Combinatorial Mathematics, Proceedings of the Third Australian Conference* (A. P. Street and W. D. Wallis, eds.), 156–170. *Lecture Notes in Mathematics* **452**. Springer-Verlag, Berlin.

LANCASTER, H. O. (1965). Pairwise statistical independence. *Ann. Math. Statist.* **36** 1313–1317.

LEVENSHTEIN, V. I. (1964). Application of the Hadamard matrices to a problem in coding. *Problems of Cybernetics* **5** 166–184.

LONGYEAR, J. Q. (1976). Criteria for a Hadamard matrix to be skew-equivalent. *Canad. J. Math.* **28** 1216–1223.

LONGYEAR, J. Q. (1977). Order 24 Hadamard matrices of character at least 3. To appear

MACWILLIAMS, F. J. and SLOANE, N. J. A. (1977). *The Theory of Error-Correcting Codes.* North-Holland, New York.

MANN, H. B. and MCFARLAND, R. L. (1973). On Hadamard difference sets. In *A Survey of Combinatorial Theory* (J. N. Srivastava, ed.) 333–334. North-Holland, Amsterdam.

MARGOLIN, B. H. (1968). Orthogonal main effect 2^n3^m designs and two-factor interaction aliasing. *Technometrics* **10** 559–573.

MCCARTHY, D., STANTON, R. G. and VANSTONE, S. (1976). On an extremal class of (r,λ)-designs related to a problem of Doehlert and Klee. *Ars Combinatoria* **2** 305–317.

MOOD, A. M. (1946). On Hotelling's weighing problem. *Ann. Math. Statist.* **17** 432–446.

MORIGUTI, S. (1954). Optimality of orthogonal designs. *Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs.* **3** 75–98.

NEWMAN, M. (1971). Invariant factors of combinatorial matrices. *Israel J. Math.* **10** 126–130.

NORMAN, C. W. (1976). Non-isomorphic Hadamard designs. *J. Combinatorial Theory (A)* **21** 336–344.

PALEY, R. E. A. C. (1933). On orthogonal matrices. *J. Math. and Physics* **12** 311–320.

PAYNE, S. E. (1973). A maximal determinant problem. Aerospace Research Laboratories, 73–1095 (U.S.A.).

PESOTAN, H. and RAGHAVARAO, D. (1975). Embedded Hadamard matrices. *Utilitas Math.* **8** 99–110.

PESOTAN, H., RAGHAVARAO, D. and RAKTOE, B. L. (1977). Further contributions to embedded Hadamard matrices. *Utilitas Math.* **12** 241–246.

PLACKETT, R. L. and BURMAN, J. P. (1946). The design of optimum multifactorial experiments. *Biometrika* **33** 305–325.

PLOTKIN, M. (1960). Binary codes with specified minimum distances. *IEEE Trans. Information Theory* **6** 445–450.

PLOTKIN, M. (1972). Decomposition of Hadamard matrices. *J. Combinatorial Theory (A)* **13** 127–130.

POSNER, E. C. (1968). Combinatorial structures in planetary reconnaissance. In *Error Correcting Codes* (H. B. Mann, ed.) 15–46. Wiley, New York.

RAGHAVARAO, D. (1971). *Construction and Combinatorial Problems in Design of Experiments.* Wiley, New York.

RAGHAVARAO, D. and PESOTAN, H. (1977). Embedded ($Sn \pm In$)-matrices. *Utilitas Math.* **11** 227–236.

RUDVALIS, A. (1971). (v,k,λ)-graphs and polarities of (v,k,λ)-designs. *Math. Z.* **120** 224–230.

RUTLEDGE, W. A. (1952). Quaternions and Hadamard matrices. *Proc. Amer. Math. Soc.* **3** 625–630.

RYSER, H. J. (1963). *Combinatorial Mathematics.* (Carus Mathematical Monograph, No. 14). Wiley, New York.

SCARPIS, U. (1898). Sui Determinants di valore massime. *Rend. R. Inst. Lombardo Sci. e Lett* (2) **31** 1441–1446.

SCHMIDT, K. W. (1970). Lower bounds for maximal (0,1)-determinants. *SIAM J. Appl. Math.* **19** 440–442.

SCHMIDT, K. W. (1973). A bound for a 4^k-order maximal (0,1)-determinants. *SIAM Review* **15** 673–674.

SCHMIDT, K. W. and WANG, E. T. H. (1977). The weights of Hadamard matrices. *J. Combinatorial Theory (A)* **23** 257–263.

SEBERRY, J. (1978). A computer listing of Hadamard matrices. *International Conference on Combinatorial Theory, Canberra. Proceedings.* Springer-Verlag, Berlin. To appear.

SHRIKHANDE, S. S. and BHAGWANDAS. (1970). A note on embedding for Hadamard matrices. In *Essays in Probability and Statistics*, 673–688. Univ. of North Carolina Press.

SHRIKHANDE, S. S. and SINGH, N. K. (1962). On a method of constructing symmetrical balanced incomplete block designs. *Sankhyā (A)* **24** 25–32.

SINGHI, N. M. (1974). (19,9,4) Hadamard designs and their residual designs. *J. Combinatorial Theory (A)* **16** 241–252.

