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THE GENERAL MANOVA PROBLEM[1]

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This paper treats a generalization of the classical MANOVA testing problem. The problem is reduced via invariance considerations and a new test statistic is proposed. This new test is shown to be a unique locally best invariant test and locally minimax.

1. Introduction and summary. The problem considered in this paper is related to the growth curve model described in Potthoff and Roy (1964). In a canonical form due to Olkin and Gleser (1970), the model is as follows:

$$Z: m \times p \sim N(\tilde{\Theta}, I_m \otimes \Sigma)\,, \qquad V: p \times p \sim W_p(\Sigma, n)\,; \qquad (n \geqq p)\,, \tag{1.1}$$
$$Z \text{ and } V \text{ are independent and } \Sigma \text{ is positive definite.}$$

Further, if $\tilde{\Theta}$ is partitioned as

$$\tilde{\Theta} = \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{21} & \Theta_{22} & \Theta_{23} \end{pmatrix} \end{array} \begin{array}{c} m_1 \\ m_2 \end{array} \quad \begin{array}{c} m_1 + m_2 = m \\ p_1 + p_2 + p_3 = p \end{array}, \tag{1.2}$$

it is assumed that $\Theta_{13} = 0$ and $\Theta_{23} = 0$. Let

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}.$$

As usual, (1.1) means that the m rows of Z are independent p-dimensional multivariate normal with a common covariance matrix Σ and V has a Wishart distribution with expectation $n\Sigma$. Under this model, the following testing problem will be studied:

$$H: \Theta_{12} = 0 \quad \text{versus} \quad K: \Theta_{12} \neq 0\,. \tag{1.3}$$

This problem will be called the general MANOVA (GMANOVA) problem since it reduces to the usual MANOVA problem when $p_1 = p_3 = 0$. Some of the technical difficulties in studying the above problem are caused by the following: (i) the sufficient statistic (Z, V) is not complete under K and, (ii) the restrictions $\Theta_{13} = 0$ and $\Theta_{23} = 0$ and the hypothesis $H: \Theta_{12} = 0$ are all nonlinear in the natural parameter space of the distributions of Z and V when these distributions are written as exponential families.

In contrast to the usual MANOVA problem, very little work has been done on the GMANOVA problem using invariance. References to work on this

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model prior to that of Potthoff and Rao (1964) and Roy (1965) can be found in Rao (1967), Gleser and Olkin (1970) and the papers referred to below. Khatri (1966) derived the LRT (likelihood ratio test) by using a conditional argument, and proposed some other related tests based on similarities between this problem and the usual MANOVA problem. Also Gleser and Olkin (1970) extended their previous work (1966) and derived the LRT for the above problem using the invariance principle. Kiefer and Schwartz (1965) briefly treated the problem in the case $\Theta_{23} \neq 0$ and proposed a noninvariant Bayes test. When $m_2 = 0$, their problem is identical with our problem, but their test is still noninvariant. Recently Fujikoshi (1973) proved the unconditional monotonicity of the power functions of the LRT and the related tests proposed by Khatri.

We summarize our work. In Section 2, following Gleser and Olkin (1970), a group leaving the problem is chosen. Under very mild conditions the LRT is always invariant under any group leaving a problem invariant (Lehmann (1959) page 252 or Eaton (1972) Chapter 7). Hence the LRT is not discarded by restricting attention to invariant tests. Based on this fact, Gleser and Olkin used invariance to derive the LRT. In doing so they chose a convenient group under which a maximal invariant is analytically tractable. As they mentioned, there is a larger group which leaves the problem invariant, but a tractable maximal invariant could not be found. In this paper we shall restrict our attention to the class of tests invariant under the larger group. Let us call a test in that class fully invariant and the group the full group. Since a maximal invariant under the full group is not readily computable, the smaller group is utilized to describe the class of fully invariant tests and to analyze the problem. Under the smaller group a maximal invariant we choose consists of 4 random matrices (T_1, T_2, T_3, T_4), where the joint distribution of (T_2, T_3, T_4) does not depend on any parameters. In Section 3, it is shown that the class of fully invariant tests based on (T_1, T_2) alone forms an essentially complete class among fully invariant tests. The elimination of (T_3, T_4) not only makes it easier to study the problem but also makes our problem analogous to the usual MANOVA problem so that we can use some known results for the MANOVA problem with (and sometimes without) modifications. In the subsequent sections the analysis of the problem is based on the statistic (T_1, T_2).

In Section 4, the LRT and the related tests proposed by Khatri (1966) are reviewed. These tests are fully invariant tests based on T_1 alone, and when $m_1 = 1$, the related tests are equivalent to the LRT. Giri (1968) has shown that when $m_1 = 1$, the LRT is the unique UMPI (uniformly most powerful invariant) test in the class of conditional level α tests. This class contains not only all fully invariant tests based on T_1 alone but also some fully invariant tests based on (T_1, T_2). However, as has been shown in Giri (1961) (1962), even in the case $m_1 = 1$, the LRT (or the related tests) cannot be UMPI in the class of all fully invariant tests based on (T_1, T_2). This fact is in contrast to the case of the usual MANOVA problem in which the case $m_1 = 1$ reduces to the

Hotelling T^2-problem. The case $m_1 = 1$ in our problem has been called the modified Hotelling T^2-problem by Olkin and Shrikhande (1954) and has been treated in various forms. For example, Cochran and Bliss (1948) and Rao (1949) treated it in relation to the covariate discriminant analysis, and Stein (1969), based on the principle of conditionality, reported the results due to Giri (1961).

