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A SIMPLE PROOF OF A CLASSICAL THEOREM WHICH CHARACTERIZES THE GAMMA DISTRIBUTION

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The following result of Lukacs is known: let X_1, X_2 be independent, positive random variables, having the nondegenerate distributions P_1 and P_2. Suppose that X_1/X_2 and $X_1 + X_2$ are independent. Then P_1 and P_2 are gamma distributions with the same scale parameter. Lukacs' original deduction requires details from complex analysis. Here a simpler proof is given. Instead of P_1 and P_2 two other probability measures μ_1 and μ_2 are shown to be determined by the independence properties of X_1 and X_2. It is possible to express P_i and μ_i by each other, and μ_i is chosen such that all moments of μ_i are finite ($i = 1, 2$). Thus the proof reduces to a straightforward calculation of moments.

1. Introduction and summary. When a statistician has to decide what probability law describes a real chance situation, he sometimes will be helped by the fact that certain properties of the real phenomena are consistent with only a certain class of distributions. He can refer to mathematical studies of probability distributions from characteristic properties. The following result of Lukacs [1] is well known: let X_1, X_2 be independent, positive, nondegenerate random variables. Then X_1 and X_2 are gamma distributed with the same scale parameter if and only if X_1/X_2 and $X_1 + X_2$ are independent.

Reading Lukacs' original deduction, a statistician who is interested in the proof has to handle Fourier transformations and specific details from complex analysis. We present in this paper a new, technically simple proof of the "if" part of Lukacs' theorem, the "only if" part being elementary anyway. The method consists of a straightforward calculation of moments. It is based on the fact that certain probability distributions on $\mathbb{R}$, the gamma distributions belonging to them, are determined uniquely by their moments: see, e.g., [2], Theorem 5.5.1.

2. New proof of Lukacs' theorem. The following notation will be used:

$d(\nu, \alpha)$ is the function on $\mathbb{R}$ which takes the value of $x^{\nu-1}e^{-\alpha x}$ for $x \in (0, \infty)$ and of 0 for $x \in (-\infty, 0]$ ($\nu, \alpha \in (0, \infty)$);

$\Gamma(\nu, \alpha)$ is the gamma distribution, given by the density $(\Gamma(\nu))^{-1} \cdot \alpha^\nu \cdot d(\nu, \alpha)$ ($\nu, \alpha \in (0, \infty)$);

λ: Lebesgue measure on $\mathbb{R}$.

The problem is to deduce the following

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THEOREM. *Let X_1, X_2 be independent, positive, nondegenerate random variables. Suppose that X_1/X_2 and $X_1 + X_2$ are independent. Then there are $\nu_1, \nu_2, \alpha \in (0, \infty)$ such that $\Gamma(\nu_i, \alpha)$ is the distribution of X_i $(i = 1, 2)$.*

PROOF. By P_i we shall mean the distribution of X_i $(i = 1, 2)$. Let $f(x) = 1_{(0,\infty)}(x) \cdot e^{-x}$ $(x \in \mathbb{R})$ and $(i = 1, 2)$ $c_i = E(f(X_i)) = \int f(x)P_i(dx)$. For $i = 1, 2$ we can define a probability measure μ_i on $\mathbb{R}$ such that the function $(1/c_i) \cdot f(x)$ is the density of μ_i relative to P_i. The nth moment of μ_i, say $m_n^{(i)}$ $(n = 0, 1, \cdots)$, is finite for any n $(i = 1, 2)$. (We have used the fact that the function $x^n \cdot f(x)$ is bounded for $n = 0, 1, \cdots$.) Now, the independence of X_1 and X_2 and of X_1/X_2 and $X_1 + X_2$ imposes some relations on the $m_n^{(i)}$. For $n, k = 0, 1, \cdots$ we obtain:

$$\begin{aligned} E[X_1^n \cdot (X_1 + X_2)^k \cdot f(X_1 + X_2)] &= E[(1 + X_2/X_1)^{-n} \cdot (X_1 + X_2)^{n+k} \cdot f(X_1 + X_2)] \\ &= E(1 + X_2/X_1)^{-n} \cdot E[(X_1 + X_2)^{n+k} \cdot f(X_1 + X_2)] \\ &= E(1 + X_2/X_1)^{-n} \cdot \textstyle\sum_{\nu=0}^{n+k} \binom{n+k}{\nu} E[X_1^\nu \cdot f(X_1)] \cdot E[X_2^{n+k-\nu} f(X_2)] . \end{aligned}$$

Dividing this equation with $k = 1$ by the corresponding equation with $k = 0$; noticing that $(1/c_i) \cdot E[X_i^\nu \cdot f(X_i)] = m_\nu^{(i)}$ $(\nu = 0, 1, \cdots; i = 1, 2)$, we get

$$\frac{m_{n+1}^{(1)}}{m_n^{(1)}} + m_1^{(2)} = \frac{\sum_{\nu=0}^{n+1} \binom{n+1}{\nu} m_\nu^{(1)} m_{n+1-\nu}^{(2)}}{\sum_{\nu=0}^{n} \binom{n}{\nu} m_\nu^{(1)} m_{n-\nu}^{(2)}} \qquad n = 1, 2, \cdots . \tag{1}$$