SINGHI, N. M. (1975). Non-isomorphic solutions of $(4t + 3, 2t + 1, t)$ designs. *Geometriae Dedicata* **4** 387–402.

SLOANE, N. J. A. (1972). A survey of constructive coding theory and a table of binary codes of highest known rate. *Discrete Math.* **3** 265–294.

SLOANE, N. J. A. and HARWIT, M. (1976). Masks for Hadamard transform optics, and weighing designs. *Applied Optics* **15** 107–114.

SPENCE, E. (1967). A new class of Hadamard matrices. *Glasgow Math. J.* **8** 59–62.

SPENCE, E. (1972). Hadamard designs. *Proc. Amer. Math. Soc.* **32** 29–31.

SPENCE, E. (1975a). Hadamard matrices from relative difference sets. *J. Combinatorial Theory (A)* **19** 287–300.

SPENCE, E. (1975b). Skew-Hadamard matrices of the Goethals-Seidel type. *Canad. J. Math.* **27** 555–560.

SPENCE, E. (1977). Skew-Hadamard matrices of order $2(q + 1)$. *Discrete Math.* **18** 79–85.

SPENCER, J. (1967). Hadamard matrices in many variables. Rand Document D-15925-PR, Rand Corporation.

STANTON, R. G, and KALBFLEISCH, J. G. (1968). Quasi-symmetric balanced incomplete block designs. *J. Combinatorial Theory* **4** 391–396.

STANTON, R. G. and MULLIN, R. C. (1969). Uniqueness theorems in balanced incomplete block designs. *J. Combinatorial Theory* **7** 37–48.

STANTON, R. G. and MULLIN, R. C. (1976). On the nonexistence of a class of circulant balanced weighing matrices. *SIAM J. Appl. Math.* **30** 98–102.

STIFFLER, J. J. and BAUMERT, L. D. (1961). Inequivalent Hadamard matrices. *Jet Propulsion Lab. Research Summary* 36–39, **1** 28–30. J. P. L., Pasadena, California.

STORER, T. (1967). *Cyclotomy and Difference Sets.* Markham, Chicago.

STORER, T. (1971). Hurwitz on Hadamard designs. *Bull. Austral. Math. Soc.* **4** 109–112.

STREET, A. P. and WALLIS, W. D. (1977). *Combinatorial Theory: An Introduction.* Charles Babbage Research Center. Winnipeg, Canada.

SYLVESTER, J. J. (1867). Thoughts on inverse orthogonal matrices, simultaneous sign-successions, and tesselated pavements in two or more colors, with applications to Newton's rule, ornamental tile-work, and the theory of numbers. *Phil. Mag.* (4) **34** 461–475.

SZEKERES, G. (1969). A new class of symmetric block designs. *J. Combinatorial Theory* **6** 219–221.

SZEKERES, G. (1971). Cyclotomy and complementary difference sets. *Acta. Arith.* **18** 349–353.

TAI, M. H., HARWIT, M. and SLOANE, N. J. (1975). Errors in Hadamard spectroscopy or imaging caused by imperfect masks. *Applied Optics* **14** 2678–2686.

TAUSSKY, O. (1971). (1,2,4,8)-sums of squares and Hadamard matrices. In *Proceedings of Symposia in Pure Mathematics XIX*: *Combinatorics.* 229–234. Amer. Math. Soc., Providence.

TURYN, R. J. (1965). Character sums and difference sets. *Pacific J. Math.* **15** 319–346.

TURYN, R. J. (1968). Sequences with small correlation. In *Error Correcting Codes* (H. B. Mann, ed.) 195–228. Wiley, New York.

TURYN, R. J. (1971). On *C*-matrices of arbitrary powers. *Canad. J. Math.* **23** 531–535.

TURYN, R. J. (1972). An infinite class of Williamson matrices. *J. Combinatorial Theory (A)* **12** 319–321.

TURYN, R. J. (1974). Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression, and surface wave encoding. *J. Combinatorial Theory (A)* **16** 313–333.

TURYN, R. J. (1975). Personal communication to A. Hedayat.

TURYN, R. J. and STORER, J. (1961). On binary sequences. *Proc. Amer. Math. Soc.* **12** 394–399.

VIJAYAN, K. (1976). Hadamard matrices and submatrices. *J. Austral. Math. Soc.* **22A** 469–475.

WALLIS, J. (1969a). A class of Hadamard matrices. *J. Combinatorial Theory* **6** 40–44.

WALLIS, J. (1969b). A note of a class of Hadamard matrices. *J. Combinatorial Theory* **6** 222–223.

WALLIS, J. (1970). Hadamard designs. *Bull. Austral. Math. Soc.* **2** 45–54.

WALLIS, J. (1971a). Some $(1, -1)$ matrices. *J. Combinatorial Theory (B)* **10** 1–11.

WALLIS, J. (1971b). Amicable Hadamard matrices. *J. Combinatorial Theory (A)* **11** 296–298.

WALLIS, J. (1971c). A skew-Hadamard matrix of order 92. *Bull. Austral. Math. Soc.* **5** 203–204.