In Sections 5 and 6, local properties of tests are considered. For a discussion of the local properties of tests, the reader is referred to Giri and Kiefer (1964). In Section 5, we derive a unique LBI (locally best invariant) test:

$$\operatorname{tr}(I + T_2)^{-1}(a_0[T_1(I + T_1)^{-1}] - I) > k .$$

Hence it is locally uniformly better than any other fully invariant test and so is admissible among fully invariant tests. When the class of tests is restricted to the class of fully invariant tests based on T_1 alone, Pillai's test, $\operatorname{tr}[T_1(I + T_1)^{-1}] > k$, is a unique LBI test in that class. In Section 6, the LBI test is shown to be locally minimax. In this sense it will be an alternative test to the non-invariant Bayes test proposed by Kiefer and Schwartz (1965). The arguments here are similar to Giri and Kiefer (1964), Schwartz (1965) and Giri (1968). For the representation of the probability ratio of the distributions of a maximal invariant, a theorem due to Wijsman (1967) is used.

In this paper, R^n denotes an Euclidean n-space, $Gl(n)$ the group of $n \times n$ nonsingular matrices, $\mathscr{O}(n)$ the group of $n \times n$ orthogonal matrices and $\mathscr{S}(n)$ the set of $n \times n$ positive definite matrices. The symbol $\sim$ following a random matrix as in (1.1) reads "is (or be) distributed as."

2. Invariance. Here the problem is reduced through invariance. Let

$$\mathscr{A} = \left\{ \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \in Gl(p) \,\middle|\, A_{ii} \in Gl(p_i)\ (i = 1, 2, 3) \right\}$$

$$\mathscr{F} = \left\{ F : m \times p \,\middle|\, F = \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{pmatrix} F_{11} & 0 & 0 \\ F_{21} & F_{22} & 0 \end{pmatrix} \end{array} \begin{array}{c} m_1 \\ m_2 \end{array} \right\}$$

$$\tilde{\mathscr{O}} = \left\{ \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \in \mathscr{O}(m) \,\middle|\, P_i \in \mathscr{O}(m_i)\ (i = 1, 2) \right\} ,$$

and consider the group $\mathscr{G} = \tilde{\mathscr{O}} \times \mathscr{A} \times \mathscr{F}$ with the group operation defined by

$$(Q_2, A_2, F_2) \circ (Q_1, A_1, F_1) = (Q_2 Q_1, A_1 A_2, Q_2 F_1 A_2 + F_2)$$

where $Q_i \in \tilde{\mathscr{O}}$, $A_i \in \mathscr{A}$, $F_i \in \mathscr{F}$ $(i = 1, 2)$. Then $\mathscr{G}$ leaves the problem invariant under the action

$$g(Z, V) = (PZA + F, A'VA) \qquad \text{and} \qquad g(\tilde{\Theta}, \Sigma) = (P\tilde{\Theta}A + F, A'\Sigma A) ,$$

where $g = (P, A, F) \in \mathscr{G}$. Since an analytical tractable maximal invariant is difficult to find under $\mathscr{G}$, we consider a smaller group $\mathscr{H} = \mathscr{A} \times \mathscr{F}$, which is isomorphic to the subgroup $\{I_m\} \times \mathscr{A} \times \mathscr{F}$ of the group $\mathscr{G}$. Hence $\mathscr{H}$ also leaves the problem invariant. Now partition (Z, V) as follows:

$$(2.1)\qquad Z = \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \end{pmatrix} \end{array} \begin{array}{c} \\ m_1 \\ m_2 \end{array} \qquad V = \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \end{array} \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array}$$

PROPOSITION 2.1 (Gleser and Olkin (1970)). *A maximal invariant under the group $\mathscr{H}$ is $s(Z, V) = (s_1(Z, V), s_2(Z, V))$ where*

$$(2.2)\qquad s_1(Z, V) = (Z_{12}, Z_{13}) \begin{pmatrix} V_{22} & V_{23} \\ V_{32} & V_{33} \end{pmatrix}^{-1} (Z_{12}, Z_{13})'$$

$$s_2(Z, V) = \begin{pmatrix} Z_{13} \\ Z_{23} \end{pmatrix} V_{33}^{-1} \begin{pmatrix} Z_{13} \\ Z_{23} \end{pmatrix}' .$$

Under $\mathscr{H}$, a maximal invariant parameter is $\gamma \equiv \Theta_{12}\Sigma_{22.3}^{-1}\Theta_{12}'$ and under $\mathscr{G}$, a maximal invariant parameter is the set of the ordered characteristic roots of γ, say $\delta_1 \geqq \delta_2 \geqq \cdots \geqq \delta_{m_1}$, where Σ is partitioned in the same way as V is partitioned in (2.1) *and $\Sigma_{22.3} = \Sigma_{22} - \Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32}$.*

In their statement of Proposition 2.1, Gleser and Olkin imposed the conditions $m \leqq p_3$ and $m_1 \leqq p_2$, but this is unnecessary.

Let $\mathscr{D}(\mathscr{H})$ and $\mathscr{D}(\mathscr{G})$ be the classes of all level α tests invariant under $\mathscr{H}$ and under $\mathscr{G}$ respectively, where $0 < \alpha < 1$. A test will be called $\mathscr{G}$-invariant (or $\mathscr{H}$-invariant) if it is in $\mathscr{D}(\mathscr{G})$ (or $\mathscr{D}(\mathscr{H})$). An alternative maximal invariant under $\mathscr{H}$ which is more convenient for our study is

$$t(Z, V) = (t_1(Z, V), t_2(Z, V), t_3(Z, V), t_4(Z, V))$$

where

$$(2.3)\qquad T_1 \equiv t_1(Z, V) = XV_{22.3}^{-1}X' : m_1 \times m_1 ;$$

$$(2.4)\qquad X \equiv (I + T_2)^{-\frac{1}{2}}(Z_{12} - Z_{13}V_{33}^{-1}V_{32}) : m_1 \times p_2 ;$$

$$(2.5)\qquad T_2 \equiv t_2(Z, V) = Z_{13}V_{33}^{-1}Z_{13}' : m_1 \times m_1 ;$$

$$(2.6)\qquad T_3 \equiv t_3(Z, V) = Z_{23}V_{33}^{-1}Z_{23}' : m_2 \times m_2 ; \qquad \text{and}$$

$$(2.7)\qquad T_4 \equiv t_4(Z, V) = Z_{13}V_{33}^{-1}Z_{23}' : m_1 \times m_2 .$$

