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EXACT SLOPES OF CERTAIN MULTIVARIATE TESTS OF HYPOTHESES[1]

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Bahadur and Raghavachari have formulated the likelihood ratio method of finding the exact slope of a sequence of test statistics which does not require explicit estimation of large deviation probabilities. The method is described herein, and readily verifiable conditions under which it may be invoked are given. The method is then used to find the exact slopes of certain sequences of test statistics arising in multivariate analysis.

1. Introduction. Let (S, A) be a sample space of infinitely many independent and identically distributed observations $s = (x_1, x_2, \cdots)$ on a random variable x, the distribution of which is determined by a parameter θ taking values in a set Θ. For Θ_0 a given subset of Θ, we wish to test the null hypothesis that some θ in Θ_0 obtains. For each n, let $T_n(s)$ be a real valued A-measurable function depending on s through $x_1, x_2, \cdots, x_n$ only, such that in testing the null hypothesis, large values of T_n are significant. For any θ and t, let

$$F_n(t, \theta) = P_\theta(T_n(s) < t)$$

and

$$G_n(t) = \inf\{F_n(t, \theta) : \theta \in \Theta_0\} .$$

Then the level attained by T_n is defined as

$$L_n(s) = 1 - G_n(T_n(s)) .$$

The rate at which L_n tends to zero when a given nonnull θ obtains is considered by Bahadur (1960, 1967, 1971) as a measure of the asymptotic efficiency of the sequence of test statistics $\{T_n\}$ against that θ. If there exists a function $c(\theta)$ defined for $\theta \in \Theta - \Theta_0$ such that

$$\lim_{n\to\infty} n^{-1} \log L_n = -\tfrac{1}{2}c(\theta) \quad [P_\theta] ,$$

the sequence of test statistics $\{T_n\}$ is said to have exact slope $c(\theta)$ when θ obtains.

It is in general a nontrivial matter to determine the exact slope of a given sequence $\{T_n\}$. One useful method (from Bahadur (1967)) is as follows:

THEOREM 1.1. *Suppose that*

$$(1.1) \qquad \lim_{n\to\infty} T_n(s) = b(\theta) \quad [P_\theta]$$

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for each $\theta \in \Theta - \Theta_0$, *where* $-\infty < b(\theta) < \infty$, *and that*

$$\lim_{n\to\infty} n^{-1} \log L_n(t) = -g(t) \tag{1.2}$$

for each t *in an open interval which includes each value of* $b(\theta)$, *and* g *is a positive continuous function on that interval. Then the exact slope of* $\{T_n\}$ *exists for each nonnull* θ *and equals* $2g(b(\theta))$.

Under certain regularity conditions, the likelihood ratio statistic is an optimal statistic in the sense of exact slopes. Suppose that for each θ the distribution of the single observation x has a density $f(x, \theta)$ with respect to a σ-finite measure μ. For any θ and θ_0 in Θ, define the Kullback–Liebler information number $K(\theta, \theta_0)$ by

$$K(\theta, \theta_0) = E_\theta \log [f(x, \theta)/f(x, \theta_0)] , \tag{1.3}$$

and let

$$J(\theta) = \inf_{\theta_0 \in \Theta_0} K(\theta, \theta_0) . \tag{1.4}$$

Denote by λ_n the likelihood ratio statistic for testing $H_0: \theta \in \Theta_0$ based on $(x_1, \cdots, x_n)$, and let $\hat{T}_n(x) = -n^{-1} \log \lambda_n(s)$. Then, Bahadur (1965) proved:

(i) if c is the exact slope of a sequence of test statistics $\{T_n\}$, then $c(\theta) \leqq 2J(\theta)$ for each nonnull θ;

(ii) the exact slope of $\{\hat{T}_n\}$ exists and equals $2J(\theta)$ for each nonnull θ.

The sequence $\{T_n\}$ is said to be asymptotically optimal if its exact slope exists and equals $2J(\theta)$ for each nonnull θ. It is noteworthy that the exact slopes of certain statistics generally believed to be equivalent to likelihood ratio statistics are in reality less than $2J(\theta)$ for most nonnull values of θ (cf. Abrahamson (1965), Gupta (1972)).

The results of Bahadur (1965) have been generalized and refined in certain directions by Bahadur and Raghavachari (1971). This latter paper formulates a new method, here called the likelihood ratio (LR) method, of finding the exact slope of a given sequence. The likelihood ratio method, which does not require explicit estimation of large deviation probabilities, is described in Section 2. In addition, a set of readily verifiable conditions under which the method may be invoked are given in that section. These conditions serve to unify the theoretical considerations concerning the likelihood ratio method found in Bahadur and Raghavachari (1971). The application of these conditions is demonstrated in Section 3 with certain examples from multivariate analysis. Examples 3.1 and 3.2 have appeared previously, but with different methodology; the results of Examples 3.3, 3.4, and 3.5 are believed new.

2. The likelihood ratio method. Consider the $S = \{s\}$, $s = (x_1, x_2, \cdots, \text{ad inf})$, A, and $\{P_\theta : \theta \in \Theta\}$ of Section 1. For each n let B_n be the sub-σ-field of A induced by the mapping $s \to (x_1, \cdots, x_n)$. Then a B_n-measurable statistic $T_n(s)$ is a statistic which depends on s only through $(x_1, \cdots, x_n)$ and is A-measurable. For each n

let there be given a σ-field C_n such that

$$C_n \subset B_n \qquad n = 1, 2, \cdots . \tag{2.1}$$

We have $B_n \subset B_{n+1}$ for each n, but monotonicity (or any property other than (2.1)) is not assumed for the sequence $\{C_n\}$.

