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ADMISSIBLE REPRESENTATION OF ASYMPTOTICALLY OPTIMAL ESTIMATES

BY H. STRASSER

University of Bayreuth

A sequence of medians of posterior distributions is approximately median unbiased of order $o(n^{-1})$ iff the prior density is equal to the square root of Fisher's information function. It is shown that in this case the sequence of medians of posterior distributions is even an optimum sequence of estimates within the class of all estimator sequences being approximately median unbiased of order $o(n^{-1})$. The result is proved by showing equivalence with an expansion of an optimum sequence given by Pfanzagl. In the case of a location parameter family the Bayesian representation is admissible.

1. Introduction. Let $(\Omega, \mathscr{A})$ be a sample space and $P_\theta \mid \mathscr{A}$, $\theta \in \Theta \subseteqq \mathbb{R}$, a family of probability measures. A measurable mapping $S_n \colon \Omega^n \to \Theta$ is called *estimate* of θ for the sample size n. Recall that S_n is *median unbiased* if

$$P_\theta^n\{S_n \geqq \theta\} \geqq \tfrac{1}{2} \quad \text{and} \quad P_\theta^n\{S_n \leqq \theta\} \geqq \tfrac{1}{2} \quad \text{for all} \quad \theta \in \Theta .$$

A median unbiased estimate T_n is called *optimal* within the class of all median unbiased estimates if

$$P_\theta^n\{\theta - c < T_n < \theta + c\} \geqq P_\theta^n\{\theta - c < S_n < \theta + c\}$$

for all $\theta \in \Theta$, $c > 0$, and any further median unbiased estimate S_n.

It is well known that for certain families with monotone likelihood ratios there exist optimal median unbiased estimates (e.g., cf. Lehmann (1959) and Pfanzagl (1970a)). In general, however, optimal median unbiased estimates need not exist. This is the reason why the weaker concept of approximately median unbiased estimates has been introduced by Pfanzagl (1970b, 1973 and 1975).

DEFINITION 1. A sequence (S_n) of estimates is (*approximately*) *median unbiased* of order $o(n^{-1})$ if for every compact $K \subseteqq \Theta$

$$\inf_{\theta \in K} P_\theta^n\{S_n \geqq \theta\} \geqq \tfrac{1}{2} - o(n^{-1})$$
$$\inf_{\theta \in K} P_\theta^n\{S_n \leqq \theta\} \geqq \tfrac{1}{2} - o(n^{-1}) .$$

Let $\mathscr{C}$ be the class of all sequences (T_n) being median unbiased of order $o(n^{-1})$.

DEFINITION 2. A sequence $(T_n) \in \mathscr{C}$ is called *asymptotically optimal* in $\mathscr{C}$ of order $o(n^{-1})$ if

$$P_\theta^n\{\theta - tn^{-\frac{1}{2}} < T_n < \theta + tn^{-\frac{1}{2}}\} \geqq P_\theta^n\{\theta - tn^{-\frac{1}{2}} < S_n < \theta + tn^{-\frac{1}{2}}\} - o(n^{-1})$$

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for any further sequence $(S_n) \in \mathscr{C}$ and uniformly for $t > 0$ and $\theta \in K$ where $K \subseteqq \Theta$ is compact.

If there exists any optimal sequence in $\mathscr{C}$ then there exist infinitely many different sequences which are optimal. A particular sequence in $\mathscr{C}$ which is optimal is called a representation of an optimal sequence. (More exactly we should call it a representation of the equivalence class of optimal sequences.)

The problem of existence of optimal sequences in $\mathscr{C}$ has been solved by Pfanzagl (1975). In this paper Pfanzagl describes the optimal sequences in $\mathscr{C}$ by their asymptotic expansions of order $o(n^{-1})$ around some initial estimates. By way of illustration, it turns out that maximum likelihood estimates have to be improved by a bias correction of magnitude n^{-1} in order to become an optimal sequence in $\mathscr{C}$.

In the present paper we give a Bayesian representation of optimal sequences in $\mathscr{C}$. The main result is as follows. Consider a sequence (μ_n) of approximate medians of posterior distributions. If the prior distribution is chosen in such a way that (μ_n) is an element of $\mathscr{C}$ then (μ_n) is optimal in $\mathscr{C}$. It has been shown previously by Peers and Welch (1963) that $(\mu_n) \in \mathscr{C}$ iff the prior density equals the square root of Fisher's information function. A related result is due to Hartigan (1965). In general, however, this function does not define a probability measure and conditions are required which imply consistency of posterior distributions. In this respect our conditions are weaker than Hartigan's.

Let us stress that we are not interested in Bayesian representations for reasons of Bayesian philosophy. The true reason is that Bayes estimates (for finite priors and sometimes even for σ-finite priors) are fairly good from the finite sample point of view. We will illustrate this fact for the particular case of a location parameter family.

Assume that $(\Omega, \mathscr{A}) = (\mathbb{R}, \mathscr{B})$, $\Theta = \mathbb{R}$, and that $\{P_\theta\}_{\theta \in \Theta}$ is a location family having Lebesgue densities $h_\theta(\omega) = h(\omega - \theta)$, $(\omega, \theta) \in \mathbb{R}^2$. If the equation

$$\int_{-\infty}^{x} \prod h(\omega_i - \sigma)\, d\sigma = \tfrac{1}{2} \int_{-\infty}^{+\infty} \prod h(\omega_i - \sigma)\, d\sigma\,, \qquad x \in \mathbb{R}\,,$$

admits a unique solution $\mu_n(\boldsymbol{\omega})$ for every $\boldsymbol{\omega} \in \mathbb{R}^n$, then μ_n is an (exactly) median unbiased estimate of θ. It is known that μ_n is the best equivariant estimate of θ for the loss function $L(t, \theta) = |t - \theta|$, and under weak conditions on h it is even an admissible estimate (Fox and Rubin (1964)). Nothing seems to be known about the role of μ_n within the class of all median unbiased estimates.

The main result of the present paper implies that under appropriate regularity conditions on h the sequence (μ_n) is asymptotically optimal of order $o(n^{-1})$. Thus, in the case of a location parameter family we have obtained an admissible representation of an optimal sequence in $\mathscr{C}$. This result could be an answer to a question posed by Pfanzagl (1975), page 35: "Why should we study the behaviour of Bayes or Pitman estimates if know that none of these can serve better than the estimators of our essentially complete class?" In this paper Pfanzagl describes the elements of an asymptotically complete class of order $o(n^{-1})$ by their asymptotic expansions of order $o(n^{-1})$. In view of our results an answer to Pfanzagl's question could be as follows: Bayes or Pitman estimates might lead to equivalent representations of elements of Pfanzagl's complete class which are superior to asymptotic expansions as far as finite sample size properties are concerned.

