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INTERVAL ESTIMATION FOR THE UNBALANCED CASE OF THE ONE-WAY RANDOM EFFECTS MODEL

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Interval estimation of variance components is studied for the unbalanced one-way random effects model. An easily calculated function, W, of the harmonic mean of the class sizes and of the sample variance of the class means is found to be important. The exact distribution of W is found and is shown to be excellently approximated by a chi-square distribution. The random variable W is used to construct interval estimates for (i) the between classes variance component and (ii) the ratio of the variance components and thus for the intraclass correlation and heritability. For most one-way unbalanced designs use of these approximate interval estimators will work very well.

1. Introduction. In many experiments the suitable model is the one-way variance component model: $y_{ij} = \mu + a_i + e_{ij}$, $i = 1, \cdots, a$, $j = 1, \cdots, n_i$ with $a_i \sim N(0, \sigma_a^2)$, $e_{ij} \sim N(0, \sigma_e^2)$ and all a_i, e_{ij} independent. In this paper attention is directed to the interval estimation of the quantities σ_a^2, σ_a^2/σ_e^2 and $\sigma_a^2/(\sigma_a^2 + \sigma_e^2)$. For the balanced case there are several approximate procedures for constructing confidence intervals on σ_a^2 and there are exact confidence intervals for σ_a^2/σ_e^2 and $\sigma_a^2/(\sigma_a^2 + \sigma_e^2)$.

There are no published methods for constructing confidence intervals for σ_a^2 in the unbalanced case which do not in fact assume the data are balanced. Wald (1940) developed an exact procedure for constructing interval estimates of σ_a^2/σ_e^2 and $\sigma_a^2/(\sigma_a^2 + \sigma_e^2)$ but the method requires the numerical solution of two nonlinear equations and is difficult to carry out.

In this paper a statistic is presented which enables us to adjust confidence interval formulae developed for the balanced case so that they hold for most unbalanced designs arising in practice.

2. The balanced case. In the traditional analysis of variance for the balanced case with a classes, with $n_1 = n_2 = \cdots = n_a = r$ and with $n = ra$ it is well known that $S_a/(r\sigma_a^2 + \sigma_e^2) \sim \chi^2(a-1)$, $S_w/\sigma_e^2 \sim \chi^2(n-a)$ with S_a, S_w independent and thus $M_a/(r\sigma_a^2 + \sigma_e^2)/M_w/\sigma_e^2 \sim F(a-1, n-a)$ where S_a, M_a, S_w, M_w are the among classes sum of squares, among classes mean square, within classes sum of squares and within classes mean square respectively. Inference on the variance components is based on these distributional properties. Confidence intervals for σ_a^2/σ_e^2 and $\sigma_a^2/(\sigma_a^2 + \sigma_e^2)$ which are exact are readily derived.

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The problem of constructing confidence intervals for σ_a^2 has been investigated by several authors. (See, for example, Bartlett, 1953; Bulmer, 1957; Huitson, 1955; Satterthwaite, 1941; Tukey, 1951; Williams, 1962.) The only exact methods available use artificial data to accomplish the exactness. Two approximate procedures which appear (Boardman, 1974) to give the specified $(1-\alpha)$ 100% coverage are Bulmer's formula (Bulmer, 1957) and the Williams–Tukey–Bross formula (Bross, 1950; Tukey, 1951; Williams, 1962).

3. The unbalanced case. In the traditional analysis of variance table for the unbalanced case $E(M_a) = \sigma_e^2 + k_0\sigma_a^2$ where $k_0 = (n^2 - \Sigma n_i^2)/n(a-1)$ with $n = \Sigma n_i$.

In this case it can be shown that $S_a/(k_0\sigma_a^2 + \sigma_e^2)$ has a $\chi^2(a-1)$ distribution if and only if $\sigma_a^2 = 0$. Since the methods for constructing interval estimates of σ_a^2 for the balanced case make use of the fact that $S_a/(r\sigma_a^2 + \sigma_e^2)$ has a χ^2 distribution, those methods are not applicable to the statistics arising from the AOV in the unbalanced case. For some unbalanced designs $S_a/(k_0\sigma_a^2 + \sigma_e^2)$ will be approximately distributed as a $\chi^2(a-1)$ variable and use of the formulae for balanced data will be adequate; however, the approximation to be given in this paper gives a considerably better approximation for a much wider scope of one-way designs.