Similarly, it can be derived that the expression on the right-hand side of (1) is also equal to $m_{n+1}^{(2)}/m_n^{(2)} + m_1^{(1)}$. Hence we have proved that substitution of $m_n^{(i)}$ for $M_n^{(i)}$ in the following equation provides a valid relation:

$$\frac{M_{n+1}^{(1)}}{M_n^{(1)}} + M_1^{(2)} = \frac{M_{n+1}^{(2)}}{M_n^{(2)}} + M_1^{(1)} \qquad n = 1, 2, \cdots . \tag{2}$$

Using this result, we can eliminate $m_{n+1}^{(2)}$ from (1). We obtain that, with $m_n^{(i)}$ for $M_n^{(i)}$, the equation (3) holds:

$$\begin{aligned} &M_{n+1}^{(1)} \cdot \textstyle\sum_{\nu=1}^{n-1} \binom{n}{\nu} M_\nu^{(1)} M_{n-\nu}^{(2)} \\ &\quad = M_n^{(1)} \cdot \textstyle\sum_{\nu=1}^{n} \binom{n+1}{\nu} M_\nu^{(1)} M_{n+1-\nu}^{(2)} + M_n^{(1)} M_n^{(2)} \cdot (M_1^{(2)} - M_1^{(1)}) \\ &\quad\quad - M_1^{(2)} \cdot M_n^{(1)} \cdot \textstyle\sum_{\nu=0}^{n} \binom{n}{\nu} M_\nu^{(1)} M_{n-\nu}^{(2)} \qquad n = 1, 2, \cdots . \end{aligned} \tag{3}$$

Since P_1 is not a degenerate distribution, neither is μ_1. Hence $m_2^{(1)} > (m_1^{(1)})^2$, which implies the existence of $\nu_1 \in (0, \infty)$ such that $(\nu_1 + 1)/\nu_1 = m_2^{(1)} \cdot (m_1^{(1)})^{-2}$. Define $\alpha' = \nu_1 \cdot (m_1^{(1)})^{-1}$, $\nu_2 = \alpha' \cdot m_1^{(2)}$ and choose two independent, positive random variables $\tilde{X}_1$, $\tilde{X}_2$ having the distributions $\Gamma(\nu_1, \alpha')$ and $\Gamma(\nu_2, \alpha')$, respectively. Let $\tilde{f}(x) = 1$ $(x \in \mathbb{R})$ and $\tilde{c}_1 = \tilde{c}_2 = 1$. Using now $\tilde{X}_1, \tilde{X}_2, \tilde{f}, \tilde{c}_1, \tilde{c}_2$ instead of X_1, X_2, f, c_1, c_2 and writing $\tilde{m}_n^{(i)}$ instead of $m_n^{(i)}$, we can exactly repeat our deduction given above: we find that, with $M_n^{(i)} = \tilde{m}_n^{(i)}$, (2) and (3) hold. (One detail of our argumentation must be modified: the functions $x^n \cdot \tilde{f}(x)$ are not bounded. But the finiteness of $\tilde{m}_n^{(i)}$, the nth moment of $\Gamma(\nu_i, \alpha')$, is known.) Hence both the $m_n^{(i)}$ and $\tilde{m}_n^{(i)}$ satisfy the same system of equations which implies

$m_n^{(i)} = \tilde{m}_n^{(i)}$ $(n = 1, 2, \cdots; i = 1, 2)$. At first, well-known formulas give $\tilde{m}_1^{(1)} = \nu_1/\alpha'$, $\tilde{m}_1^{(2)} = \nu_2/\alpha'$, $\tilde{m}_2^{(1)} = \nu_1(\nu_1 + 1)\cdot(\alpha')^{-2}$; therefore, $\tilde{m}_1^{(1)} = m_1^{(1)}$, $\tilde{m}_1^{(2)} = m_1^{(2)}$, $\tilde{m}_2^{(1)} = m_2^{(1)}$. But under the assumption $M_n^{(i)} > 0$ $(n = 1, 2, \cdots; i = 1, 2)$, (2) and (3) result in a calculus to determine all $M_n^{(i)}$ out of $M_1^{(1)}$, $M_1^{(2)}$, and $M_2^{(1)}$. Since a gamma distribution cannot have the same moments as any other distribution on $\mathbb{R}$, we have $\mu_i = \Gamma(\nu_i, \alpha')$ $(i = 1, 2)$. Put $g(x) = 1_{(0,\infty)}(x)\cdot e^x$ $(x \in \mathbb{R})$. Combining the relations

$$\frac{dP_i}{d\mu_i} = c_i \cdot g \qquad \text{and} \qquad \frac{d\mu_i}{d\lambda} = (\Gamma(\nu_i))^{-1}\alpha'^{\nu_i}\cdot d(\nu_i, \alpha') \qquad i = 1, 2,$$

we get $dP_i/d\lambda = a_i \cdot d(\nu_i, \alpha' - 1)$, where a_i is a constant factor $(i = 1, 2)$. Hence $\int d(\nu_i, \alpha' - 1)\, d\lambda < \infty$, $\alpha' - 1 > 0$, and $P_i = \Gamma(\nu_i, \alpha)$ with $\alpha = \alpha' - 1$ $(i = 1, 2)$.

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