2. Notations. Let $(\Omega, \mathscr{A})$ be a measurable space and $P_\theta \mid \mathscr{A}$, $\theta \in \Theta$, a family of probability measures. The n-fold product of $(\Omega, \mathscr{A})$ is denoted by $(\Omega^n, \mathscr{A}^n)$ and a single element of Ω^n by $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_n)$. Let $\Theta \subseteq \mathbb{R}$ be an open interval and let $\mathscr{B}$ be the Borel-σ-algebra of $\mathbb{R}$.

Assume that the family $\{P_\theta\}_{\theta \in \Theta}$ is dominated by a σ-finite measure $\mu \mid \mathscr{A}$ and let $h_\theta \equiv dP_\theta/d\mu$, $\theta \in \Theta$. For all $\omega \in \Omega$, $\theta \in \Theta$, $j = 1, 2, \cdots$ define

$$l(\omega, \theta) = \log h_\theta(\omega)$$

$$l_j(\omega, \theta) = \frac{d^j}{d\theta^j} l(\omega, \theta)$$

$$x_j^{(n)}(\boldsymbol{\omega}, \theta) = n^{-\frac{1}{2}} \sum_{i=1}^n (l_j(\omega_i, \theta) - E_\theta(l_j(\cdot, \theta))) .$$

Moreover, denote

$$L_{ijkm}(\theta) = E_\theta(l_1(\cdot, \theta)^i l_2(\cdot, \theta)^j l_3(\cdot, \theta)^k l_4(\cdot, \theta)^m) .$$

For L_{ij00} or L_{i000} we will write L_{ij} or L_i, respectively. Recall that $L_2 = -L_{01}$ is Fisher's information function.

A positive measure $\lambda \mid \mathscr{B} \cap \Theta$ is called Borel measure if $\lambda(K) < \infty$ for every compact $K \subseteq \Theta$. If λ has a positive Lebesgue density p, let $\Lambda = \log p$.

Let Φ denote the standard normal distribution and φ its density. If G is a distribution function on $\mathbb{R}$ then $G(B)$, $B \in \mathscr{B}$, denotes the value of the Borel measure of B defined by G.

Let P, Q denote probability measures on $(\Omega, \mathscr{A})$. Then

$$\|P - Q\| \equiv \sup_{A \in \mathscr{A}} |P(A) - Q(A)|$$

is called variational distance of P and Q.

3. Results. The present section contains the results of the paper. Regularity conditions and proofs are collected in Sections 4 and 5.

DEFINITION 3. Let $\lambda \mid \mathscr{B} \cap \Theta$ be a Borel measure. For those $\boldsymbol{\omega} \in \Omega^n$, $n \in \mathbb{N}$, for which

$$0 < \int_\Theta \frac{dP_\theta^n}{d\mu^n}(\boldsymbol{\omega})\lambda(d\theta) < \infty ,$$

the *posterior distribution* $R_{n,\boldsymbol{\omega}} \mid \mathscr{B} \cap \Theta$ is defined by

$$R_{n,\boldsymbol{\omega}}(B \cap \Theta) = \frac{\int_{B \cap \Theta} \frac{dP_\theta^n}{d\mu^n}(\boldsymbol{\omega})\lambda(d\theta)}{\int_\Theta \frac{dP_\theta^n}{d\mu^n}(\boldsymbol{\omega})\lambda(d\theta)} , \qquad B \in \mathscr{B}.$$

The posterior distribution $R_{n,\boldsymbol{\omega}}$ depends on the prior distribution λ, the sample size $n \in \mathbb{N}$ and the sample $\boldsymbol{\omega} \in \Omega^n$.

Our first result states that posterior distributions concentrate on arbitrary neighbourhoods of the true parameter value. If λ is a probability measure there are previous results of this kind in Le Cam (1953) and Schwartz (1965). Estimates of the speed of convergence are given in Strasser (1976a). The present result covers also the case of σ-finite measures λ. A similar result has been given by Hartigan (1965), Theorem 1. Hartigan's condition II 2 requires boundedness of $\omega \mapsto \int h_\sigma(\omega)\lambda(d\sigma)$ which implies

$$\int E_\theta(h_\sigma)\lambda(d\sigma) < \infty \,. \tag{3.1}$$

This condition is stronger than our condition

$$\int \exp(E_\theta(\log h_\sigma))\lambda(d\sigma) < \infty \,. \tag{3.2}$$

This is easily seen from $h_\sigma = \exp(\log h_\sigma)$ and applying Jensen's inequality. A slightly more general condition than (3.1) is used by Bickel and Yahav (1967) (condition A′2.2, page 274). The same generalization could be applied to our condition (3.2).

EXAMPLE 1. Let $\Omega = \mathbb{R}$, $\mathscr{A} = \mathscr{B}$ and h an absolutely continuous probability density. If $h_\theta(\omega) = h(\omega - \theta)$, $(\omega, \theta) \in \mathbb{R}^2$, then Fisher's information function L_2 is constant. It can easily be seen that the Lebesgue measure (having constant Lebesgue density) satisfies conditions (3.1) and (3.2).

EXAMPLE 2. Let $\Omega = (0, \infty)$, $\mathscr{A} = \mathscr{B} \cap (0, \infty)$ and h an absolutely continuous probability density on $(0, \infty)$. Assume that h is strongly unimodal (i.e., $\log h$ is concave) and has finite expectation. Define $h_\theta(\omega) = \theta^{-1}h(\omega\theta^{-1})$, $(\omega, \theta) \in (0, \infty)^2$. If $\{P_\theta\}_{\theta\in\Theta}$ has finite Fisher's information L_2 then $L_2^{\frac{1}{2}}(\theta) = c\theta^{-1}$ for some $c > 0$. It can easily be shown that the measure λ with Lebesgue density $L_2^{\frac{1}{2}}$ satisfies condition (3.2).

THEOREM 1. *Assume that conditions* (i)—(iv) *are satisfied. Assume that* $\lambda | \mathscr{B} \cap \Theta$ *is a Borel measure satisfying conditions* (j) *and* (jjj). *Let* $\delta > 0$ *be arbitrary and let* $U_\theta = \{\sigma \in \Theta : |\sigma - \theta| < \delta\}$, $\theta \in \Theta$. *Then uniformly on every compact* $K \subseteqq \Theta$

$$P_\theta^n \left\{0 < \int_\Theta \frac{dP_\sigma^n}{d\mu^n}\lambda(d\sigma) < \infty\right\} \geqq 1 - o(n^{-1})$$

and

$$P_\theta^n\{R_{n,\boldsymbol{\omega}}(U_\theta) \geqq 1 - Ce^{-cn}\} \geqq 1 - o(n^{-1}) \,.$$

Here and in the following C, c denote finite, positive constants depending on the compact set $K \subseteqq \Theta$.