Here $(I + T_2)^{-\frac{1}{2}} \in \mathscr{S}(m_1)$ satisfies $[(I + T_2)^{-\frac{1}{2}}]^2 = (I + T_2)^{-1}$. With this choice, any $\mathscr{H}$-invariant test is a function of $t(Z, V)$ and in terms of $t(Z, V)$ the class of $\mathscr{G}$-invariant tests $\mathscr{D}(\mathscr{G})$ is described. Since $\mathscr{H}$ is a subgroup of $\mathscr{G}$, any $\mathscr{G}$-invariant test is $\mathscr{H}$-invariant, so $\mathscr{D}(\mathscr{G}) \subset \mathscr{D}(\mathscr{H})$. Consequently a $\mathscr{G}$-invariant test is a function of the maximal invariant $t(Z, V)$ and so any test ϕ in $\mathscr{D}(\mathscr{G})$ can be written as

$$(2.8)\qquad \phi(Z, V) = \phi_0(t_1(Z, V), t_2(Z, V), t_3(Z, V), t_4(Z, V))$$

for some ϕ_0 defined on the space of (T_1, T_2, T_3, T_4). Since for $g = (P, A, F) \in \mathscr{G}$

and

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix},$$

$t(g(Z, V)) = (P_1 t_1(Z, V)P_1', P_1 t_2(Z, V)P_1', P_2 t_3(Z, V)P_2', P_1 t_4(Z, V)P_2')$, it follows immediately from Theorem 2 in Lehmann (1959) page 218 that the class of $\mathscr{G}$-invariant tests $\mathscr{D}(\mathscr{G})$ can be specified as the set of all tests ϕ in $\mathscr{D}(\mathscr{H})$ such that whenever ϕ is expressed in the form of (2.8), ϕ_0 satisfies

$$\phi_0(P_1 t_1 P_1', P_1 t_2 P_1', P_2 t_3 P_2', P_1 t_4 P_2') = \phi_0(t_1, t_2, t_3, t_4) \tag{2.9}$$

for any $t_i \in \mathscr{T}_i (i = 1, \cdots, 4)$ and for any $P_i \in \mathscr{O}(m_i) (i = 1, 2)$ where $\mathscr{T}_i$ denotes the range space of T_i. By the definition of T_i, the ranges can be regarded as $\mathscr{T}_i = R^{l_i}$ where $l_1 = l_2 = m_1(m_1 + 1)/2$, $l_3 = m_2(m_2 + 1)/2$ and $l_4 = m_1 m_2$. Hence along with $T \equiv (T_1, T_2, T_3, T_4)$, $\mathscr{D}(\mathscr{H})$ is the class of all level α tests defined on the space $\mathscr{T}_1 \times \mathscr{T}_2 \times \mathscr{T}_3 \times \mathscr{T}_4$ and $\mathscr{D}(\mathscr{G})$ is the class of tests ϕ_0 in $\mathscr{D}(\mathscr{H})$ satisfying (2.9). Since the distribution of T depends on (Θ, Σ) only through the maximal invariant parameter γ, the power function of ϕ in $\mathscr{D}(\mathscr{H})$ can be written as $\pi(\phi, \gamma) \equiv E_\gamma \phi(T_1, T_2, T_3, T_4)$, and for ϕ in $\mathscr{D}(\mathscr{G})$ it can be written as $\pi(\phi, \delta) = E_\delta \phi(T_1, T_2, T_3, T_4)$ since $\delta = (\delta_1, \delta_2, \cdots, \delta_{m_1})$ is a maximal invariant parameter under $\mathscr{G}$.

3. Essentially complete classes. It is now shown that the class of $\mathscr{H}$-invariant tests based on (T_1, T_2) alone forms an essentially complete class in $\mathscr{D}(\mathscr{H})$ and with this result, the class of $\mathscr{G}$-invariant tests based on (T_1, T_2) alone forms an essentially complete class in $\mathscr{D}(\mathscr{G})$. For this purpose, the relations among the statistics X and T_i's defined in (2.3)—(2.7) are considered. First it is well to recall

DEFINITION. Let U_1, U_2 and U_3 be random matrices. Then U_1 and U_2 are said to be conditionally independent given U_3 if

$$P(U_1 \in A_1, U_2 \in A_2 \mid U_3) = \textstyle\prod_{i=1}^2 P(U_i \in A_i \mid U_3)$$

for Borel sets A_i's on the space of U_i's $(i = 1, 2)$ where $P(\cdot \mid U_3)$'s denote versions of conditional probability given U_3.

LEMMA 3.1. (1) *Given (Z_{13}, V_{33}), the conditional distribution of X is $N((I_{m_1} + T_2)^{-\frac{1}{2}}\Theta_{12}, I_{m_1} \otimes \Sigma_{22.3})$ and hence the conditional distribution of X given (Z_{13}, V_{33}) depends on (Z_{13}, V_{33}) only through T_2.*

(2) *$V_{22.3} \sim W_{p_2}(\Sigma_{22.3}, n_1)$ and $V_{22.3}$ is independent of (X, T_2, T_3, T_4) where $V_{22.3} = V_{22} - V_{23} V_{33}^{-1} V_{32}$ and $n_1 = n - p_3$.*

(3) *Given (Z_{13}, V_{33}), T_1 and (T_3, T_4) are conditionally independent.*

(4) *The joint distribution of (T_2, T_3, T_4) does not depend on γ.*

The proof is straightforward and so omitted.