Let θ_0 and θ be points in Θ_0 and $\Theta - \Theta_0$ respectively, and consider testing the simple null hypothesis θ_0 against the simple alternative θ. Assume that P_{θ_0} and P_θ are mutually absolutely continuous on C_n, and let $\rho_n(s) = \rho_n(s, \theta, \theta_0)$ be a C_n-measurable function such that $0 < \rho_n < \infty$ and $dP_\theta = \rho_n dP_{\theta_0}$ on C_n ($n = 1, 2, \cdots$). Note that ρ_n is the LR statistic for testing θ_0 against θ when the sample space is (S, C_n). Note also that if $C_n = B_n$, then, in the notation of Section 1, $\rho_n = \prod_{i=1}^n f(x_i, \theta)/\prod_{i=1}^n f(x_i, \theta_0)$.

For each n, let $\hat{L}_n(s)$ be the level attained by ρ_n in testing θ_0, i.e., $\hat{L}_n(s) = \{P_{\theta_0}(\rho_n \geqq t)\}_{t=\rho_n(s)}$.

LEMMA 2.1. *Let c be a positive constant. A necessary and sufficient condition that $\{\rho_n\}$ have exact slope c when θ obtains*, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \log \hat{L}_n(s) = -\tfrac{1}{2}c \quad [P_\theta] \tag{2.2}$$

is that

$$\lim_{n\to\infty} \frac{1}{n} \log \rho_n(s) = \tfrac{1}{2}c \quad [P_\theta] . \tag{2.3}$$

In this case, if T_n is any C_n-measurable statistic and L_n is the level attained by T_n ($n = 1, 2, \cdots$) we have

$$\liminf_{n\to\infty} \frac{1}{n} \log L_n(s) \geqq -\tfrac{1}{2}c \quad [P_\theta] . \tag{2.4}$$

Special interest attaches to the case when, despite (2.1), (2.2) holds with $c = K(\theta, \theta_0)$ defined by (1.3). The following lemma states that for this special value of c it suffices to verify a part of (2.3).

LEMMA 2.2. *If $c = K(\theta, \theta_0)$, (2.3) is equivalent to*

$$\liminf_{n\to\infty} \frac{1}{n} \log \rho_n(s) \geqq \tfrac{1}{2}c \quad [P_\theta] . \tag{2.5}$$

Lemmas 2.1 and 2.2 are immediate consequences of certain results of Bahadur and Raghavachari (1971).

We now show how Lemmas 2.1 and 2.2 can sometimes be used to find the exact slope of a *given* sequence $\{T_n\}$ such that T_n is B_n-measurable for $n = 1, 2, \cdots$. Suppose the following conditions are satisfied.

C1. For each n and θ, the distribution of T_n has a density function, $g_n(t, \theta)$ say, with respect to Lebesgue measure, with $0 \leqq g_n \leqq \infty$.

C2. For each $\theta \in \Theta - \Theta_0$ there exists a $\theta_0 \in \Theta_0$ such that

$$K(\theta, \theta_0) = J(\theta) . \tag{2.6}$$

C3. For each n, T_n has an exact null distribution, i.e., $g_n(t, \theta_0) = g_n(t, \theta^0)$ for all θ^0 in Θ_0.

C4. For each n and θ in $\Theta - \Theta_0$, the distributions of T_n under θ and θ_0 are mutually absolutely continuous.

This condition implies that

$$h_n(t, \theta) = \frac{g_n(t, \theta)}{g_n(t, \theta_0)} \tag{2.7}$$

is well defined and $0 < h_n < \infty$ for all relevant values of t.

C5. For each θ in $\Theta - \Theta_0$ and each n, $h_n(t, \theta)$ defined by (2.7) is strictly increasing in t.

C6. For each θ in $\Theta - \Theta_0$,

$$\liminf_{n\to\infty} \frac{1}{n} \log h_n(T_n(s), \theta) \geqq J(\theta) \quad [P_\theta] . \tag{2.8}$$

LEMMA 2.3. *If conditions* C1—C6 *are satisfied, then* $\{T_n\}$ *has exact slope* $2J(\theta)$ *when* θ *obtains and so is optimal.*

To establish this, let C_n be the σ-field induced by the mapping $s \to T_n(s)$. Then (2.1) is satisfied. Choose and fix $\theta \in \Theta - \Theta_0$, and then choose and fix θ_0 in Θ_0 such that (2.6) holds. It follows from C1, C4, and (2.7) that $h_n(T_n(s), \theta)$ is a version of $\rho_n(s, \theta, \theta_0)$; hence, by (2.6), (2.8) is (2.5) with $c = K(\theta, \theta_0)$. It follows from Lemmas 2.1 and 2.2 that $\{\rho_n(s, \theta, \theta_0)\}$ has exact slope $2J(\theta)$ in testing θ_0 against θ. But then from C3 and C5 it follows that, for every n and every s, the level attained by $\rho_n(s, \theta, \theta_0)$ in testing θ_0 equals the level attained by T_n in testing Θ_0. Hence $\{T_n\}$ has exact slope $2J(\theta)$ against θ.

