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ON α-RESOLVABILITY AND AFFINE α-RESOLVABILITY OF INCOMPLETE BLOCK DESIGNS

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A necessary condition for the α-resolvability of an incomplete block design is obtained. Further, a necessary and sufficient condition for an α-resolvable incomplete block design to be affine α-resolvable is obtained in terms of the largest characteristic root of NN' other than rk, where N is the incidence matrix of the design.

1. Introduction. Bose [3] introduced the concept of resolvability and affine resolvability in designs and proved that a necessary condition for the resolvability of a BIBD is $b \geqq v + r - 1$ and that a necessary and sufficient condition for a resolvable BIBD to be affine resolvable is that $b = v + r - 1$. These results were generalized by Agrawal [2], who proved that a necessary condition for the resolvability of an incomplete block design (v, b, r, k) is that $\mu_0 \geqq k$, and that a necessary and sufficient condition for a resolvable incomplete block design to be affine resolvable is that $\mu_0 = k$, where μ_0 is the largest characteristic root of NN' other than rk, and N is the incidence matrix of the design.

The concept of resolvability and affine resolvability was generalized by Shrikhande and Raghavarao [6] to α-resolvability and affine α-resolvability. An incomplete block design with parameters v, $b = \beta t$, $r = \alpha t$, k is said to be α-resolvable if the blocks can be divided into t sets of β each, such that each treatment occurs α times in each set of blocks. Further, an α-resolvable incomplete block design with parameters v, $b = \beta t$, $r = \alpha t$, k is said to be affine α-resolvable if any two blocks belonging to the same set contain q_1 treatments in common and two blocks belonging to different sets contain q_2 treatments in common. Shrikhande and Raghavarao [6] proved that the necessary and sufficient condition for an α-resolvable BIBD to be affine α-resolvable is that $b = v + t - 1$. Raghavarao [5] and Kageyama [4] showed that a necessary condition for the α-resolvability of a BIBD is that $b \geqq v + t - 1$.

In this paper, we derive a necessary condition for the α-resolvability of any incomplete block design (v, b, r, k) and a necessary and sufficient condition for an α-resolvable incomplete block design to be affine α-resolvable. These conditions are derived in terms of the largest characteristic root μ_0 of NN' other than rk, where N is the incidence matrix of the design. Conditions for α-resolvability and affine α-resolvability for known designs can be easily obtained as particular cases of the general result proved here.

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2. Main results. Let N be the incidence matrix of an incomplete block design (v, b, r, k) and μ_0 be the largest characteristic root of NN', other than rk. We need the following theorem due to Agrawal [1] in the derivation of the results of this section.

THEOREM 2.A. *If N is the incidence matrix of a connected incomplete block design (v, b, r, k) and μ_0 is the largest characteristic root of NN' other than rk, then l_{ij}, the number of common treatments between any two blocks i and j ($i \neq j = 1, 2, \cdots, b$), satisfies*

$$\max(0, 2k - v, k - \mu_0) \leqq l_{ij} \leqq \min[k, \mu_0 - k + 2(rk - \mu_0)b^{-1}], \tag{2.1}$$

and if for some i and j ($i \neq j$), $l_{ij} = k - \mu_0$, then

$$l_{ip} - l_{jp} = 0 \qquad p \neq i, j;\ p = 1, 2, \cdots, b;\ i \neq j = 1, 2, \cdots, b. \tag{2.2}$$

We consider here an α-resolvable incomplete block design with parameters $v, b = \beta t, r = \alpha t, k$. Let N be its incidence matrix and μ_0 be the largest characteristic root of NN' other than rk. Denote the jth block in the ith set by B_{ij}, $j = 1, 2, \cdots, \beta$ and $i = 1, 2, \cdots, t$. The number of common treatments between B_{11} and B_{ij} is denoted by l_{ij}. We now prove the following theorems.