4. Development of the new statistic. The results were obtained by use of diagonalization methods such as those used by Hultquist and Atzinger (1972) and others including Spjotvoll (1967) who used such methods on this model in finding an optimum test for hypotheses about σ_a^2/σ_e^2. Here transformations were applied in an attempt to arrive as close as possible to chi-square statistics corresponding to those found in the balanced case. We present here only the essential points. For a more complete discussion see Thomas (1976). Our model written in matrix notation is: $Y = \mu j_n + ZA + E$ where $A \sim N(\phi_a, \sigma_a^2 I)$ and $E \sim N(\phi_n, \sigma_e^2 I)$. We have $E(Y) = \mu j_n$ and $V(Y) = \sigma_a^2 ZZ' + \sigma_e^2 I$.

Our first step is to employ the transformation $R = [R_1/R_2]$ where $R_1 = \text{diag}\,((1/n_1)j'_{n_1}, \cdots, (1/n_a)j'_{n_a}) = (Z'Z)^{-1}Z'$ and R_2 consists of $n-a$ orthonormal rows in the orthogonal complement of the row space of $(Z'Z)^{-1}Z'$. It follows that $E(R_1Y) = R_1\mu j_n, = \mu j_a$, $E(R_2Y) = \phi_{n-a}$ and

$$V\begin{bmatrix} R_1Y \\ \hdashline R_2Y \end{bmatrix} = \sigma_a^2 RZZ'R' + \sigma_e^2 RR' = \sigma_a^2\begin{bmatrix} I & \Phi \\ \Phi & \Phi \end{bmatrix} + \sigma_e^2\begin{bmatrix} (Z'Z)^{-1} & \Phi \\ \Phi & I \end{bmatrix}$$

thus $Y'R_2'R_2Y \sim \chi^2(n-a)$ and R_1Y, R_2Y are independent.

We next form the $a \times a$ orthogonal matrix H where $H = [h_1/H_2]$ with $h_1 = (1/a^{\frac{1}{2}})j_a'$ and H_2 consists of $a-1$ orthonormal rows in the orthogonal complement of h_1. We then have $E(HR_1Y) = E(Hj_a) = [\mu a^{\frac{1}{2}} \mid \phi'_{a-1}]'$ and $V(HR_1Y) = HR_1\Sigma R_1'H' = \sigma_a^2 I + \sigma_e^2 H(Z'Z)^{-1}H'$.

Consider H_2R_1Y. We have $E(H_2R_1Y) = \phi_{a-1}$ and $V(H_2R_1Y) = \sigma_a^2 I + \sigma_e^2 H_2(Z'Z)^{-1}H_2'$. Let P be an $(a-1) \times (a-1)$ orthogonal matrix which diagonalizes $H_2(Z'Z)^{-1}H_2'$. Let $PH_2(Z'Z)^{-1}H_2'P' = \Lambda = \text{diag}\,(\lambda_1, \cdots, \lambda_{a-1})$ where $\lambda_1, \cdots, \lambda_{a-1}$ are the eigenvalues of $H_2(Z'Z)^{-1}H_2'$.

We now have $PH_2R_1Y \sim N_{a-1}(\phi, \sigma_a^2 I + \sigma_e^2\Lambda)$ with PH_2R_1Y and R_2Y independent. If P_i denotes the ith row of P

$$v_i = P_iH_2R_1Y \sim N(0, \sigma_a^2 + \lambda_i\sigma_e^2), \qquad i = 1, \cdots, a-1;$$

$$\frac{v_i^2}{\sigma_a^2 + \lambda_i\sigma_e^2} \sim \chi^2(1) \qquad \text{and} \qquad W^* = \sum_{i=1}^{a-1} \frac{v_i^2}{\sigma_a^2 + \lambda_i\sigma_e^2} \sim \chi^2(a-1).$$

Note that $\mathrm{tr}\,[H_2(Z'Z)^{-1}H_2'] = \mathrm{tr}\,[(Z'Z)^{-1}H_2'H_2] = \mathrm{tr}\,[(Z'Z)^{-1}[I - J/a]] = \sum_{i=1}^{a}(1/n_i) - (1/a)j'(Z'Z)^{-1}j = ((a-1)/a)\sum_1^a 1/n_i$. So $\bar{\lambda} = \Sigma\lambda_i/(a-1) = (1/a)\Sigma(1/n_i)$ in general and $\bar{\lambda} = 1/\tilde{n}$ where $\tilde{n} = a/\Sigma 1/n_i$ is the harmonic mean of the n_i.

The formulae for constructing confidence intervals for σ_a^2 in the balanced case make use of the facts that

(a) $S_a/(r\sigma_a^2 + \sigma_e^2) \sim \chi^2(a-1)$ and

(b) S_a, S_w are independent. The statistic W^* although it has a chi-square distribution and is independent of S_w cannot be put in a form similar to the form in (a) above.