The verification of (2.8) can sometimes be accomplished as follows.

LEMMA 2.4. *Suppose we can find a constant* $b(\theta)$ *such that*

$$\liminf_{n\to\infty} T_n(s) \geqq b(\theta) \quad [P_\theta] \tag{2.9}$$

and we can also find a function $\xi(t, \theta)$ *such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log h_n(t, \theta) \geqq \xi(t, \theta) \tag{2.10}$$

for all t *(at least in a neighborhood of* $t = b(\theta)$*)*,

$$\xi(b(\theta), \theta) = J(\theta) , \tag{2.11}$$

and such that $\xi(\cdot, \theta)$ *is continuous in* t *at* $t = b(\theta)$*. Then condition* C6 *is satisfied.*

Clearly, Lemma 2.4 can apply to a given case only if both b and the function ξ are as large as possible consistent with (2.9) and (2.10). In practice we scale each T_n if necessary so that $T_n(s) \to b(\theta)$ $[P_n]$ for each θ, where $b = 0$ on Θ_0 and > 0 on $\Theta - \Theta_0$, and then try to find $\lim n^{-1} \log h_n(t, \theta)$, or at least a continuous function ξ such that (2.10) and (2.11) are satisfied.

3. Examples from multivariate analysis.

EXAMPLE 3.1: *testing for a given mean vector, covariance known.* Let X be the p-dimensional Euclidean space of points x, and let $\Theta = \{\theta = (\theta_1, \cdots, \theta_p)', -\infty < \theta_i < \infty, 1 \leqq i \leqq p\}$. Suppose that when θ obtains x has a multivariate normal distribution with mean θ and known covariance matrix Σ. We wish to test the null hypothesis

$$H_0: Ex = \theta_0$$

against the alternative

$$H_1: Ex \neq \theta_0$$

where θ_0 is specified. We may easily show in this case that for $\theta \neq \theta_0$,

$$J(\theta) = K(\theta, \theta_0) = \tfrac{1}{2}(\theta - \theta_0)'\Sigma^{-1}(\theta - \theta_0) .$$

For each n, let

$$T_n(s) = (\bar{x}_n - \theta_0)'\Sigma^{-1}(\bar{x}_n - \theta_0) ,$$

where $\bar{x}_n = n^{-1} \sum_{i=1}^n x_i$. T_n is distributed under the null hypothesis as n^{-1} times a central χ^2 variate with p degrees freedom and under the alternative as n^{-1} times a noncentral χ^2 variate with p degrees freedom and noncentrality parameter $n\lambda$, where $\lambda = \lambda(\theta) = (\theta - \theta_0)'\Sigma^{-1}(\theta - \theta_0)$. Abrahamson (1965) proved that $\{T_n\}$ has exact slope $2J(\theta)$ against every θ and so is asymptotically optimal by noting that

$$T_n(s) \to (\theta - \theta_0)'\Sigma^{-1}(\theta - \theta_0) \quad [P_\theta]$$

and then showing that, with $F_n(t)$ the null distribution function of T_n,

$$n^{-1} \log [1 - F_n(t)] \to -\tfrac{1}{2}t$$

for each $t > 0$. Since this limit is continuous in t, it follows from Theorem 1.1 that $\{T_n\}$ has exact slope $2J(\theta)$. This result will now be obtained by the likelihood ratio method of Section 2.

Choose and fix a nonnull θ; since the null hypothesis is simple, $J(\theta) = K(\theta, \theta_0)$. Let $g_n(t, \theta_0)$ and $g_n(t, \theta)$ be the probability densities of $T_n(s)$ under θ_0 and θ respectively (Lehmann (1959), page 312), and let $h_n(t, \theta) = g_n(t, \theta)/g_n(t, \theta_0)$ for $t > 0$. Then

$$h_n(t, \theta) = \sum_{j=0}^\infty e^{-n\lambda/2} \frac{(n\lambda/2)^j}{j!} \left(\frac{nt}{2}\right)^j \frac{\Gamma(p/2)}{\Gamma((p + 2j)/2)} \tag{3.1}$$

is a strictly increasing function of t. Conditions C1 through C5 of Lemma 2.3 thus hold; we shall demonstrate C6 by means of Lemma 2.4. Note that

$$T_n(s) \to \lambda \quad [P_\theta] , \tag{3.2}$$

so (2.9) is satisfied with $b(\theta) = (\theta - \theta_0)'\Sigma^{-1}(\theta - \theta_0)$. For (2.10), let j_n for each n be the positive integer such that $\frac{1}{2}n\lambda < j_n \leqq \frac{1}{2}n\lambda + 1$. Since each term in the series expansion (3.1) of $h_n(\cdot, \theta)$ is positive,

$$h_n(t, \theta) \geqq e^{-n\lambda/2} \frac{(n\lambda/2)^{j_n}}{j_n!} \left(\frac{nt}{2}\right)^{j_n} \frac{\Gamma(p/2)}{\Gamma(p/2 + j_n)} . \tag{3.3}$$

By an application of Stirling's formula in (3.3),

$$\liminf_{n\to\infty} n^{-1} \log h_n(t, \theta) \geqq \frac{\lambda}{2} \left[1 + \log\left(\frac{t}{\lambda}\right)\right] .$$

But in view of (3.2), (2.10) holds; consequently by Lemma 2.3, $\{T_n\}$ does indeed have exact slope $2J(\theta)$ against θ.