THEOREM 2.1. *A necessary condition for a connected incomplete block design $(v, b = \beta t, r = \alpha t, k)$ to be α-resolvable is that*

$$\mu_0 \geqq k(b - r)/(b - t).$$

PROOF. Suppose the design is α-resolvable. Then by (2.1), we have

$$k - \mu_0 \leqq l_{1j} \qquad j = 2, 3, \cdots, \beta. \tag{2.3}$$

Adding (2.3) over $j = 2, 3, \cdots, \beta$ and dividing by $(\beta - 1)$, we get

$$k - \mu_0 \leqq \textstyle\sum_{j=2}^{\beta} l_{1j}/(\beta - 1). \tag{2.4}$$

Now, since the design is α-resolvable, every treatment occurs α times in each set and hence $\sum_{j=2}^{\beta} l_{1j} = k(\alpha - 1)$. Thus, from (2.4), we obtain

$$k - \mu_0 \leqq k(\alpha - 1)/(\beta - 1),$$

which gives

$$\mu_0 \geqq k(b - r)/(b - t). \tag{2.5}$$

THEOREM 2.2. *A necessary and sufficient condition for an α-resolvable connected incomplete block design $(v, b = \beta t, r = \alpha t, k)$ to be affine α-resolvable is that* (i) $\mu_0 = k(b - r)/(b - t)$, (ii) *k^2/v is a positive integer and* (iii) *$k - \mu_0$ is a positive integer.*

PROOF. (i) *Necessity.* Suppose the α-resolvable incomplete block design is affine α-resolvable. Then clearly we have

$$q_1 = \frac{\sum_{j=2}^{\beta} l_{1j}}{(\beta - 1)} = \frac{k(r - t)}{(b - t)}, \tag{2.6}$$

$$q_2 = \frac{\sum_{j=1}^{\beta} l_{ij}}{\beta} = \frac{k\alpha}{\beta} = \frac{k^2}{v} \qquad i = 2, 3, \cdots, t. \tag{2.7}$$

From (2.6) and (2.7), it follows that $k(r-t)/(b-t)$ and k^2/v are positive integers. Now, if the incidence matrix of this design is N, then we can write $N'N$ as

$$N'N = I_t \otimes (A + B - C) + E_{tt} \otimes C\,,$$

where $A = [k(b-r)/(b-t)]I_\beta$, $B = [k(r-t)/(b-t)]E_{\beta\beta}$, and $C = (k^2/v)E_{\beta\beta}$, I_β is an identity matrix of order β, $E_{\beta\beta}$ a $\beta \times \beta$ matrix with all elements unity and $\otimes$ denotes the Kronecker product of matrices. Then one can easily verify that the characteristic roots of $N'N$ are rk, $k(b-r)/(b-t)$ and 0 with respective multiplicities 1, $b-t$, and $t-1$. Since the nonzero characteristic roots of NN' and $N'N$ are same, it follows that the nonzero characteristic roots of NN' are rk and $k(b-r)/(b-t)$. Hence we get

$$\mu_0 = k(b-r)/(b-t)\,. \tag{2.8}$$

In view of (2.8), (2.6) becomes $q_1 = k - \mu_0$ and hence it follows that $k - \mu_0$ is a positive integer.

(ii) *Sufficiency*. Suppose for the α-resolvable incomplete block design, $\mu_0 = k(b-r)/(b-t)$. Then $k - \mu_0 = k(r-t)/(b-t) = \sum_{j=2}^{\beta} l_{1j}/(\beta - 1) = \bar{l}$, say. From (2.1), we obtain

$$k(r-t)/(b-t) \leqq l_{1j}\,, \qquad j = 2, 3, \cdots, \beta\,, \tag{2.9}$$

i.e., $\bar{l} \leqq l_{1j}$, which is only possible if $\bar{l} = l_{1j}$, $j = 2, 3, \cdots, \beta$, since $\sum (l_{1j} - \bar{l}) = 0$. Hence, we get