The result of Theorem 1 leads to the question whether a similar assertion is true when U_θ is replaced by a sequence of shrinking neighbourhoods. The first result in this direction is due to Le Cam (1953). Estimates for the speed of convergence can be found in Strasser (1976a). Lemma 1 is a related result which is concerned with neighbourhoods centered at

$$X_{n,\theta}(\boldsymbol{\omega}) = \theta + (nL_2(\theta))^{-1} \sum_{i=1}^n \frac{d}{d\theta} \log h_\theta(\omega_i)\,, \qquad \boldsymbol{\omega} \in \Omega^n,\ \theta \in \Theta\,.$$

For some $a > 1$ denote

$$W_n(\boldsymbol{\omega}, \theta) = \{\sigma \in \Theta : |\sigma - X_{n,\theta}(\boldsymbol{\omega})| < (nL_2(\theta))^{-\frac{1}{2}}(\log n)^a\}\,.$$

Recall that $X_{n,\theta}$ is the beginning of the asymptotic expansion of any asymptotically efficient estimate around θ.

LEMMA 1. *Assume that conditions* (i)—(ix), (jj) *and* (jjj) *are satisfied. Let* $a > 1$ *and* $r > 0$ *be arbitrary. Then*

$$P_\tau^n\{\theta \in W_n(\boldsymbol{\omega}, \theta)\} \geqq 1 - o(n^{-1})$$

and

$$P_\tau^n\{R_{n,\boldsymbol{\omega}}(W_n(\boldsymbol{\omega}, \theta)) \geqq 1 - Cn^{-r}\} \geqq 1 - o(n^{-1})$$

uniformly for $|\tau - \theta| \leqq (\log n)n^{-\frac{1}{2}}$.

It should be noted that the assertion of Lemma 1 is true even if the true parameter value τ deviates slightly from the parameter value θ which is the starting point for the expansion $X_{n,\theta}$. The same holds for all results which follow.

DEFINITION 4. Let $F_{n,\boldsymbol{\omega}}^\theta | \mathscr{B}$ be the Borel measure which is induced by $R_{n,\boldsymbol{\omega}} | \mathscr{B} \cap \Theta$ and the mapping $T_{n,\boldsymbol{\omega}}^\theta : \Theta \to \mathbb{R}$ defined by

$$T_{n,\boldsymbol{\omega}}^\theta(\sigma) = (nL_2(\theta))^{\frac{1}{2}}(\sigma - X_{n,\theta}(\boldsymbol{\omega}))\,.$$

Those transformed posterior distributions $F_{n,\boldsymbol{\omega}}^\theta$ will be approximated by probability measures whose densities relative to the normal distribution are polynomials.

DEFINITION 5. Let

$$P_n(s) = 1 + n^{-\frac{1}{2}} \sum_{i=1}^3 \gamma_{1i} s^i + n^{-1} \sum_{j=1}^6 \gamma_{2j} s^j\,, \qquad s \in \mathbb{R}\,,$$

be polynomials with coefficients $\gamma_{kl} = \gamma_{kl}(n, \boldsymbol{\omega}, \theta)$ defined below (Remark 1). Then

$$G_{n,\boldsymbol{\omega}}^\theta(B) \equiv \frac{\int_B P_n(\boldsymbol{\omega}, \theta)(s)\Phi(ds)}{\int_{\mathbb{R}} P_n(\boldsymbol{\omega}, \theta)(s)\Phi(ds)}\,, \qquad B \in \mathscr{B}\,.$$

The next theorem is related to the famous result of Bernstein and von Mises which approximates posterior distributions by normal distributions. Numerous authors have contributed to this subject (e.g., Le Cam (1953), Bickel and Yahav (1967)). Estimates for the speed of convergence are given in Strasser (1976a), and Hipp and Michel (1976). Asymptotic expansions of the posterior distribution functions are due to Johnson (1970). Our Theorem 2 shows that the asymptotic expansions $G_{n,\boldsymbol{\omega}}^\theta$ approximate the posterior distributions $F_{n,\boldsymbol{\omega}}^\theta$ with respect to the variational distance.

THEOREM 2. *Assume that conditions* (i)—(ix), (jj) *and* (jjj) *are satisfied. Let* $K \subseteq \Theta$ *be compact. Then*

$$P_\tau^n\{\sup_{B\in\mathscr{B}} |F^\theta_{n,\omega}(B) - G^\theta_{n,\omega}(B)| \leqq C(\log n)^c n^{-\frac{3}{2}}\} \geqq 1 - o(n^{-1})$$

uniformly for $\theta \in K$, $|\tau - \theta| \leqq (\log n)n^{-\frac{1}{2}}$.

REMARK 1. The coefficients γ_{ij} depend on $x_k = x_k^{(n)}(\boldsymbol{\omega}, \theta)$, $1 \leqq k \leqq 3$, and on $L_{ijkm}(\theta)$. We give these coefficients in a coded form in order to avoid complicated expressions.

$$\begin{aligned}
\gamma_{11} &= a_{11} + b_{11}\\
\gamma_{12} &= a_{12}\\
\gamma_{13} &= a_{13}\\
\gamma_{21} &= a_{21} + b_{21}\\
\gamma_{22} &= a_{22} + b_{22} + \tfrac{1}{2}a_{11}^2 + a_{11}b_{11} + \tfrac{1}{2}b_{11}^2\\
\gamma_{23} &= a_{23} + a_{11}a_{12} + a_{12}b_{11}\\
\gamma_{24} &= a_{24} + \tfrac{1}{2}a_{12}^2 + a_{11}a_{13} + a_{13}b_{11}\\
\gamma_{25} &= a_{12}a_{13}\\
\gamma_{26} &= \tfrac{1}{2}a_{13}^2
\end{aligned}$$

$$\begin{aligned}
a_{11} &= x_1x_2L_2^{-\frac{3}{2}} + \tfrac{1}{2}x_1^2L_{001}L_2^{-\frac{5}{2}}\\
a_{12} &= \tfrac{1}{2}x_2L_2^{-1} + \tfrac{1}{2}x_1L_{001}L_2^{-2}\\
a_{13} &= \tfrac{1}{6}L_{001}L_2^{-\frac{3}{2}}\\
a_{21} &= \tfrac{1}{2}x_1^2x_3L_2^{-\frac{5}{2}} + \tfrac{1}{6}x_1^3L_{0001}L_2^{-\frac{7}{2}}\\
a_{22} &= \tfrac{1}{2}x_1x_3L_2^{-2} + \tfrac{1}{4}x_1^2L_{0001}L_2^{-3}\\
a_{23} &= \tfrac{1}{6}x_3L_2^{-\frac{3}{2}} + \tfrac{1}{6}x_1L_{0001}L_2^{-\frac{5}{2}}\\
a_{24} &= \tfrac{1}{24}L_{0001}L_2^{-2}\\
b_{11} &= \Lambda'L_2^{-\frac{1}{2}}\\
b_{21} &= x_1\Lambda''L_2^{-\frac{3}{2}}\\
b_{22} &= \tfrac{1}{2}\Lambda''L_2^{-1}\,.
\end{aligned}$$