EXAMPLE 3.2: *testing for a given mean vector, covariance unknown.* Let X be the p-dimensional Euclidean space of points x, and Θ be the set of all points $\{(\mu, \Sigma) : \mu \in R^p, \Sigma^{(p\times p)}$ positive definite$\}$. Suppose that when $\theta = (\mu, \Sigma)$ obtains, x is multivariate normally distributed with mean μ and covariance Σ. We wish to test the null hypothesis

$$H_0 : \mu = \mu_0$$

against the alternative

$$H_1 : \mu \neq \mu_0$$

where μ_0 is specified but Σ is arbitrary positive definite. Denote by θ_0 the point (μ_0, Ψ) in the parameter subspace Θ_0. Then

$$K(\theta, \theta_0) = -\tfrac{1}{2} \log |\Sigma| + \tfrac{1}{2} \log |\Psi| - \tfrac{1}{2}p + \tfrac{1}{2} \operatorname{tr} [\Psi^{-1}(\Sigma + (\mu - \mu_0)(\mu - \mu_0)')] .$$

It follows that, for any $\theta = (\mu, \Sigma)$,

$$\begin{aligned} J(\theta) = \inf_{\theta_0 \in \Theta_0} K(\theta, \theta_0) &= -\tfrac{1}{2} \log |\Sigma| + \tfrac{1}{2} \log |\Sigma + (\mu - \mu_0)(\mu - \mu_0)'| \\ &= \tfrac{1}{2} \log |I + \Sigma^{-1}(\mu - \mu_0)(\mu - \mu_0)'| \\ &= \tfrac{1}{2} \log [1 + (\mu - \mu_0)'\Sigma^{-1}(\mu - \mu_0)] \\ &= \tfrac{1}{2} \log (1 + \lambda) , \end{aligned}$$

where $\lambda = \lambda(\theta) = (\mu - \mu_0)'\Sigma^{-1}(\mu - \mu_0)$.

Given n independent observations $x_1, \cdots, x_n$, let $U_n = \sum_{j=1}^n (x_j - \bar{x}_n)(x_j - \bar{x}_n)'$ and $V_n = n^{\frac{1}{2}}(\bar{x}_n - \mu_0)$. Then $U_n \sim W(\Sigma, p, n-1)$, $V_n \sim N(n^{\frac{1}{2}}(\mu - \mu_0), \Sigma)$, and U_n and V_n are independent. The likelihood ratio test of the null hypothesis is equivalent to the test that rejects H_0 for small values of $W_n(s) = |U_n|/|U_n + V_n V_n'|$ (Eaton (1972), Proposition 9.121), or, equivalently, to the test that rejects H_0 for large values of $T_n(s) = 1 - W_n(s)$. We shall now establish the asymptotic optimality of $\{T_n\}$ by utilizing the likelihood ratio method of Section 2.

Choose and fix a nonnull $\theta = (\mu, \Sigma)$. Let $\theta_0 = (\mu, \Sigma + (\mu - \mu_0)(\mu - \mu_0)')$; then $J(\theta) = K(\theta, \theta_0)$. Let $g_n(t, \theta)$ and $g_n(t, \theta_0)$ be the probability densities of T_n under θ and θ_0 respectively; since T_n is invariant under nonsingular linear transformations of the individual observations $x_1, \cdots, x_n$, it has an exact null distribution depending only on μ_0. Let $h_n(t, \theta) = g_n(t, \theta)/g_n(t, \theta_0)$ for $0 < t < 1$. Since

$W_n \sim 1/(1 + (p/(n-p))F_{p,n-p})$ under H_0 and $\sim 1/(1 + (p/(n-p))F_{p,n-p}(n\lambda))$ under H_1 (Eaton (1972), Proposition 9.129), we find that

$$(3.4) \qquad h_n(t, \theta) = \sum_{j=0}^{\infty} \frac{e^{-n\lambda/2}(n\lambda/2)^j \Gamma(n/2 + j)\Gamma(p/2)}{j!\, \Gamma(n/2)\Gamma(p/2 + j)} t^j, \qquad 0 < t < 1,$$

and hence that $h_n(t, \theta)$ is strictly increasing in t. Accordingly, we see that conditions C1 — C5 of Lemma 2.3 hold. We may use the same approach as in the previous example to conclude by Lemma 2.4 that

$$(3.5) \qquad \liminf_{n\to\infty} n^{-1} \log h_n(T_n(s), \theta) \geqq \tfrac{1}{2} \log(1 + \lambda).$$

C6 is satisfied; $\{T_n\}$ therefore has exact slope $2J(\theta)$ against θ. But θ being arbitrary, $\{T_n\}$ is asymptotically optimal.