Considering that $\bar{\lambda} = 1/\tilde{n}$ we conclude that the λ_i are usually small and it would seem that W^* will often differ little from

$$W = \sum_{i=1}^{a-1} \frac{v_i^2}{\sigma_a^2 + \bar{\lambda}\sigma_e^2}.$$

We propose that the random variable W will have approximately a $\chi^2(a-1)$ distribution and would thus be suitable for use in constructing confidence intervals for σ_a^2.

Since P is orthogonal $\Sigma v_i^2 = \Sigma Y'R_1'H_2'P_i'P_iH_2R_1Y = Y'R_1'H_2'H_2R_1Y$. We see that it is not necessary to calculate P in order to find W. In summation notation

$$Y'R_1'H_2'H_2R_2Y = \sum_{i=1}^{a} \bar{y}_i^2 - \frac{1}{a}[\sum_{i=1}^{a} \bar{y}_i]^2 = (a-1)S_{\bar{y}}^2$$

where $S_{\bar{y}}^2$ is the sample variance for the treatment means, and thus $W = (a-1)S_{\bar{y}}^2/(\sigma_a^2 + \bar{\lambda}\sigma_e^2)$. Also $Y'R_2'R_2Y = S_w$, the analysis of variance within-classes sum of squares.

5. The distribution of W. The exact distribution of W given values of σ_a^2 and σ_e^2 can be displayed using a theorem due to Robbins and Pitman (1949).

Letting λ_m = minimum of the λ_i, $i = 1, \cdots, a-1$, and $b = (\sigma_a^2 + \lambda_m\sigma_e^2)/(\sigma_a^2 + \bar{\lambda}\sigma_e^2)$ the cumulative distribution of W is given by

$$P(W < w) = \sum_{j=0}^{\infty} c_j F_{a-1+2j}(w/b)$$

where $F_v(x)$ is the cumulative chi-square distribution,

$$c_j = \sum_{i_1+\cdots+i_r=j} c_{1,i_1} \cdot c_{2,i_2} \cdot \cdots \cdot c_{r,i_r} \qquad \text{with}$$

$$c_{i,j} = b_i^{-\frac{1}{2}} \frac{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+j-1)}{j!}\left[1 - \frac{1}{b_i}\right]^j,$$

$$(c_{i,0} = b_i^{-\frac{1}{2}}), \quad i = 1, \cdots, r;$$

and where $b_j = (\sigma_a^2 + \lambda_i\sigma_e^2)/(\sigma_a^2 + \lambda_m\sigma_e^2)$.

As previously stated W should have approximately a $\chi^2(a-1)$ distribution. To examine this nine designs (given in Table 1) representing a wide spectrum of unbalancedness were selected for study. For each design ten values of the ratio σ_a^2/σ_e^2 were chosen: .25, .50, .75, 1.0, 2.0, 3.0, 4.0, 6.0, 8.0, 10., thus creating ninety experimental situations. For each situation $\Pr(W < \chi_\alpha^2)$ was calculated for ten values of α where $\Pr(\chi^2 < \chi_\alpha^2) = \alpha$. These values are: .005, .010, .025, .050, .100, .900, .950, .975, .990, .995.

TABLE 1

Design	Number of classes	Values of n_i
1	3	5, 10, 15
2	6	5, 10, 15, 20, 25, 30
3	10	5, 10, 15, 20, 25, 30, 35, 40, 45, 50
4	3	2, 2, 100
5	3	10, 50, 500
6	10	2, 2, 2, 2, 2, 2, 2, 2, 2, 100
7	6	10, 10, 50, 50, 500, 500
8	4	4, 5, 6, 7
9	3	10, 20, 40

Although the accuracy of the approximation depends to some extent on the values of α and σ_a^2/σ_e^2, for 84% of the situations we have $|\Pr(W < \chi_\alpha^2) - \alpha| < .0005$. For 98% of all situations $|\Pr(\chi^2_{\alpha/2} < W < \chi^2_{1-\alpha/2}) - (1-\alpha)| < .005$ and with $1-\alpha = .95$ for all but two of the 90 situations $|\Pr(\chi^2_{.025} < W < \chi^2_{.975}) - .95| < .005$. Summarizing our observations we conclude that the χ^2 distribution is an excellent approximation to the distribution of W affording (at least for $\sigma_a^2 \geqq .25\sigma_e^2$) two or more places of accuracy for calculating $\Pr(\chi^2_{\alpha/2} < W < \chi^2_{1-\alpha/2})$ for a wide spectrum of unbalanced designs.