DEFINITION 6. A sequence of $\mathscr{A}^n$-measurable functions $\mu_n\colon \Omega^n \to \Theta$ is called a sequence of (*approximate*) *medians of the posterior distributions* $R_{n,\omega}$ if for every compact $K \subseteq \Theta$

$$\sup_{\theta\in K} P_\theta^n\{|R_{n,\omega}\{\sigma \leqq \mu_n\} - \tfrac{1}{2}| \leqq C(\log n)^c n^{-\frac{3}{2}}\} \geqq 1 - o(n^{-1})\,.$$

Lemma 2 gives an asymptotic expansion of medians of posterior distributions for arbitrary regular prior densities p. The expansion is given as a polynomial in $x_1^{(n)}$, $x_2^{(n)}$, $x_3^{(n)}$. Related expansions around the maximum likelihood estimate have been given by Johnson (1970).

LEMMA 2. *Assume that conditions* (i)—(ix), (jj) *and* (jjj) *are satisfied. Let* (μ_n)

be a sequence of medians of the posterior distributions. Then there exist polynomials

$$\begin{aligned} C_{n,\theta} = a_0 &+ n^{-\frac{1}{2}}a_1 + n^{-1}a_2 \\ &+ x_1 \\ &+ n^{-\frac{1}{2}}(\textstyle\sum_i b_i x_i + \sum_{j,k} b_{jk} x_j x_k) \\ &+ n^{-1}(\textstyle\sum_i c_i x_i + \sum_{j,k} c_{jk} x_j x_k + \sum_{p,q,r} c_{pqr} x_p x_q x_r) \end{aligned}$$

such that for every compact $K \subseteqq \Theta$

$$P_\tau^n\{|n^{\frac{1}{2}}L_2(\theta)(\mu_n - \theta) - C_{n,\theta}| \leqq C(\log n)^c n^{-\frac{3}{2}}\} \geqq 1 - o(n^{-1})$$

uniformly for $\theta \in K$, $|\tau - \theta| \leqq (\log n)n^{-\frac{1}{2}}$.

REMARK 2. The coefficients of the polynomials in Lemma 2 are

$$\begin{aligned} a_1 &= \tfrac{1}{3}L_{001}L_2^{-1} + \Lambda' \\ b_{11} &= \tfrac{1}{2}L_{001}L_2^{-2} \\ b_{12} &= L_2^{-1} \\ c_1 &= \tfrac{1}{3}L_{0001}L_2^{-2} + \Lambda''L_2^{-1} + \tfrac{2}{3}L_{001}^2L_2^{-3} + \Lambda'L_{001}L_2^{-2} \\ c_2 &= \tfrac{2}{3}L_{001}L_2^{-2} + \Lambda'L_2^{-1} \\ c_3 &= \tfrac{1}{3}L_2^{-1} \\ c_{111} &= \tfrac{1}{6}L_{0001}L_2^{-3} + \tfrac{1}{2}L_{001}^2L_2^{-4} \\ c_{112} &= \tfrac{3}{2}L_{001}L_2^{-3} \\ c_{113} &= \tfrac{1}{2}L_2^{-2} \\ c_{122} &= L_2^{-2}\,. \end{aligned}$$

Coefficients not specified are zero.

Lemma 3 gives the bias correction for (μ_n) which is needed to obtain median unbiasedness of order $o(n^{-1})$. The proof is based on Edgeworth expansions which require additional regularity conditions to exclude lattice distributions. The same result has been obtained previously by Peers and Welch (1963) (without stating any regularity conditions or uniformity assertions).

LEMMA 3. *Assume that conditions* (i)—(xii), (jj) *and* (jjj) *are satisfied. Let* (μ_n) *be a sequence of medians of the posterior distributions. Then*

$$P_\theta^n\{\mu_n \geqq \theta - n^{-1}r_n(\theta)\} = \tfrac{1}{2} + o(n^{-1})$$

uniformly on every compact $K \subseteqq \Theta$ iff

$$r_n = L_2^{-2}(L_{11} + \tfrac{1}{2}L_3 - \Lambda'L_2)\,.$$

It follows that the bias correction vanishes iff $p = L_2^{\frac{1}{2}}$ (use $L_2' = 2L_{11} + L_3$ and $L_{001} = -3L_{11} - L_3$). Thus we obtain a characterization of those prior distributions for which (μ_n) is median unbiased of order $o(n^{-1})$.

COROLLARY 1. *Assume that conditions* (i)—(xii) *are satisfied and let* (μ_n) *be a sequence of medians of posterior distributions.*

(1) *If* $p = L_2^{\frac{1}{2}}$ *satisfies conditions* (jj) *and* (jjj) *then* (μ_n) *is median unbiased of order* $o(n^{-1})$.

(2) *If an arbitrary prior density p satisfies conditions* (jj) *and* (jjj) *and if* (μ_n) *is median unbiased of order* $o(n^{-1})$ *then* $p = L_2^{\frac{1}{2}}$.

The next theorem is our main result. It states that if $p = L_2^{\frac{1}{2}}$ then medians of the posterior distributions are optimal in $\mathscr{C}$. Pfanzagl (1973) proves for certain asymptotic expansions that the abovementioned result holds with $o(n^{-\frac{1}{2}})$ instead of $o(n^{-1})$. In another paper Pfanzagl (1975) announces that for the same asymptotic expansions the result is even true of order $o(n^{-1})$. Therefore we have to show that medians of posterior distributions for $p = L_2^{\frac{1}{2}}$ are equivalent of order $o(n^{-1})$ with Pfanzagl's expansions. Let us denote those expansions by $B_0^{(n)}$.

THEOREM 3. *Assume that* (i)—(xii) *are satisfied and that* $p = L_2^{\frac{1}{2}}$ *fulfills conditions* (jj) *and* (jjj). *Then for every compact* $K \subseteqq \Theta$

$$\sup_{\theta\in K} |P_\theta^n\{\mu_n \geqq \theta + tn^{-\frac{1}{2}}\} - P_\theta^n\{B_0^{(n)} \geqq \theta + tn^{-\frac{1}{2}}\}| = o(n^{-1})$$
$$\sup_{\theta\in K} |P_\theta^n\{\mu_n \leqq \theta - tn^{-\frac{1}{2}}\} - P_\theta^n\{B_0^{(n)} \leqq \theta - tn^{-\frac{1}{2}}\}| = o(n^{-1})$$

uniformly for $t \geqq 0$.