The next lemma, which is due to a referee, is essential to our main results below. These results were originally established (Kariya (1975)) by a lengthy conditional argument.

LEMMA 3.2. *The statistic* (T_1, T_2) *is sufficient for the family of distributions of* T.

PROOF. First it is shown that T_1 and (T_3, T_4) are conditionally independent given T_2. From (2) it sufficies to show that X and (T_3, T_4) are conditionally independent given T_2. However, this follows easily from (1) and (3) in Lemma 3.1. Second, using the conditional independence of T_1 and (T_3, T_4) given T_2, the conditional distribution of (T_3, T_4) given (T_1, T_2) is the same as the conditional distribution of (T_3, T_4) given T_2. But the latter is parameter free by (4). This completes the proof.

THEOREM 3.1. (1) *The class of* $\mathscr{H}$*-invariant tests based on* (T_1, T_2) *only forms an essentially complete class in* $\mathscr{D}(\mathscr{H})$.

(2) *The class of* $\mathscr{G}$*-invariant tests based on* (T_1, T_2) *only forms an essentially complete class in* $\mathscr{D}(\mathscr{G})$.

PROOF. From Lemma 3.2 (1) is obvious. To show (2), we apply Theorem 3.1 in Hall, Wijsman and Ghosh (1965) page 600. To verify Assumption A required in their theorem, regard the Borel field on the space $\mathscr{T}_1 \times \mathscr{T}_2 \times \mathscr{T}_3 \times \mathscr{T}_4$ of T as the σ-field $\mathscr{A}$ in their paper, the Borel field on the space $\mathscr{T}_1 \times \mathscr{T}_2$ of the sufficient statistic (T_1, T_2) as $\mathscr{A}_S$ there, and the group $\tilde{\mathscr{O}} = \mathscr{O}(m_1) \times \mathscr{O}(m_2)$ as the group G there where $\tilde{\mathscr{O}}$ acts on T by $g(T) = (P_1T_1P_1', P_1T_2P_1', P_2T_3P_2', P_1T_4P_2')$ for $g = (P_1, P_2) \in \tilde{\mathscr{O}}$. Then (i) in the assumption is clear and (ii) follows from an application of Theorem 4 in Lehmann (1959) page 225. This completes the proof.

Let us denote by $\mathscr{C}(\mathscr{G})$ the class of $\mathscr{G}$-invariant level α tests based on (T_1, T_2) alone. By Theorem 3.1 (2), we can restrict our attention to the class $\mathscr{C}(\mathscr{G})$. However it is still difficult to find a tractable maximal invariant under $(T_1, T_2) \to (PT_1P', PT_2P')$ for $P \in \mathscr{O}(m_1)$ and so (T_1, T_2) is used for the analysis of the problem.

With (T_3, T_4) eliminated, the problem is rather analogous to the usual MANOVA problem although there is the additional variable T_2 which is related to T_1. If a value of T_2 is given, then the problem is exactly the same as the usual MANOVA problem. Therefore conditional on T_2, all the results in the MANOVA problem can be applied whenever the corresponding tests are chosen. The next lemma follows from Lemma 3.1 (1) but is important.

LEMMA 3.3. T_1 *and* T_2 *are independent under the null hypothesis* H.

4. The LRT and the related tests proposed. The LRT with the following critical region $\mathscr{K}_4$ is due to Gleser and Olkin (1970).

$$(4.1)\qquad \mathscr{K}_4\colon l(Z, V) = \frac{|I + Z_{13}V_{33}^{-1}Z_{13}'|}{\left|I_{m_1} + (Z_{12}, Z_{13})\begin{pmatrix} V_{22} & V_{23} \\ V_{32} & V_{33}\end{pmatrix}^{-1}(Z_{12}, Z_{13})'\right|} < k_4 .$$

By the definition of T_1 (4.1) can be written as

$$(4.2)\qquad \mathscr{K}_4\colon l(Z, V) = |I + T_1|^{-1} < k_4 .$$

Hence in our terms the LRT is a $\mathscr{G}$-invariant test based on T_1 alone. It will be shown that the class of $\mathscr{G}$-invariant tests based on T_1 alone is not an essentially complete class. Khatri (1966) proposed the following tests:

(i) Roy's maximum root test with the critical region $\mathscr{K}_1$:

$$\mathscr{K}_1\colon \mathrm{ch}_1(T_1) = \mathrm{ch}_1(XV_{22.3}^{-1}X') > k_1 .$$

(ii) Lawley–Hotelling's trace test with the critical region $\mathscr{K}_2$:

$$\mathscr{K}_2\colon \mathrm{tr}\,(T_1) = \mathrm{tr}\,(XV_{22.3}^{-1}X') > k_2 .$$

(iii) Pillai's trace test with the critical region $\mathscr{K}_3$:

$$\mathscr{K}_3\colon \mathrm{tr}\,[T_1(I+T_1)^{-1}] = \mathrm{tr}\,[X(X'X + V_{22.3})^{-1}X'] > k_3$$

where $\mathrm{ch}_1(A)$ denotes the maximum latent root of A. By Lemma 3.3, T_1 and T_2 are independent under H and therefore the cut-off points k_i's $(i = 1, \cdots, 4)$ can be taken to be independent of T_2. Consequently these are unconditional $\mathscr{G}$-invariant tests based on T_1 alone. Let $\mathscr{C}_0(\mathscr{G})$ be the class of tests in $\mathscr{C}(\mathscr{G})$ which depend on T_1 alone and let

$$\mathscr{C}_1(\mathscr{G}) = \{\phi \in \mathscr{C}(\mathscr{G}) | E_0[\phi(T_1, T_2)|T_2] \leqq \alpha, \text{ a.e. } T_2\} ,$$

where $E[\cdot \mid T_2]$ denotes the conditional expectation given T_2. Clearly $\mathscr{C}_0(\mathscr{G}) \subset \mathscr{C}_1(\mathscr{G})$.