Remark. The optimality of the likelihood ratio criterion can be proved in an alternative, yet similar, manner. It is well known that the likelihood ratio statistic is equivalent to the statistic $T_n' = (n-1)(\bar{x}_n - \mu_0)' U_n^{-1}(\bar{x}_n - \mu_0)$, which is n^{-1} times the usual Hotelling's T^2 statistic. Since the distribution of T^2 is (proportional to) F, the ratio of the alternative to the null distribution of T_n' is the ratio of a noncentral to a central F distribution. The verification of the asymptotic optimality of $\{T_n'\}$ can be demonstrated by straightforward modification of the arguments presented in Example 7.3 of Bahadur and Raghavachari (1971). An interesting consequence of this is a large deviation probability estimate for the Hotelling's T^2 statistic. Since $\{T_n'\}$ has exact slope $\log(1 + (\mu - \mu_0)'\Sigma^{-1}(\mu - \mu_0))$ against $\theta = (\mu, \Sigma)$, and since $T_n' \to (\mu - \mu_0)'\Sigma^{-1}(\mu - \mu_0)\ [P_\theta]$, we see immediately that

$$(3.6) \qquad \lim_{n\to\infty} n^{-1} \log P_\theta(T_n' \geqq t) = -\tfrac{1}{2} \log(1 + t),$$

a result found by Killeen, Hettmansperger, and Sievers (1972) by an entirely different argument.

Example 3.3: *testing independence of one variate from a set of variates.* Let C be the p-dimensional Euclidean space of points $x = (x_1, \cdots, x_p)'$, and let Θ be the set of all points $\theta = (\mu, \Sigma)$, where $\mu \in R^p$ and $\Sigma^{(p\times p)} = \begin{pmatrix} \sigma_{11} & \sigma_{21}' \\ \sigma_{21} & \Sigma_{22} \end{pmatrix}$ is positive definite, σ_{11} denoting a scalar. Suppose that when θ obtains x is multivariate normally distributed with mean μ and covariance Σ. We wish to test the null hypothesis that x_1 is independent of $x_2, \cdots, x_p$.

Let $\theta = (\mu, \Sigma) \in \Theta$, and let $\theta_0 = (\nu, \begin{pmatrix} \Psi_{11} & 0 \\ 0 & \Psi_{22} \end{pmatrix}) \in \Theta_0$. Then

$$K(\theta, \theta_0) = -\tfrac{1}{2} \log |\Sigma| + \tfrac{1}{2} \log |\Psi_{11}| + \tfrac{1}{2} \log |\Psi_{22}| - \tfrac{1}{2} p + \tfrac{1}{2} \operatorname{tr} \Psi^{-1}[\Sigma + (\mu - \nu)(\mu - \nu)'].$$

It follows that

$$\begin{aligned} J(\theta) &= \inf_{\theta_0 \in \Theta_0} K(\theta, \theta_0) \\ &= -\tfrac{1}{2} \log |\Sigma| + \tfrac{1}{2} \log \sigma_{11} + \tfrac{1}{2} \log |\Sigma_{22}| \\ &= -\tfrac{1}{2} \log (1 - \sigma_{12}\Sigma_{22}^{-1}\sigma_{21}/\sigma_{11}) \\ &= -\tfrac{1}{2} \log (1 - \rho^2) \end{aligned}$$

where $\rho = \rho(\theta)$ is the multiple correlation between x_1 and $x_2, \cdots, x_p$ when θ obtains.

For each $n \geqq 2$, let $T_n(s) = R_n^2$, the square of the sample multiple correlation based on the first n observations of x. The likelihood ratio test of

$$H_0: \rho = 0$$

against the alternative

$$H_1: \rho > 0$$

when μ and Σ are unspecified is equivalent to the test that rejects H_0 for large values of $T_n(s)$ (Anderson (1958), Theorem 4.4.2). We shall establish the asymptotic optimality of $\{T_n\}$ using the likelihood ratio method.

Choose and fix a nonnull $\theta = (\mu, \binom{\sigma_{11}\ \sigma_{21}'}{\sigma_{12}\ \Sigma_{22}})$, and let $\theta_0 = (\mu, \binom{\sigma_{11}\ 0}{0\ \Sigma_{22}})$. Then $J(\theta) = K(\theta, \theta_0)$. Let $g_n(t, \theta)$ and $g_n(t, \theta_0)$ be the probability densities of $T_n(s)$ under θ and θ_0 respectively, and let $h_n(t, \theta) = g_n(t, \theta)/g_n(t, \theta_0)$. We have (Anderson (1958), page 95)

$$h_n(t, \theta) = (1 - \rho^2)^{(n-1)/2} \sum_{j=0}^{\infty} (\rho^2)^j \frac{t^j \Gamma((p-1)/2)}{j!\, \Gamma((p-1)/2 + j)} \frac{\Gamma^2((n-1)/2 + j)}{\Gamma^2((n-1)/2)};$$

clearly, h_n is a strictly increasing function of t. Conditions C1—C5 of Lemma 2.3 hold; C6 may be established as in the previous examples, using the technique provided by Lemma 2.4. We may show that

$$\liminf_{n\to\infty} n^{-1} \log h_n(t, \theta) \geqq \tfrac{1}{2} \log (1 - \rho^2) - \log (1 - \rho t^{\frac{1}{2}}) .$$

But since

$$T_n(s) \to \rho^2 \quad [P_\theta] ,$$

(2.8) follows from Lemma 2.4; thus $\{T_n\}$ has exact slope $2J(\theta)$ against every θ, and so is asymptotically optimal.