4. Regularity conditions. Let $\lambda \mid \mathscr{B} \cap \Theta$ be an absolutely continuous Borel measure with positive and continuous density p. Denote $\Lambda = \log p$.

(j) For every $\delta > 0$ and every compact $K \subseteqq \Theta$

$$\inf_{\theta\in K} \lambda\{\sigma \in \Theta : |\sigma - \theta| < \delta\} > 0 .$$

(jj) Λ is twice differentiable and the second derivative satisfies for every compact $K \subseteqq \Theta$

$$|\Lambda''(\sigma) - \Lambda''(\tau)| \leqq c_K|\sigma - \tau| \qquad \text{if} \quad |\sigma - \tau| < e_K\,, \quad \sigma, \tau \in K\,.$$

(jjj) For every compact $K \subseteqq \Theta$

$$\sup_{\theta\in K} \int_\Theta \exp(E_\theta(l_\sigma))\lambda(d\sigma) < \infty\,.$$

The following regularity conditions deal with log-likelihood functions.

(i) $\theta \mapsto P_\theta$ is continuous with respect to the supremum metric.

(ii) For every $\omega \in \Omega$, $\theta \mapsto l(\omega, \theta)$ is continuous on $\bar{\Theta}$.

(iii) For every $\theta \in \Theta$ there exists a neighbourhood W_θ of θ such that

$$\sup_{\tau\in W_\theta} E_\tau(\sup_{\sigma\in W_\theta} |l_\sigma|^3) < \infty\,.$$

(iv) For every $\theta \in \bar{\Theta}$ there exists a neighbourhood U_θ of θ such that for every neighbourhood U of θ, $U \subseteqq U_\theta$, and every compact $K \subseteqq \Theta$

$$\sup_{\tau\in K} E_\tau(|\sup_{\sigma\in U} l_\sigma|^3) < \infty\,.$$

(v) $l \mapsto l(\omega, \theta)$ is four times differentiable on Θ for every $\omega \in \Omega$ and $L_1(\theta) = 0$.

(vi) For every $\theta \in \Theta$ there exists a neighbourhood U_θ of θ such that

$$\inf_{\tau\in U_\theta} L_2(\tau) > 0\,.$$

(vii) For every $\theta \in \Theta$ there exists a neighbourhood U_θ of θ such that

$$\sup_{\tau_1, \delta \in U_\theta} E_\tau(|l_i(\delta)|^{k_i}) < \infty , \qquad 1 \leqq i \leqq 4 ,$$

where $k_1 = 5$, $k_2 = 5$, $k_3 = 3$, $k_4 = 4$.

(viii) For every $\theta \in \Theta$ there exists a neighbourhood U_θ of θ and a function $m(\cdot, \theta)$ such that

(a) $$|l_4(\omega, \tau) - l_4(\omega, \delta)| \leqq |\tau - \delta| m(\omega, \theta) , \qquad \omega \in \Omega,\ (\tau, \delta) \in \Theta^2 .$$

(b) $$\sup_{\tau \in U_\theta} E_\tau(|m(\cdot, \theta)|^3) < \infty .$$

(ix) For every $\theta \in \Theta$ there exists a neighbourhood U_θ and a function $k(\cdot, \theta)$ such that

(a) $$\left|\frac{h(\omega, \delta)}{h(\omega, \tau)} - 1\right| \leqq |\delta - \tau| k(\omega, \theta) , \qquad \omega \in \Omega,\ (\delta, \tau) \in \Theta^2 .$$

(b) $$\sup_{\tau \in U_\theta} E_\tau(|k(\cdot, \tau)|^3) < \infty .$$

Let $\tau \in \Theta$, $\theta \in \Theta$. Let $Q_{\tau,\theta} | \mathscr{B}^3$ be the probability measure which is induced by P_τ and

$$(l_1(\theta) - E_\tau(l_1(\theta)),\ l_2(\theta) - E_\tau(l_2(\theta)),\ l_3(\theta) - E_\tau(l_3(\theta))) .$$

Let $Q_{\tau,\theta}^n$ be the n-fold product measure of $Q_{\tau,\theta}$ and let $Q_{\tau,\theta}^{(n)}$ be the distribution of the mapping $\mathbb{R}^{3n} \to \mathbb{R}^3$ defined by

$$[(\xi_{1i}, \xi_{2i}, \xi_{3i})]_{1 \leqq i \leqq n} \mapsto n^{-\frac{1}{2}} \sum_{i=1}^n (\xi_{1i}, \xi_{2i}, \xi_{3i})$$

under $Q_{\tau,\theta}^n$.

(x) For every $\theta \in \Theta$ there exists a neighbourhood U_θ such that

$$\limsup_{\|u\| \to \infty} \sup_{\tau, \delta \in U_\theta} |\textstyle\int \exp(i \sum_{j=1}^3 u_j \xi_j) Q_{\tau,\delta}(d\xi)| < 1 .$$

(xi) The covariance matrix of $Q_{\theta,\theta}$ is positive definite for every $\theta \in \Theta$.

(xii) For every $\theta \in \Theta$ there exists a neighbourhood U_θ such that

$$\lim_{a \to \infty} \sup_{\tau, \delta \in U_\theta} \textstyle\int \|\xi\|^4 1_{\{\|\xi\| > a\}} Q_{\tau,\delta}(d\xi) = 0 .$$

5. Proofs.

PROOF OF THEOREM 1. For every $\tau \in \bar{\Theta}$, every compact $K \subseteq \Theta$ and every $\varepsilon > 0$ there exists a neighbourhood V_τ of τ such that uniformly for $\theta \in K$

$$(5.1) \qquad P_\theta^n \left\{\boldsymbol{\omega} \in \Omega^n : \sup_{\sigma \in V_\tau} \left(\frac{1}{n} \sum_{i=1}^n l(\omega_i, \sigma) - E_\theta(l_\sigma)\right) < \varepsilon\right\} \geqq 1 - o(n^{-1}) .$$

The proof of (5.1) is almost the same as for part one of Lemma 6 in Michel and Pfanzagl (1970). The only difference consists in using a stronger version of the law of large numbers, e.g., Lemma 1 in Pfanzagl (1973).