The next theorem is a generalized version of the result in the Hotelling T^2-problem and has been proved by Giri (1962).

THEOREM 4.1. *When* $m_1 = 1$, *the* LRT *is a* UMPI (*uniformly most powerful invariant*) *test in* $\mathscr{C}_1(\mathscr{G})$, *and the power function is strictly increasing in* γ.

It is noted that when $m_1 = 1$ the tests $\mathscr{K}_i$'s are all equivalent. As will be shown in Theorem 5.1, the LRT is not UMPI in the class $\mathscr{C}(\mathscr{G})$ even when $m_1 = 1$. In fact, there exists no UMPI test in $\mathscr{C}(\mathscr{G})$.

5. Locally best invariant (LBI) test $\mathscr{K}_5$. In this section we derive a unique LBI test in $\mathscr{C}_\alpha(\mathscr{G})$, the class of $\mathscr{G}$-invariant *size* α tests in $\mathscr{C}(\mathscr{G})$ and a unique LBI test in $\mathscr{C}_{0\alpha}(\mathscr{G})$, the class of $\mathscr{G}$-invariant size α tests based on T_1 alone.

For technical reasons, we consider the testing problem in terms of (X, Y, T_2) where $Y'Y = V_{22.3}$, $Y\colon n_1 \times p_2 \sim N(0, I_{n_1} \otimes \Sigma_{22.3})$ and Y is independent of (X, T_2). Recall that given T_2,

$$X \sim N((I + T_2)^{-\frac{1}{2}}\Theta_{12}, I_{m_1} \otimes \Sigma_{22.3}) , \qquad \text{and} \qquad n_1 = n - p_3 \geqq p_2 .$$

The group $G = \mathscr{O}(m_1) \times \mathscr{O}(n_1) \times Gl(p_2)$ leaves the testing problem $H\colon \Theta_{12} = 0$ versus $K\colon \Theta_{12} \neq 0$ invariant under the action

$$g(X, Y, T_2) = (QXB', PYB', QT_2Q') \qquad \text{for} \quad g = (Q, P, B) \in G ,$$

and a maximal invariant under G is a maximal invariant under $\mathscr{G}$. Thus the class of size α tests invariant under G, say $\mathscr{C}_\alpha(G)$, is equal to $\mathscr{C}_\alpha(\mathscr{G})$. Let $\Delta = \Delta(\delta) = \sum_{i=1}^{m} \delta_i$ where $\delta = (\delta_1, \delta_2, \cdots, \delta_{m_1})$ is the maximal invariant parameter in Proposition 2.1. The following theorems are main results here and the proof of Theorem 5.1 is given later.

THEOREM 5.1. *There is a* $\Delta_0 > 0$ *such that on the set* $\{\delta \mid \Delta(\delta) < \Delta_0\}$ *the power function of any test* ϕ *in* $\mathscr{C}_\alpha(\mathscr{G})$ *is given by*

$$\pi(\phi, \delta) = \alpha + B(\phi)\Delta + o(\Delta)\,, \tag{5.1}$$

where

$$B(\phi) = a_1 E_0\{\phi(T_1, T_2)\,\mathrm{tr}\,[(I + T_2)^{-1}(a_0[T_1(I + T_1)^{-1}] - I)]\}\,, \tag{5.2}$$

$o(\Delta)$ *is uniform in* $\phi \in \mathscr{C}_\alpha(\mathscr{G})$, $a_0 = (m_1 + n_1)/p_2$, $a_1 = (2m_1)^{-1}$ *and* E_0 *denotes the expectation under* H. *That is,*

$$\lim_{\Delta\to 0} \sup_\phi |[\pi(\phi, \delta) - \alpha - B(\phi)\Delta]/\Delta| = 0\,.$$

Further the test with the critical region

$$\mathscr{K}_5 \colon \mathrm{tr}\,(I + T_2)^{-1}[a_0 T_1(I + T_1)^{-1} - I] > k_5$$

is the unique LBI *test in* $\mathscr{C}_\alpha(\mathscr{G})$ *and so it is admissible in* $\mathscr{C}_\alpha(\mathscr{G})$.

THEOREM 5.2 (Schwartz (1967)). *Pillai's test* $\mathscr{K}_3$ *is the unique* LBI *test in* $\mathscr{C}_{0\alpha}(\mathscr{G})$.

PROOF. Since the conditional problem given $T_2 = t_2$ is exactly the same as the MANOVA problem and since $\mathrm{tr}\,(I + t_2)^{-1} \leqq m_1$, the result follows immediately from Schwartz's (1967).

In the case $p_3 = 0$ where T_2 vanishes, the LBI test $\mathscr{K}_5$ in Theorem 5.1 is reduced to Pillai's test $\mathscr{K}_3$, and in the case $p_1 = p_3 = 0$ where our problem is the MANOVA problem, both $\mathscr{K}_3$ and $\mathscr{K}_5$ are naturally reduced to Pillai's test in the MANOVA problem. The reader is referred to John (1971) and Sugiura (1972) for the LBI tests in other problems.