$$l_{1j} = \bar{l} = k - \mu_0 \qquad j = 2, 3, \cdots, \beta\,. \tag{2.10}$$

Thus, two blocks belonging to the same set contain $q_1 = k - \mu_0$ treatments in common. Therefore the two blocks B_{ij} and $B_{ij'}$, $j \neq j' = 1, 2, \cdots, \beta$ and $i = 1, 2, \cdots, t$ contain the same number of treatments $q_1 = k - \mu_0$ in common. Therefore, using (2.2), we see that the blocks B_{ij} and $B_{ij'}$ contain the same number of treatments in common with B_{11}, i.e.,

$$l_{ij} = l_{ij'}\,, \qquad i = 2, 3, \cdots, t;\ j \neq j' = 1, 2, \cdots, \beta\,.$$

Now, clearly $\sum_{j=1}^{\beta} l_{ij} = k\alpha$, i.e., $\beta l_{ij} = k\alpha$, i.e., $l_{ij} = k\alpha/\beta = k^2/v$, $i = 2, 3, \cdots, t$; $j = 1, 2, \cdots, \beta$. This proves that two blocks belonging to different sets contain k^2/v treatments in common. Hence, the design is affine α-resolvable.

From the sufficiency part of the above theorem, we easily get the following theorem.

THEOREM 2.3. *In an affine α-resolvable connected incomplete block design, any two blocks belonging to the same set contain $(k - \mu_0)$ treatments in common and any two blocks belonging to different sets contain k^2/v treatments in common.*

We now derive some useful results. We consider a connected design. Let the distinct characteristic roots of NN' be rk, $\mu_1, \mu_2, \cdots, \mu_p$ with respective multiplicities $1, a_1, a_2, \cdots, a_p$. Then we have

$$rk + \sum_{i=1}^{p} a_i \mu_i = vr\,. \tag{2.11}$$

Let us consider all the characteristic roots other than rk and put them equal to zero except one characteristic root, μ, say, with multiplicity a. Then, from (2.11), we obtain

$$\mu = r(v - k)/a . \tag{2.12}$$

Then, using Theorems 2.1 and 2.2, we have

COROLLARY 2.1. *A necessary condition for a connected incomplete block design* $(v, b = \beta t, r = \alpha t, k)$, *having only one nonzero characteristic root of* NN' *other than* rk, *with multiplicity* a, *to be* α*-resolvable is that* $b \geqq t + a$.

COROLLARY 2.2. *A necessary and sufficient condition for a connected* α*-resolvable incomplete block design* $(v, b = \beta t, r = \alpha t, k)$ *having only one nonzero characteristic root of* NN' *other than* rk *with multiplicity* a, *to be affine* α*-resolvable is that* (i) $b = t + a$ *and* (ii) k^2/v *is a positive integer.*

Using Corollary 2.1, one can easily derive the necessary conditions for the α-resolvability of BIB designs, singular and semiregular group divisible designs, certain triangular, L_i and rectangular designs. The necessary and sufficient conditions for the affine α-resolvability of the above designs follow from the application of Corollary 2.2.

REFERENCES

[1] AGRAWAL, H. L. (1964). On the bounds of the number of common treatments between blocks of certain two associate PBIB designs. *Calcutta Statist. Assoc. Bull.* **13** 76–79.

[2] AGRAWAL, H. L. (1965). A note on incomplete block designs. *Calcutta Statist. Assoc. Bull.* **14** 80–83.

[3] BOSE, R. C. (1942). A note on the resolvability of incomplete block designs. *Sankhyā* **6** 105–110.

[4] KAGEYAMA, S. (1973). On μ-resolvable and affine μ-resolvable balanced incomplete block designs. *Ann. Statist.* **1** 195–203.

[5] RAGHAVARAO, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments.* Wiley, New York.

[6] SHRIKHANDE, S. S. and RAGHAVARAO, D. (1964). Affine α-resolvable incomplete block designs. *Contributions to Statistics.* Volume presented to Professor P.C. Mahalanobis on his 70th birthday. Pergamon Press, Calcutta.

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