Remark. In the special case $p = 2$, an alternative proof of the asymptotic optimality of $\{T_n\}$ is available. Let $T_n'(s) = (n-2)^{\frac{1}{2}}|r_n|/(1 - r_n^2)^{\frac{1}{2}}$, where r_n is the bivariate sample correlation coefficient. Clearly, T_n and T_n' are equivalent statistics. T_n' is distributed under the null hypothesis as the absolute value of a t-statistic with $n - 2$ degrees freedom. If $\rho \neq 0$, then $n^{-\frac{1}{2}} T_n' \to |\rho|/(1 - \rho^2)^{\frac{1}{2}}$ $[P_\theta]$. Further, upon noting that the square of Student's t is distributed as Hotelling's T^2, we have from (3.6) that

$$n^{-1} \log P_{\theta_0}(|T_n'| \geqq n^{\frac{1}{2}} x) \to -\tfrac{1}{2} \log (1 + x^2) .$$

It follows from Theorem 1.1 that the exact slope of $\{T_n'\}$, and hence of $\{T_n\}$, is

$$\log\left(1 + \frac{\rho^2}{1 - \rho^2}\right) = -\log (1 - \rho^2) .$$

Example 3.4: *testing independence of two sets of variates.* Let C be the p-dimensional Euclidean space of points x and Θ be the set of all points $\theta = (\mu, \Sigma)$, where $\mu \in R^p$ and $\Sigma^{(p\times p)}$ is positive definite. Suppose that when θ obtains, x has a multivariate normal distribution with mean μ and covariance Σ. Suppose in addition that we partition $\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{21}'\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$, where Σ_{11} is of rank g, Σ_{22} is of rank h, and $g + h = p$. We wish to test the null hypothesis

$$H_0: \Sigma_{21} = 0$$

against the alternative

$$H_1: \Sigma_{21} \neq 0 ,$$

when μ and Σ are unspecified.

Let θ_0 denote the point $(\nu, \begin{pmatrix}\Psi_{11} & 0\\ 0 & \Psi_{22}\end{pmatrix})$ of Θ_0. Then

$$K(\theta, \theta_0) = -\tfrac{1}{2}\log|\Sigma| + \tfrac{1}{2}\log|\Psi_{11}| + \tfrac{1}{2}\log|\Psi_{22}| \\ - \tfrac{1}{2}p + \tfrac{1}{2}\operatorname{tr}\Psi^{-1}[\Sigma + (\mu - \nu)(\mu - \nu)'] .$$

Hence

$$J(\theta) = \inf K(\theta, \theta_0) = -\tfrac{1}{2}\log|\Sigma_{22\cdot 1}|/|\Sigma_{22}| .$$

Suppose in addition that Σ_{21} is of rank 1—that is, Σ_{21} can be written as uv', where $u \in R^h$ and $v \in R^g$. Then

$$\begin{aligned}|\Sigma_{22\cdot 1}|/|\Sigma_{22}| &= |I - \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}| \\ &= |I - \Sigma_{22}^{-1}uv'\Sigma_{11}^{-1}vu'| \\ &= (1 - v'\Sigma_{11}^{-1}vu'\Sigma_{22}^{-1}u) \\ &= 1 - \lambda^2 ,\end{aligned}$$

where $\lambda^2 = \lambda^2(\theta) = v'\Sigma_{11}^{-1}vu'\Sigma_{22}^{-1}u$. Then we may conveniently express $J(\theta)$ as $-\tfrac{1}{2}\log(1 - \lambda^2)$.

For each n, let $S_n = \sum_{j=1}^n (x_j - \bar{x}_n)(\bar{x}_j - \bar{x}_n)'$; we write S_n as

$$\begin{pmatrix} S_{11}^{(n)} & S_{12}^{(n)} \\ S_{21}^{(n)} & S_{22}^{(n)} \end{pmatrix} ,$$

where $S_{11}^{(n)}$ is $g \times g$, $S_{22}^{(n)}$ is $h \times h$. Let $U_n(s) = |S_n|/|S_{11}^{(n)}| \cdot |S_{22}^{(n)}|$. The likelihood ratio test of $H_0: \Sigma_{21} = 0$ versus $H_1: \Sigma_{21} \neq 0$ rejects H_0 for small values of $U_n(s)$, or equivalently, for large values of $T_n(s) = 1 - U_n(s)$ (Eaton (1972), Proposition 10.123). We shall establish the asymptotic optimality of $\{T_n\}$ by the likelihood ratio method.

Choose and fix a nonnull $\theta = (\mu, \Sigma)$, and let $\theta_0 = (\mu, \begin{pmatrix}\Sigma_{11} & 0\\ 0 & \Sigma_{22}\end{pmatrix})$. Then $J(\theta) = K(\theta, \theta_0)$. Under the null hypothesis,

$$U_n \sim \textstyle\prod_{i=1}^h W_i ,$$

where $W_1, \cdots, W_h$ are independent, and $W_i \sim 1/1 + (g/(n - 1 - p + i))F_{g,n-1-p+i}$; under the alternative, the same distributional result obtains, with the exception that

$$W_h \sim 1/1 + \frac{g}{n - 1 - p + h} F_{g,n-1-p+h}(\delta) ,$$

where δ is a random variable, $\delta \sim \lambda^2/(1-\lambda^2)\chi^2_{n-1}$ (Eaton (1972), Propositions 10.130, 10.137). After some algebraic simplification, we may write the ratio of the probability densities of T_n under the alternative and the null hypothesis as

$$h_n(t\,|\,\theta, \delta) = \sum_{j=0}^{\infty} e^{-\delta/2} \frac{(\delta/2)^j}{j!} \frac{\Gamma((n-1)/2+j)}{\Gamma((n-1)/2)} \frac{\Gamma(g/2)}{\Gamma(g/2+j)} t^j\,, \qquad 0 < t < 1$$

and so

$$h_n(t, \theta) = \sum_{j=0}^{\infty} \left(\frac{c}{1+c}\right)^j (1+c)^{-(n-1)/2} \frac{\Gamma(g/2)}{\Gamma(g/2+j)} \frac{\Gamma^2((n-1)/2+j)}{\Gamma^2((n-1)/2)} \frac{t^j}{j!}\,,$$

where $c = \lambda^2/(1-\lambda^2)$. Conditions C1—C5 of Lemma 2.3 are readily verified; the verification of condition C6 is handled exactly as in the previous examples. The asymptotic optimality of $T_n(s)$ follows immediately.