In order to prove Theorem 5.1, some lemmas are needed. Let $T = t(X, Y, T_2)$ be a maximal invariant under G. Let $f(x, y, t_2 \mid \Theta_{12}, \Sigma_{22.3})$ be the conditional density of (X, Y) given $T_2 = t_2$, that is,

$$\begin{aligned} f(x, y, t_2 \mid \Theta_{12}, \Sigma_{22.3}) = C(\Sigma_{22.3})\,\mathrm{etr}\,[&-\tfrac{1}{2}\Sigma_{22.3}^{-1}(x'x + y'y) \\ &+ \Sigma_{22.3}^{-1}\Theta_{12}'(I + t_2)^{-\frac{1}{2}}x - \tfrac{1}{2}\Theta_{12}\Sigma_{22.3}^{-1}\Theta_{12}'(I + t_2)^{-1}]\,. \end{aligned} \tag{5.3}$$

Further for $g = (Q, P, B) \in G$ and with $M = m_1 + n_1$, define

$$H(x, y, t_2 \mid \Theta_{12}, \Sigma_{22.3}) = \int_G f(g(x, y, t_2) \mid \Theta_{12}, \Sigma_{22.3})|BB'|^{M/2}\nu(dg)\,, \tag{5.4}$$

where $\nu(dg) = \nu_1(dQ)\nu_2(dP)\nu_3(dB)$ and ν_1, ν_2 and ν_3 are left invariant measures on $\mathscr{O}(m_1)$, $\mathscr{O}(n_1)$ and $Gl(p_2)$ respectively such that $\nu_1(\mathscr{O}(m_1)) = 1$ and $\nu_2(\mathscr{O}(n_1)) = 1$. The next lemma is an application of Theorem 4 in Wijsman (1967).

LEMMA 5.1. *The probability ratio of the distributions of* T *under the alternative and the null hypotheses is given by*

$$P = \frac{P^T(dt \mid \Theta_{12}, \Sigma_{22.3})}{P^T(dt \mid 0, \Sigma_{22.3})}(t(x, y, t_2)) = \frac{H(x, y, t_2 \mid \Theta_{12}, \Sigma_{22.3})}{H(x, y, t_2 \mid 0, \Sigma_{22.3})} \tag{5.5}$$

where $P^T(dt \mid \Theta_{12}, \Sigma_{22.3})$ *is the distribution of* T *under* $(\Theta_{12}, \Sigma_{22.3})$.

PROOF. From Theorem 4 in Wijsman (1967), it is sufficient to show that the space of (X, Y, T_2) is a Cartan G-space. From Proposition 1.3.3 in Palais (1961) it is sufficient to show that the space, say $\mathscr{Y}$, of Y is a Cartan $Gl(p_2)$-space under $y \to yB'$ where $B \in Gl(p_2)$ and $y \in \mathscr{Y}$. The space $\mathscr{Y}$ can be restricted to the set of $n_1 \times p_2$ matrices ($n_1 \geqq p_2$) that are of maximal rank, excluding a set of Lebesgue measure 0. For any $y \in \mathscr{Y}$, since y is of maximal rank p_2, $yB' = y$ implies $B = I$. That is, no $B \in Gl(p_2)$ except the identity leaves any y fixed. Hence it follows from Theorem 1.1.3 in Palais (1961) that $\mathscr{Y}$ is a Cartan $Gl(p_2)$-space. This completes the proof.

Now choose $A_0 \in G_T^+(p_2)$ such that $A_0'\Sigma_{22.3}A_0 = I$ and let $\xi = \Theta_{12}A_0$ so that $\Theta_{12}\Sigma_{22.3}^{-1}\Theta_{12}' = \xi\xi'$, where $G_T^+(p)$ denotes the set of $p \times p$ lower triangular matrices with positive diagonal elements. Then the distribution of T under $(\Theta_{12}, \Sigma_{22.3})$ is the same as under (ξ, I), so the ratio R in (5.5) remains unchanged when $(\Theta_{12}, \Sigma_{22.3})$ is replaced by (ξ, I). Let $B_0 \in G_T^+(p_2)$ such that $B_0(x'x + y'y)B_0' = I$ and let $g_0 = (I, I, B_0)$. From the left invariance of ν, substituting $gg_0 = (Q, P, BB_0)$ for g in (5.5) leaves the ratio R the same. Then after cancellation of constants, the numerator of R with this substitution can be expressed as

$$(5.6)\qquad \int_{Gl(p_2)} |BB'|^{M/2}\,\mathrm{etr}\,(-\tfrac{1}{2}BB')\nu_3(dB) \int_{\mathscr{O}(m_1)} \mathrm{etr}\,[\xi'QvB' - \tfrac{1}{2}\xi'Q(I + t_2)^{-1}Q'\xi]\nu_1(dQ)$$

in which $v = (I + t_2)^{-\frac{1}{2}}xB_0'$. To evaluate R near $\Delta(\delta) = 0$, the next lemma is used.

LEMMA 5.2. (James (1964) equations (22) and (23) with $k = 1$). *Let A and B be $p \times p$ matrices and let ν be the invariant probability measure on $\mathscr{O}(p)$. Then*

$$\int_{\mathscr{O}(p)} \mathrm{tr}\, AQBQ'\nu(dQ) = \mathrm{tr}\, A\, \mathrm{tr}\, B/p \qquad \textit{and} \qquad \int_{\mathscr{O}(p)} (\mathrm{tr}\, AQ)^2\nu(dQ) = \mathrm{tr}\, AA'/p\,.$$

LEMMA 5.3 *Let $v = (I + t_2)^{-\frac{1}{2}}xB_0'$. Then the ratio R in* (5.5) *is evaluated as*

$$(5.7)\qquad R = 1 + [(2m_1p_2)^{-1}(m_1 + n_1)\,\mathrm{tr}\, v'v - (2m_1)^{-1}\,\mathrm{tr}\,(I + t_2)^{-1}]\Delta + o(\Delta)$$

where $o(\Delta)$ is uniform in (x, y, t_2) and $\Delta < \Delta_0$ for some Δ_0.