We prove that uniformly for $\theta \in K$

$$(5.2) \qquad P_\theta^n \left\{\boldsymbol{\omega} \in \Omega^n : 0 < \int \frac{dP_\sigma^n}{d\mu^n}(\boldsymbol{\omega}) \lambda(d\sigma) < \infty\right\} \geqq 1 - o(n^{-1}) .$$

Moreover there exists a compact interval $K_0 \subseteqq \Theta$ and $\alpha > 0$ such that uniformly for $\theta \in K$

$$(5.3) \qquad P_\theta{}^n\{\boldsymbol{\omega} \in \Omega^n : R_{n,\boldsymbol{\omega}}(K_0) \geqq 1 - Ce^{-\alpha n}\} \geqq 1 - o(n^{-1}) .$$

Let $K \subseteqq \Theta$ be compact. Let θ_0, θ_1 be such that $\bar{\Theta} = [\theta_0, \theta_1]$. Choose neighbourhoods $U_0 \equiv U_{\theta_0}$, $U_1 \equiv U_{\theta_1}$ such that $(\overline{U_0 \cup U_1}) \cap K = \emptyset$. Since $(\theta, \sigma) \mapsto E_\theta(l_\sigma) - E_\theta(l_\theta)$ is upper semicontinuous and negative on $K \times (U_0 \cup U_1)$ there exist $\varepsilon > 0$, $\alpha > 0$ such that

$$\sup_{\theta \in K} \sup_{\sigma \in U_0 \cup U_1} E_\theta(l_\sigma) - E_\theta(l_\theta) + \varepsilon \leqq -\alpha .$$

Choose $V_0 \equiv V_{\theta_0} \subseteqq U_0$, $V_1 \equiv V_{\theta_1} \subseteqq U_1$, according to (5.1). Let $K_0 \subseteqq \Theta$ be a compact interval such that $K \subseteqq \mathring{K}_0$, $\bar{\Theta} \backslash K_0 \subseteqq V_0 \cup V_1$. Then

$$\sup_{\sigma \in V_0 \cup V_1} \frac{1}{n} \sum_{i=1}^n l(\omega_i, \sigma) - E_\theta(l_\sigma) < \varepsilon$$

implies for every $\theta \in K$

$$\begin{aligned} \int_{\Theta \backslash K_0} & \exp\left(\textstyle\sum_{i=1}^n l(\omega_i, \sigma) - nE_\theta(l_\theta)\right)\lambda(d\sigma) \\ & \leqq \int_{\Theta \backslash K_0} \exp(n(E_\theta(l_\sigma) - E_\theta(l_\theta) + \varepsilon))\lambda(d\sigma) \\ & \leqq \exp(-\alpha(n-1) - E_\theta(l_\theta) + \varepsilon) \int_\Theta \exp(E_\theta(l_\sigma))\lambda(d\sigma) \\ & \leqq C \exp(-\alpha n) . \end{aligned}$$

Now (5.1) proves (5.2) and (5.3).

It remains to show that

$$P_\theta{}^n\{R_{n,\boldsymbol{\omega}}(U_\theta \mid K_0) \geqq 1 - Ce^{-cn}\} \geqq 1 - o(n^{-1})$$

uniformly for $\theta \in K$. Since $K \subseteqq \mathring{K}_0$ and $\lambda(K_0) < \infty$ this assertion is equivalent with Theorem 1 in Strasser (1976a). □

PROOF OF LEMMA 1. According to conditions (vii) and (viii)(b) there exist neighbourhoods V_θ such that uniformly for $\tau \in V_\theta$

$$P_\tau{}^n\{n^{-\frac{1}{2}}|x_2{}^{(n)}(\cdot, \theta)| \leqq \tfrac{1}{4}L_2(\theta)\} \geqq 1 - o(n^{-1})$$

and

$$P_\tau{}^n\{n^{-1} \textstyle\sum_{i=1}^n m(\omega_i, \theta) \leqq \tfrac{1}{4}L_2(\theta)\} \geqq 1 - o(n^{-1}) .$$

Moreover we have

$$P_\tau{}^n\{|x_1{}^{(n)}| < C \log n\} \geqq 1 - o(n^{-1})$$

uniformly for $|\tau - \theta| \leqq (\log n)n^{-\frac{1}{2}}$. Let $U_\theta = \{\sigma \in \Theta : |\sigma - \theta| < \delta\}$, $\delta > 0$. Since

$$P_\tau{}^n\{\|R_{n,\boldsymbol{\omega}} - R_{n,\boldsymbol{\omega}}(\cdot \mid U_\theta)\| < Ce^{-cn}\} \geqq 1 - o(n^{-1})$$

uniformly for $|\tau - \theta| < (\log n)n^{-\frac{1}{2}}$ we may restrict our attention to $R_{n,\boldsymbol{\omega}}(\cdot \mid U_\theta)$.

Expanding likelihood ratios yields

$$\begin{aligned} \log \frac{dP^n_{\theta + tn^{-\frac{1}{2}}}}{dP_\theta{}^n}(\boldsymbol{\omega}) &= tx_1{}^{(n)}(\boldsymbol{\omega}) + \tfrac{1}{2}n^{-\frac{1}{2}}t^2x_2{}^{(n)}(\boldsymbol{\omega}) - \tfrac{1}{2}t^2L_2 \\ &\quad + \tfrac{1}{2}n^{-1}t^2 \textstyle\sum_{i=1}^n (l_2(\omega_i, \theta_n(\boldsymbol{\omega}, t)) - l_2(\omega_i, \theta)) \\ &= tx_1{}^{(n)} - \tfrac{1}{2}t^2L_2 + \tfrac{1}{2}t^2R_n(t) \end{aligned}$$

where $|\theta_n(\boldsymbol{\omega}, t) - \theta| \leqq tn^{-\frac{1}{2}}$ and

$$P_\tau{}^n\{|R_n(t)| \leqq \tfrac{1}{2}L_2 \text{ for all } |t| < \delta n^{\frac{1}{2}}\} \geqq 1 - o(n^{-1})$$

uniformly for $|\tau - \theta| \leqq (\log n)n^{-\frac{1}{2}}$.

Condition (jj) implies that p is bounded away from zero on bounded intervals. Let $p_n(t) = p(\theta + tn^{-\frac{1}{2}})$, $t \in \mathbb{R}$.

Recall that $X_{n,\theta} = \theta + L_2{}^{-1}n^{-\frac{1}{2}}x_1{}^{(n)}$. Easy computations show that

$$R_{n,\omega}(W_n{}' \mid U_\theta)$$

$$\leqq \exp(\tfrac{2}{3}(x_1{}^{(n)})^2L_2{}^{-2}) \frac{\int_{|t|<\delta n^{\frac{1}{2}}, L_2^{\frac{1}{2}}|t - x_1^{(n)}L_2^{-1}| \geqq (\log n)^a} \exp\left(-\frac{1}{4}L_2\left(t - \frac{2x_1{}^{(n)}}{L_2}\right)^2\right) p_n(t)\, dt}{\int_{|t|<\delta n^{\frac{1}{2}}} \exp\left(-\frac{3}{4}L_2\left(t - \frac{2x_1{}^{(n)}}{3L_2}\right)^2\right) p_n(t)\, dt} .$$