EXAMPLE 3.5: *the modified T^2 problem.* Let X be the p-dimensional Euclidean space of points $x = (x_1', x_2')'$, where x_1 is of dimension g, x_2 is of dimension h, and $g + h = p$. Let Θ be the set of points $\theta = (\mu, \Sigma)$, where $\mu = (0', \mu_2')$, μ is of dimension p, μ_2 is of dimension h, and $\Sigma^{(p\times p)}$ is positive definite. Suppose that when θ obtains x is multivariate normally distributed with mean μ and covariance Σ. We wish to test the null hypothesis

$$H_0\colon Ex_2 = 0$$

against the alternative

$$H_1\colon Ex_2 \neq 0\,.$$

That is, knowing that $Ex_1 = 0$, we wish to ascertain whether Ex_2 is 0 also. There does not exist a uniformly most powerful test of this hypothesis (Giri (1961)); hence it is of interest to compare various possible tests using the criterion of Bahadur efficiency. Gleser (1966) derived approximate slopes (for a discussion of the concept of approximate slopes, see Bahadur (1960, 1967) or Gleser (1964)) for certain of the test statistics we shall consider; however, our likelihood ratio method allows us to calculate the exact slopes, thereby obviating the uncertainties attendant with an approximate analysis.

Let θ_0 denote the point $(0, \Psi)$ in Θ_0. Then

$$K(\theta, \theta_0) = -\tfrac{1}{2}\log|\Sigma| + \tfrac{1}{2}\log|\Psi| - \tfrac{1}{2}p + \tfrac{1}{2}\operatorname{tr}\Psi^{-1}(\Sigma + \mu\mu')\,.$$

It follows that

$$J(\theta) = \tfrac{1}{2}\log(1 + \mu_2'\Sigma_{22\cdot 1}^{-1}\mu_2)\,.$$

Given a sample of size n, let $\bar{x}$ and S denote the sample mean and covariance respectively. (Our notation contains no explicit mention of n, a point that we trust will not lead to confusion.) Partition

$$\bar{x} = \begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}, \qquad S = \begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}$$

into components of dimension g and h. Then the likelihood ratio statistic for testing H_0 may be written as (Rao (1946), Giri (1961))

$$l = [(1 + \bar{x}'S^{-1}\bar{x})/(1 + \bar{x}_1'S_{11}^{-1}\bar{x}_1)]^{-n/2} .$$

The likelihood ratio test is equivalent to the test that rejects H_0 for large values of the statistic

$$T_n^{(1)} = (n - p)(\bar{x}'S^{-1}\bar{x} - \bar{x}_1'S_{11}^{-1}\bar{x}_1)/(1 + \bar{x}_1'S_{11}^{-1}\bar{x}_1) .$$

The exact slope of $\{T_n^{(1)}\}$ follows readily upon observing that

$$n^{-1}T_n^{(1)} \rightarrow \mu_2'\Sigma_{22\cdot1}^{-1}\mu_2 \quad [P_\theta]$$

and that the null distribution of $T_n^{(1)}$ is proportional to Hotelling's T^2 with $n - g$ degrees freedom (Kshirsagar (1972), page 139). Combining these facts with the large deviation probability estimate (3.6), we conclude from Theorem 1.1 that $\{T_n^{(1)}\}$ has exact slope $\log(1 + \mu_2'\Sigma_{22\cdot1}^{-1}\mu_2)$ against $\theta = (\mu, \Sigma)$; θ being arbitrary, $\{T_n^{(1)}\}$ is thus asymptotically optimal.

Consider

$$T_n^{(2)} = n\bar{x}'S^{-1}\bar{x} ,$$

the Hotelling's T^2 statistic appropriate for testing the null hypothesis H_0: $Ex = 0$ without knowledge that $Ex_1 = 0$. Note, however, that when $\theta = ((0', \mu_2')', \Sigma)$ obtains

$$n^{-1}T_n^{(2)} \rightarrow \mu_2'\Sigma_{22\cdot1}^{-1}\mu_2 \quad [P_\theta] .$$

It follows that $\{T_n^{(2)}\}$ is another asymptotically optimal sequence of test statistics for the modified T^2 problem—we achieve asymptotic optimality without utilizing the prior information that $Ex_1 = 0$.

We may not, however, ignore the information furnished by x_1. Suppose we restrict ourselves to procedures based solely on x_2. The likelihood ratio test of H_0: $Ex_2 = 0$ in this instance is equivalent to the test based on

$$T_n^{(3)} = n\bar{x}_2'S_{22}^{-1}\bar{x}_2 ;$$

as $T_n^{(3)}$ is a Hotelling's T^2 statistic in the restricted setting, $\{T_n^{(3)}\}$ has exact slope $\log(1 + \mu_2'\Sigma_{22}^{-1}\mu_2)$ against $\theta = (\mu, \Sigma)$. The Bahadur efficiency of $\{T_n^{(3)}\}$ compared to $\{T_n^{(1)}\}$ or $\{T_n^{(2)}\}$ is strictly less than one except when x_1 and x_2 are independent under θ. Thus $\{T_n^{(3)}\}$ is not asymptotically optimal.