PROOF. We first evaluate the numerator in (5.6) of the ratio R. Write the integrand of the inner integral in (5.6) as F_1F_2 where $F_1 = \mathrm{etr}\,(\xi'QvB')$ and $F_2 = \mathrm{etr}\,(-\frac{1}{2}\xi'Q(I + t_2)^{-1}Q'\xi)$. Then F_2 can be expanded as

$$F_2 = 1 - \tfrac{1}{2}\,\mathrm{tr}\,\xi'Q(I + t_2)^{-1}Q'\xi + o(\mathrm{tr}\,\xi'Q(I + t_2)^{-1}Q'\xi)\,.$$

Since $\mathrm{tr}\,\xi'Q(I + t_2)^{-1}Q'\xi \leqq \Delta\,\mathrm{tr}\,(I + t_2)^{-1} \leqq \Delta m_1$ from $\mathrm{tr}\,\xi\xi' = \Delta$,

$$(5.8)\qquad F_2 = 1 - \tfrac{1}{2}\,\mathrm{tr}\,\xi'Q(I + t_2)^{-1}Q'\xi + o(\Delta) = 1 + G_2 + o(\Delta)\,,\quad \text{say,}$$

where $o(\Delta)$ is uniform in (x, y, t_2) and $|G_2| \leqq m_1\Delta$. For F_1, expand it as

$$F_1 = 1 + \mathrm{tr}\,\xi'QvB' + \tfrac{1}{2}(\mathrm{tr}\,\xi'QvB')^2 + o((\mathrm{tr}\,\xi'QvB')^2)\,.$$

Since for any matrices A and B such that AB is square, $(\mathrm{tr}\, AB)^2 \leqq \mathrm{tr}\, AA'\,\mathrm{tr}\, BB'$,

$(\operatorname{tr} \xi' Q v B')^2 \leqq \Delta \operatorname{tr} v B' B v' \leqq \Delta \operatorname{tr} v' v \operatorname{tr} B' B$ holds. With $v = (I + t_2)^{-\frac{1}{2}} x B_0'$,

$$\begin{aligned} \operatorname{tr} v'v &= \operatorname{tr} B_0 x'(I + t_2)^{-1} x B_0' \leqq \operatorname{tr} x B_0' B_0 x' \operatorname{tr} (I + t_2)^{-1} \\ &\leqq m_1 \operatorname{tr} B_0(x'x + y'y)B_0' = m_1 \operatorname{tr} I_{p_2} = m_1 p_2 \,. \end{aligned}$$

Hence $(\operatorname{tr} \xi' Q v B')^2 \leqq m_1 p_2 \Delta \operatorname{tr} BB'$ holds and it follows that

$$\begin{aligned} F_1 &= 1 + \operatorname{tr} \xi' Q v B' + \tfrac{1}{2}(\operatorname{tr} \xi' Q v B')^2 + o(\Delta) \operatorname{tr} BB' \\ &= 1 + G_1 + o(\Delta) \operatorname{tr} BB' \,, \quad \text{say.} \end{aligned} \tag{5.9}$$

Here $o(\Delta)$ is uniform in (x, y, t_2) and $|G_1| \leqq c\Delta^{\frac{1}{2}}(1 + \operatorname{tr} BB')$ for some constant c provided Δ is bounded above. Therefore we obtain from (5.8) and (5.9)

$$F_1 F_2 = (1 + G_1 + o(\Delta) \operatorname{tr} BB')(1 + G_2 + o(\Delta)) = 1 + G_1 + G_2 + G_3 \,, \quad \text{say.}$$

It is easily shown that

$$\int\int_{Gl(p_2) \times \mathscr{O}(m_1)} |G_3| \, |BB'|^{M/2} \operatorname{etr} (-\tfrac{1}{2} BB') \nu_1(dQ) \nu_3(dB) = o(\Delta)$$

uniformly in (x, y, t_2). Since the integration of the first term in G_1 over $\mathscr{O}(m_1)$ vanishes, the inner integral of the remaining terms is by Lemma 5.2,

$$\begin{aligned} &\int_{\mathscr{O}(m_1)} [1 + \tfrac{1}{2}(\operatorname{tr} \xi' Q v B')^2 - \tfrac{1}{2} \operatorname{tr} \xi' Q (I + t_2)^{-1} Q' \xi] \nu_1(dQ) \\ &\qquad = 1 + (2m_1)^{-1} \operatorname{tr} v B' \xi' \xi B v' - (2m_1)^{-1} \Delta \operatorname{tr} (I + t_2)^{-1} \,. \end{aligned} \tag{5.10}$$

Now to integrate this over $Gl(p_2)$ with respect to $|BB'|^{M/2} \operatorname{etr} (-\frac{1}{2} BB') \nu_3(dB)$, we write $B = SC$ with $\nu_3(dB) = \nu_4(dC) \nu_5(dS)$ where $C \in \mathscr{O}(p_2)$, $S \in G_T{}^+(p_2)$ and ν_4 and ν_5 are left invariant measures on $\mathscr{O}(p_2)$ and $G_T{}^+(p_2)$ respectively with $\nu_4(\mathscr{O}(p_2)) = 1$. (See, for example, Wijsman (1967) page 398 or Eaton (1972) pages 6.31–6.35.) Then the second term of the right hand side in (5.10) when integrated over $\mathscr{O}(p_2)$ becomes $(2m_1 p_2)^{-1} \operatorname{tr} v'v \operatorname{tr} \xi' \xi SS'$ by Lemma 5.2. Therefore the numerator (5.6) becomes

$$\begin{aligned} N = &\int_{G_T^+(p_2)} [1 + (2m_1 p_2)^{-1} \operatorname{tr} v'v \operatorname{tr} \xi' \xi SS' - (2m_1)^{-1} \Delta \operatorname{tr} (I + t_2)^{-1}] \\ &\times |SS'|^{M/2} \operatorname{etr} (-\tfrac{1}{2} SS') \nu_5(dS) + o(\Delta) \end{aligned} \tag{5.11}$$