The leading factor is bounded by $\exp(C(\log n)^2)$. The numerator is bounded from above by

$$\exp(-\tfrac{1}{4}((\log n)^a - C(\log n))^2) \int_{|t|<\delta n^{\frac{1}{2}}} p_n(t)\, dt \leqq \exp(-C(\log n)^{2a})$$

for all $r > 0$. The denominator is bounded from below by

$$\left(\int_{|t|<\delta n^{\frac{1}{2}}, |t - \frac{2}{3}x_1^{(n)}| < D} p_n(t)\, dt\right) \exp(-\tfrac{3}{4}D^2) ,$$

where $D > 0$ is chosen arbitrary. The points $\theta + tn^{-\frac{1}{2}}$ stay in bounded intervals as long as $|x_1{}^{(n)}| < C(\log n)$ which implies that $p_n(t)$ is bounded away from zero whenever $|t - \frac{2}{3}x_1{}^{(n)}| < D$. □

PROOF OF THEOREM 2. Let $K \subseteq \Theta$ be compact. Keep $\theta \in K$ fixed. The following statements hold uniformly for $\theta \in K$ and $|\tau - \theta| \leqq (\log n)n^{-\frac{1}{2}}$. For $a > 1$ and $B \in \mathscr{B}$ let

$$\psi_n(B) = \int_{W_n \cap \{(nL_2)^{\frac{1}{2}}(\sigma - \theta) - x_1^{(n)}L_2^{-\frac{1}{2}} \in B\}} \frac{dP_\sigma{}^n}{dP_\theta{}^n} \frac{p(\sigma)}{p(\theta)}\, d\sigma .$$

Then Lemma 1 implies that

$$P_\tau{}^n\left\{\sup_{B \in \mathscr{B}} \left|R_{n,\omega}\{(nL_2)^{\frac{1}{2}}(\sigma - \theta) - x_1{}^{(n)}L_2{}^{-\frac{1}{2}} \in B\} - \frac{\psi_n(B)}{\psi_n(\mathbb{R})}\right| \leqq Cn^{-2}\right\}$$

$$\geqq 1 - o(n^{-1}) .$$

We find approximations of $\psi_n(B)$ by expanding

$$\log \frac{dP^n_{\theta + tn^{-\frac{1}{2}}}}{dP_\theta{}^n} , \qquad \log \frac{p(\theta + tn^{-\frac{1}{2}})}{p(\theta)} .$$

Put $s = L_2^{\frac{1}{2}}(t - x_1L_2{}^{-1})$. Then a Taylor expansion yields

$$\begin{aligned} \frac{dP^n_{\theta+tn^{-\frac{1}{2}}}}{dP_\theta{}^n}(\boldsymbol{\omega}) &= \textstyle\sum_{k=1}^4 \frac{t^k}{k!} n^{-k/2}(\sum_{i=1}^n l_k(\omega_i, \theta)) + R_{1,n}(\boldsymbol{\omega}, t, \theta) \\ &= (tx_1 + \tfrac{1}{2}t^2L_{01}) + n^{-\frac{1}{2}}(\tfrac{1}{2}t^2x_2 + \tfrac{1}{6}t^3L_{001}) \\ &\quad + n^{-1}(\tfrac{1}{6}t^3x_3 + \tfrac{1}{24}t^4L_{0001}) + R_{1,n} + R_{2,n} \\ &= -\frac{s^2}{2} + n^{-\frac{1}{2}}A_1(s) + n^{-1}A_2(s) + Q_n{}^{(1)} + R_{1,n} + R_{2,n} \end{aligned}$$

where

$$A_1(s) = sa_{11} + s^2a_{12} + s^3a_{13}$$
$$A_2(s) = sa_{21} + s^2a_{22} + s^3a_{23} + s^4a_{24} ,$$

$Q_n^{(1)}$ does not depend on s and $R_{1,n}$, $R_{2,n}$ are residuals. (Terms like $Q_n^{(1)}$ vanish when considering the quotients $\psi_n(B)/\psi_n(\mathbb{R})$. The residual terms will be estimated below.) Another Taylor expansion yields

$$\log \frac{p(\theta + tn^{-\frac{1}{2}})}{p(\theta)} = n^{-\frac{1}{2}}B_1(s) + n^{-1}B_2(s) + Q_n^{(2)} + R_{3,n}$$

where

$$B_1(s) = sb_{11}$$
$$B_2(s) = sb_{21} + s^2b_{22} ,$$

$Q_n^{(2)}$ does not depend on s and $R_{3,n}$ is a residual term. Putting terms together we get

$$\frac{dP^n_{\theta+tn^{-\frac{1}{2}}}}{dP_\theta^n} \frac{p(\theta + tn^{-\frac{1}{2}})}{p(\theta)} = \exp\left(-\frac{s^2}{2} + Q_n^{(1)} + Q_n^{(2)}\right)$$
$$\times (1 + n^{-\frac{1}{2}}(A_1 + B_1) + n^{-1}(A_2 + B_2 + \tfrac{1}{2}(A_1 + B_1)^2) + R_n)$$

where

$$R_n = R_{1,n} + R_{2,n} + R_{3,n} + O(R_{1,n}^2 + R_{2,n}^2 + R_{3,n}^3)$$
$$+ n^{-2}O(A_2^2 + B_2^2) + n^{-\frac{3}{2}}O(|A_1|^3 + |B_1|^3)$$

provided that A_i, B_i, $R_{i,n}$, do not grow too fast. This implies

$$\frac{\psi_n(B)}{\psi_n(\mathbb{R})} = \frac{\int_{|s|<(\log n)^a, s\in B} (P_n(s) + R_n(s))\Phi(ds)}{\int_{|s|<(\log n)^a, s\in \mathbb{R}} (P_n(s) + R_n(s))\Phi(ds)} .$$

It remains to show that

$$P_\tau^n\{\sup_{|s|<(\log n)^a} |R_n(s)| \leqq C(\log n)^c n^{-\frac{3}{2}}\} \geqq 1 - o(n^{-1}) \tag{5.4}$$

and

$$P_\tau^n\{\textstyle\int_{|s|<(\log n)^a} |P_n(s)|\Phi(ds) \geqq \varepsilon\} \geqq 1 - o(n^{-1}) \tag{5.5}$$

for some $\varepsilon > 0$. In doing so we restrict our attention to points $\boldsymbol{\omega} \in \Omega^n$ where

$$|x_k^{(n)}(\boldsymbol{\omega}, \theta)| \leqq C(\log n) , \qquad 1 \leqq k \leqq 4 .$$

The P_τ^n-probabilities of those sets do not fall below $1 - o(n^{-1})$.