Note that the moment generating function of W is

$$\Phi_w(t) = \prod_{j=1}^{a-1}\left(1 - 2\frac{\sigma_a^2 + \lambda_j\sigma_e^2}{\sigma_a^2 + \bar{\lambda}\sigma_e^2}t\right)^{-\frac{1}{2}}$$

and thus $\Phi_w(t) \to (1-2t)^{-(a-1)/2}$ if all $n_i \to a$ constant value, or if all $n_i \to \infty$ or if $\sigma_a^2/\sigma_e^2 \to \infty$. We thus expect the approximation to improve if any of these latter conditions occur. The numerical study supports these conclusions and we state

RESULT I. *The distribution of* $W = (a-1)S_{\bar{y}}^2/(\sigma_a^2 + \sigma_e^2/\tilde{n})$ *is approximately a* $\chi^2(a-1)$.

6. The exact distribution of $[W/(a-1)]/[S_w/((n-a)\sigma_e^2)]$. Since

$$R_2 Y \sim N_{n-a}(\phi, \sigma_e^2 I)\,, \qquad Y'R_2'R_2Y/\sigma_e^2 = S_w/\sigma_e^2 \sim \chi^2(n-a)$$

and W and S_w are independent, it would be expected that

$$\frac{W/(a-1)}{S_w/(\sigma_e^2(n-a))} \sim \text{(approx.) } F(a-1, n-a)\,.$$

Using another theorem due to Robbins and Pitman (1949) the exact distribution of $G = [W/(a-1)]/[S_w/((n-a)\sigma_e^2)]$ can be given. The approximation of this distribution by the F distribution was investigated for each of the unbalanced designs given in Table 1. For each design $P(G < F_\alpha)$ was calculated for the same values of α and σ_a^2/σ_e^2 considered for W. The calculations indicated excellent agreement. For designs 8 and 9 there was exact agreement to three decimal places for all entries. Designs 1, 5, and 7 had maximum errors of .001 while designs 2 and 3 had .002 as the maximum error. The quantity $|P(F_{.025} < G < F_{.975}) - .95|$ was less than .005 for all cases and less than .002 for 94% of the cases.

RESULT II. *The ratio*

$$G = \frac{S_{\bar{y}}^2/(\sigma_a^2 + \sigma_e^2/\tilde{n})}{M_w/\sigma_e^2}$$

is distributed approximately as $F(a-1, n-a)$.

The W statistic is related to the balanced case by the following

COROLLARY. *If* $n_1 = n_2 = \cdots = n_a = r$ *then* $rS_{\bar{y}}^2 = M_a$, W *has exactly a* $\chi^2(a-1)$ *distribution, and* G *has exactly an* $F(a-1, n-a)$ *distribution.*

An approximation to the distribution of the statistic G with $\sigma_a^2 = 0$ has been given by Rankin (1974).

7. Confidence interval formulae for σ_a^2, σ_a^2/σ_e^2, and the intraclass correlation. Many of the confidence interval formulae for σ_a^2 developed for the balanced case can now be applied to the unbalanced case by making the substitutions $\tilde{n}$ for r and $F^* = \tilde{n}S_{\bar{y}}^2/M_w$ for F. The Williams–Tukey formula becomes

$$\frac{(a-1)M_w}{\tilde{n}\chi^2_{1-\alpha/2}}[F^* - F_{1-\alpha/2}] < \sigma_a^2 < \frac{(a-1)M_w}{\tilde{n}\chi^2_{\alpha/2}}[F^* - F_{\alpha/2}]$$

where the chi-square values are for $a - 1$ degrees of freedom. Bulmer's and the other formulae are likewise readily adjusted.

Confidence intervals for σ_a^2/σ_e^2 and the intraclass correlation can be formed using the same substitutions. For σ_a^2/σ_e^2 we have

$$\frac{1}{\tilde{n}}\left[\frac{F^*}{F_{1-\alpha/2}} - 1\right] < \frac{\sigma_a^2}{\sigma_e^2} < \frac{1}{\tilde{n}}\left[\frac{F^*}{F_{\alpha/2}} - 1\right].$$

To appraise the use of the W statistic in constructing interval estimates for the unbalanced case, data was simulated for the designs given in Table 1 for several values of σ_a^2/σ_e^2. Interval estimates for σ_a^2 were calculated for 2000 simulated experiments using Bulmer's formula and the Williams–Tukey formula. The percent coverage and the average width were observed for each experiment. The conclusion drawn from these simulations was that we would not expect exact methods to give much better results, at least for the designs considered.

Interval estimates were also constructed for σ_a^2/σ_e^2 and $\sigma_a^2/(\sigma_a^2 + \sigma_e^2)$ for the same designs and values of σ_a^2/σ_e^2. The results indicate that the formulae do give $1 - \alpha$ coverage. Considering the ease with which these interval estimates are calculated as compared to Wald's method we recommend that these formulae be used in preference to Wald's method.

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