Finally, let

$$T_n^{(4)} = n(\bar{x}'S^{-1}\bar{x} - \bar{x}_1'S_{11}^{-1}\bar{x}_1) ,$$

the $(D_{g+h}^2 - D_g^2)$ statistic proposed by Rao (1949) and investigated in greater detail recently by the Subrahmaniams (1973). The exact slope of $\{T_n^{(4)}\}$ follows from these considerations:

$$P_{\theta_0}(T_n^{(4)} \geqq nt) \leqq P_{\theta_0}(D_{g+h}^2 \geqq nt) ,$$

since D_{g+h}^2 and D_g^2, Hotelling's T^2 statistics, are both positive. Hence

$$\limsup_{n\to\infty} n^{-1} \log P(T_n^{(4)} \geqq nt) \leqq -\tfrac{1}{2}\log(1 + t) .$$

But

$$n^{-1}T_n^{(4)} \to \mu_2'\Sigma_{22\cdot1}^{-1}\mu_2 \quad [P_\theta]$$

and

$$n^{-1}D_{g+h}^2 \to \mu_2'\Sigma_{22\cdot1}^{-1}\mu_2 \quad [P_\theta]$$

imply

$$\limsup_{n\to\infty} n^{-1} \log L_n \leqq -\tfrac{1}{2}\log(1 + \mu_2'\Sigma_{22\cdot1}^{-1}\mu_2)\,,$$

where L_n is the level of $T_n^{(4)}$. On the other hand, it is well known (Bahadur (1965)) that

$$\liminf_{n\to\infty} n^{-1} \log L_n \geqq -J(\theta) = -\tfrac{1}{2}\log(1 + \mu_2'\Sigma_{22\cdot1}^{-1}\mu_2)\,.$$

Therefore $\{T_n^{(4)}\}$ has exact slope $2J(\theta)$ against θ, and so is asymptotically optimal.

On the basis of Bahadur efficiency, then, $T_n^{(3)}$ is an inefficient statistic and $T_n^{(1)}$, $T_n^{(2)}$, and $T_n^{(4)}$ are all fully efficient. Needless to say, these conclusions concern asymptotic efficiency; the actual relative performances in samples of given size may present a different picture.

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REFERENCES

[1] ABRAHAMSON, I. G. (1965). On the stochastic comparison of tests of hypothesis. Ph.D. dissertation, Univ. of Chicago.

[2] ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[3] BAHADUR, R. R. (1960). Stochastic comparison of tests. *Ann. Math. Statist.* **31** 276–295.

[4] BAHADUR, R. R. (1965). An optimal property of the likelihood ratio statistic. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **1** 13–26. Univ. of California Press.

[5] BAHADUR, R. R. (1967). Rates of convergence of estimates and test statistics. *Ann. Math. Statist.* **38** 303–324.

[6] BAHADUR, R. R. (1971). Some limit theorems in statistics. *Regional Conference Series in Applied Mathematics* **4**. SIAM, Philadelphia.

[7] BAHADUR, R. R. and RAGHAVACHARI, M. (1971). Some asymptotic properties of likelihood ratios on general sample spaces. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **1** 129–152. Univ. of California Press.

[8] EATON, M. L. (1972). *Multivariate Statistical Analysis.* Institute of Mathematical Statistics, Univ. of Copenhagen.

[9] GIRI, N. (1961). On tests with likelihood ratio criteria in some problems of multivariate analysis. Tech. Report No. 39, Dept. of Statistics, Stanford Univ.

[10] GLESER, L. J. (1964). On a measure of test efficiency proposed by R. R. Bahadur. *Ann. Math. Statist.* **35** 1537–1544.

[11] GLESER, L. J. (1966). The comparison of multivariate tests of hypothesis by means of Bahadur efficiency. *Sankhyā A* **28** 157–174.

[12] GUPTA, J. C. (1972). Probabilities of medium and large deviations and their statistical applications. Ph.D. dissertation, Univ. of Chicago.

[13] KILLEEN, T. J., HETTMANSPERGER, T. P. and SIEVERS, G. L. (1972). An elementary theorem on the probability of large deviations. *Ann. Math. Statist.* **43** 181–192.

[14] KSHIRSAGAR, A. M. (1972). *Multivariate Analysis.* Dekker, New York.

[15] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.

[16] RAGHAVACHARI, M. (1970). On a theorem of Bahadur on the rate of convergence of test statistics. *Ann. Math. Statist.* **41** 1695–1699.

[17] RAO, C. R. (1946). Tests with discriminant functions in multivariate analysis. *Sankhyā* **7** 407–413.

[18] RAO, C. R. (1949). On some problems arising out of discrimination with multiple characters. *Sankhyā* **9** 343–366.

[19] SUBRAHMANIAM, K. and SUBRAHMANIAM, K. (1973). On the distribution of $D^2_{p+q} - D_q{}^2$ statistic: percentage points and the power of the test. *Sankhyā B* **35** 51–78.

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