Choose $\delta > 0$ such that for $U_0 = \{\sigma \in \Theta : |\sigma - \theta| \leqq \delta\}$ conditions (vii) and (viii) are true. Computing the residuals we obtain

$$R_{1,n}(\boldsymbol{\omega}, t, \theta) = \frac{t^4}{24} n^{-2} \textstyle\sum_{i=1}^n (l_4'(\omega_i, \hat{\theta}_n(\boldsymbol{\omega}, t)) - l_4(\omega_i, \theta)) ,$$

$$R_{2,n}(\boldsymbol{\omega}, t, \theta) = \frac{t^4}{24} n^{-\frac{3}{2}} x_4^{(n)}(\boldsymbol{\omega}, \theta) ,$$

$$R_{3,n}(t) = \frac{t^2}{2} n^{-1}(\Lambda''(\hat{\theta}_n(t)) - \Lambda''(\theta)) ,$$

where $|\hat{\theta}_n(\omega, t) - \theta| \leqq tn^{-\frac{1}{2}}$, $|\hat{\hat{\theta}}_n(t) - \theta| \leqq tn^{-\frac{1}{2}}$; The regularity conditions imply

$$P_\tau^n\{|R_{1,n}(t)| \leqq C|t|^5 n^{-\frac{3}{2}} \text{ if } |t| < \delta n^{\frac{1}{2}}\} \geqq 1 - o(n^{-1})$$
$$P_\tau^n\{|R_{2,n}(t)| \leqq Ct^4(\log n)n^{-\frac{3}{2}} \text{ if } |t| < \delta n^{\frac{1}{2}}\} \geqq 1 - o(n^{-1})$$
$$|R_{3,n}(t)| \leqq C|t|^3 n^{-\frac{3}{2}}$$

(e.g., by Corollary 17.12 in Bhattacharya and Rao (1976)). Since $|s| < (\log n)^a$ implies $|t| \leqq C(\log n)^a$ the residuals $R_{i,n}$ are bounded by $C(\log n)^c n^{-\frac{3}{2}}$. Moreover, since $|x_k| \leqq C(\log n)$, both $O(A_2^2 + B_2^2)$ and $O(|A_1|^3 + |B_1|^3)$ do not exceed $C(\log n)^c$ as long as $|s| < (\log n)^a$. Thus we have proved (5.4). The proof of (5.5) is obvious by remarking that $|1 - P_n(s)|$ is uniformly smaller than one for $|s| < (\log n)^a$ and sufficiently large n. □

Proof of Lemma 2. From Theorem 2 we obtain that

$$P_\tau^n\{|G_{n,\omega}((nL_2)^{\frac{1}{2}}(\mu_n - \theta) - x_1 L_2^{-\frac{1}{2}}) - \tfrac{1}{2}| \geqq C(\log n)^c n^{-\frac{3}{2}}\} = o(n^{-1})$$

uniformly for $\theta \in K$, $|\tau - \theta| \leqq (\log n)n^{-\frac{1}{2}}$. Expanding $G_{n,\omega}$ around zero shows that

$$\hat{\mu}_n = \theta + (nL_2)^{-\frac{1}{2}}(x_1 L_2^{-\frac{1}{2}} + (\tfrac{1}{2} - G_{n,\omega}(0))(G'_{n,\omega}(0))^{-1})$$

satisfies

$$P_\tau^n\{n^{\frac{1}{2}}|\hat{\mu}_n - \mu_n| \geqq C(\log n)^c n^{-\frac{3}{2}}\} = o(n^{-1})$$

uniformly for $\theta \in K$, $|\tau - \theta| \leqq (\log n)n^{-\frac{1}{2}}$. Now elementary computations finish the proof. □

Proof of Lemma 3. Lemma 3 in Pfanzagl (1973b) implies

$$\begin{aligned} P_\theta^n\{(nL_2)^{\frac{1}{2}}(\mu_n - \theta) \geqq u\} &= \Phi(-u + n^{-\frac{1}{2}}L_2^{-\frac{1}{2}}a_1) \\ &\quad - n^{-\frac{1}{2}}L_2^{-\frac{3}{2}}\varphi(-u + (nL_2)^{-\frac{1}{2}}a_1) \textstyle\sum_{j=0}^{2} r_{1j}(\theta)(-u + (nL_2)^{-\frac{1}{2}}a_1)^j \\ &\quad - n^{-1}L_2^{-\frac{5}{2}}\varphi(-u) \textstyle\sum_{j=0}^{5} r_{2j}(\theta)(-u)^j + o(n^{-1}) . \end{aligned}$$

The coefficients $r_{ij}(\theta)$ can be obtained by tedious but elementary computations. (Explicit expressions are published in Strasser (1976b).) Then an application of Lemma 7 in Pfanzagl (1973a) proves the assertion. □

Proof of Theorem 3. Let $K \subseteq \Theta$ be compact. Since

$$\sup_{\theta \in K} P_\theta^n\{|\mu_n - \theta| \geqq (\log n)n^{-\frac{1}{2}}\} = o(n^{-1})$$

and

$$\sup_{\theta \in K} P_\theta^n\{|B_0^{(n)} - \theta| \geqq (\log n)n^{-\frac{1}{2}}\} = o(n^{-1})$$

we need only show that the assertion holds uniformly for $|t| \leqq (\log n)$. In Pfanzagl (1975), however, the expansion of

$$P^n_{\theta + tn^{-\frac{1}{2}}}\{B_0^{(n)} \geqq \theta\}, \qquad |t| \leqq \log n ,$$

is given instead of the expansion of

$$P_\theta^n\{B_0^{(n)} \geqq \theta - tn^{-\frac{1}{2}}\}, \qquad |t| \leqq \log n .$$

Therefore we will show that

$$\sup_{\theta \in K} |P^n_{\theta+tn^{-\frac{1}{2}}}\{\mu_n \geqq \theta\} - P^n_{\theta+tn^{-\frac{1}{2}}}\{B_0^{(n)} \geqq \theta\}| = o(n^{-1})$$

uniformly for $|t| \leqq \log n$.

We prove this assertion by arguing that the expansion of (μ_n) (given in Lemma 2) differs from the expansion of $(B_0^{(n)})$ by a term $n^{-1}Q$, where Q is a polynomial in x_1, x_2, x_3 of degree two. It is known that power functions of one-sided tests based on such expansions coincide of order $o(n^{-1})$ provided that their levels coincide of order $o(n^{-1})$ (cf. Pfanzagl (1975), pages 10f.). Since our expansions could be used for testing one-sided hypotheses at levels $\frac{1}{2} + o(n^{-1})$, this argument proves the assertion. □

An explicit proof of Theorem 2 could be based on Lemma 3 in Pfanzagl (1973b), and needs lengthy computations. In fact power functions of tests are computed which are based on medians of posterior distributions. The explicit proof for the more general case of arbitrary quantiles of posterior distributions is published in Strasser (1976b).

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MATHEMATICAL INSTITUTE
UNIVERSITY OF BAYREUTH
D-8580 BAYREUTH
POSTFACH 3008, WEST GERMANY