WALLIS, J. (1972a). On supplementary difference sets. *Aequationes Math.* **8** 242–257.

WALLIS, J. (1972b). Orthogonal $(0,1,-1)$ matrices. In *Proceedings of the First Australian Conference on Combinatorial Mathematics* (J. Wallis and W. D. Wallis, eds.) 61–84, TUNRA, Newcastle.

WALLIS, J. (1973a). Hadamard matrices of order 28m, 36m, and 44m. *J. Combinatorial Theory (A)* **15** 323–328.

WALLIS, J. (1973b). A note on amicable Hadamard matrices. *Utilitas Math.* **3** 119–125.

WALLIS, J. (1973c). Recent advances in the construction of Hadamard matrices. In *Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing* (F. Hoffman, R. B. Levow and R. S. D. Thomas, eds.) 53–89. *Utilitas Math.*, Winnipeg.

WALLIS, J. (1973d). Some matrices of Williamson type. *Utilitas Math.* **4** 147–154.

WALLIS, J. (1974). Williamson matrices of even order. In *Combinatorial Mathematics, Proceedings of the Second Australian Conference* (D. A. Holton, ed.) 132–142. *Lecture Notes in Mathematics* **403**. Springer-Verlag, Berlin.

WALLIS, J. (1975a). Construction of Williamson type matrices. *Lin. Multilin Alg.* **3** 197–207.

WALLIS, J. (1975b). On Hadamard matrices. *J. Combinatorial Theory (A)* **18** 149–164.

WALLIS, J. (1976). On the existence of Hadamard matrices. *J. Combinatorial Theory (A)* **21** 188–195.

WALLIS, J. and WHITEMAN, A. L. (1972). Some classes of Hadamard matrices with constant diagonal. *Bull. Austral. Math. Soc.* **7** 233–249.

WALLIS, W. D. (1969a). Certain graphs arising from Hadamard matrices. *Bull. Austral. Math. Soc.* **1** 325–331.

WALLIS, W. D. (1969b). On the relationship between partially balanced incomplete block designs and graphs. *Bull. Austral. Math. Soc.* **1** 425–430.

WALLIS, W. D. (1970a). A non-existence theorem for (v,k,λ)-graphs. *J. Austral. Math. Soc.* **11** 381–383.

WALLIS, W. D. (1970b). A note on quasi-symmetric designs. *J. Combinatorial Theory* **9** 100–101.

WALLIS, W. D. (1971a). Integral equivalence of Hadamard matrices. *Israel J. Math.* **10** 349–358.

WALLIS, W. D. (1971b). Some notes on integral equivalence of combinatorial matrices. *Israel J. Math.* **10** 457–464.

WALLIS, W. D. (1971c). Construction of strongly regular graphs using affine designs. *Bull. Austral. Math. Soc.* **4** 41–49. (Corrigendum: *Bull. Austral. Math. Soc.* **5** 431.)

WALLIS, W. D. (1972a). Hadamard matrices in canonical form. In *Proceedings of the First Australian Conference on Combinatorial Mathematics* (J. Wallis and W. D. Wallis, eds.) 103–106. TUNRA, Newcastle.

WALLIS, W. D. (1972b). On the number of inequivalent Hadamard matrices. In *Proceedings of the Second Manitoba Conference on Numerical Mathematics* (R. S. D. Thomas and H. C. Williams, eds.) 383–401. *Utilitas Math.*, Winnipeg.

WALLIS, W. D. (1972c). On a problem of K. A. Bush concerning Hadamard matrices. *Bull. Austral. Math. Soc.* **6** 321–326.

WALLIS, W. D. (1977). On the weights of Hadamard matrices. *Ars Combinatoria* **3** 287–291.

WALLIS, W. D., STREET, A. P. and WALLIS, J. S. (1972). *Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices.* (*Lecture Notes in Mathematics* **292**.) Springer-Verlag, Berlin.

WALLIS, W. D. and WALLIS, J. S. (1969). Equivalence of Hadamard matrices. *Israel J. Math.* **7** 122–128.

WHITEMAN, A. L. (1971). An infinite family of skew Hadamard matrices. *Pacific J. Math.* **38** 817–822.

WHITEMAN, A. L. (1972). Skew Hadamard matrices of Goethals-Seidel type. *Discrete Math.* **2** 397–405.

WHITEMAN, A. L. (1973). An infinite family of Hadamard matrices of Williamson type. *J. Combinatorial Theory (A)* **14** 334–340.

WHITEMAN, A. L. (1976). Hadamard matrices of order $4(2p + 1)$. *J. Number Theory* **8** 1–11.

WILLIAMSON, J. (1944). Hadamard's determinant theorem and the sum of four squares. *Duke Math. J.* **11** 65–81.

WILLIAMSON, J. (1946). Determinants whose elements are 0 and 1. *Amer. Math. Monthly* **53** 427–434.

WILLIAMSON, J. (1947). Note on Hadamard's determinant theorem. *Bull. Amer. Math. Soc.* **53** 608–613.

YANG, C. H. (1971). On Hadamard matrices constructible by circulant submatrices. *Math. Comp.* **25** 181–186.

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