since $|BB'|^{M/2} \operatorname{etr} (-\frac{1}{2} BB') = |SS'|^{M/2} \operatorname{etr} (-\frac{1}{2} SS')$. On the other hand, the denominator, say D, of the ratio R is obtained by setting $\xi = 0$ in (5.6) and so $D = \int_{G_T^+(p_2)} |SS'|^{M/2} \operatorname{etr} (-\frac{1}{2} SS') \nu_5(dS)$. Note that

$$D^{-1} \int_{G_T^+(p_2)} (\operatorname{tr} \xi \xi' SS') |SS'|^{M/2} \operatorname{etr} (-\tfrac{1}{2} SS') \nu_5(dS) = M \operatorname{tr} \xi \xi' \tag{5.12}$$

since the left hand side can be expressed as $E(\operatorname{tr} \xi \xi' W)$ with $W \sim W_{p_2}(I, M)$. Thus the ratio $R = N/D$ is evaluated from (5.11) and (5.12) as (5.7), completing the proof.

PROOF OF THEOREM 5.1. Let $\phi \in \mathscr{C}_\alpha(\mathscr{G}) = \mathscr{C}_\alpha(G)$. With a maximal invariant $t(x, y, t_2)$, write $\phi(x, y, t_2) = \phi_0(t(x, y, t_2))$. Then, using Lemma 5.3 and $x(x'x + y'y)^{-1} x' = T_1(I + T_1)^{-1}$, the power function of ϕ is for Δ near zero

$$\pi(\phi, \delta) = \int \phi_0(t) R(t) P^T(dt \,|\, 0, I) = \alpha + B(\phi_0)\Delta + o(\Delta)$$

where $R(t)$ is the R in (5.7) and $B(\phi_0)$ is given by (5.2). Applying the generalized Neyman–Pearson lemma and maximizing $B(\phi_0)$ with respect to $\phi_0 \in \mathscr{C}_\alpha(\mathscr{G})$ yield a unique LBI test $\mathscr{K}_5$ given in Theorem 5.1. (See Lehmann (1959) page 83.) This completes the proof.

6. Local minimaxity of the test $\mathscr{K}_5$. It is proved that the test $\mathscr{K}_5$ is locally minimax in the sense of Giri–Kiefer (1964). In Giri–Kiefer Lemma 1 states the conditions under which a given test is locally minimax as follows. Let $(\mathscr{X}, \mathscr{B})$ be a measurable space where $\mathscr{X} \subset R^l$ and $\mathscr{B}$ is a Borel σ-field. Let $p(\cdot : \delta, \xi)$ be a density with respect to a σ-finite measure μ where δ is a real parameter, ξ is nuisance parameter and the range of ξ may depend on δ. Consider a testing problem $H_0: \delta = 0$ versus $H_1: \delta = \lambda (\lambda > 0)$.

LEMMA 6.1 (Giri and Kiefer (1964)). *Under the following Assumptions the test ϕ^* is locally minimax for testing $H_0: \delta = 0$ against $\delta = \lambda$ as $\lambda \to 0$.*

ASSUMPTION 1. There exists a statistic $U(x)$ such that U is bounded and positive and has a continuous distribution function for each (δ, ξ), which is equicontinuous in (δ, ξ) for $\delta < \delta_0$, and such that $\phi^*(x) = 1$ if $U(x) > c$ and $\phi^*(x) = 0$ otherwise.

ASSUMPTION 2. $E(\phi^* \mid 0, \xi) = \alpha$ and $E(\phi^* \mid \lambda, \xi) = \alpha + h(\lambda) + g(\lambda, \xi)$ where $g(\lambda, \xi) = o(h(\lambda))$ uniformly in ξ, $h(\lambda) > 0$ for $\lambda > 0$ and $h(\lambda) = o(1)$.

ASSUMPTION 3. There exist probability measures $\eta_{0,\lambda}$, $\eta_{1,\lambda}$ on the sets $\{\delta = 0\}$ and $\{\delta = \lambda\}$ respectively for which

$$\int p(x : \lambda, \xi)\eta_{1,\lambda}(d\xi) / \int p(x : 0, \xi)\eta_{0,\lambda}(d\xi) = 1 + h(\lambda)[g(\lambda) + U(x)r(\lambda)] + B(x, \lambda),$$

where $0 < c_1 < r(\lambda) < c_2 < \infty$ for λ sufficiently small, $g(\lambda) = o(1)$ and $B(x, \lambda) = o(h(\lambda))$ uniformly in x.

Following the notation used in Section 5, we verify these conditions for the test $\mathscr{K}_5$. Instead of the group G in Section 5, the subgroup $G_0 = \mathscr{O}(m_1) \times G_T(p_2)$ is chosen, leaving the problem invariant under the action $g(X, Y, T_2) = (QXS', YS', QT_2Q')$ and $g(\Theta_{12}, \Sigma_{22.3}) = (Q\Theta_{12}S', S\Sigma_{22.3}S')$ for $g = (Q, S) \in G_0$ where $G_T(p)$ denotes the set of $p \times p$ nonsingular lower triangular matrices. As is well known, $G_T(p_2)$ satisfies the conditions in the Hunt–Stein theorem, and so does G_0. Let $\mathscr{C}_\alpha(G_0)$ be the class of G_0-invariant size α tests. By the Hunt–Stein theorem, a test which is locally minimax in $\mathscr{C}_\alpha(G_0)$ is locally minimax. Under G_0, without loss of generality, $(\Theta_{12}, \Sigma_{22.3})$ can be replaced by (ξ, I) where $\xi = \Theta_{12}A_0$ as defined in Section 5. Now we choose (Δ, ξ) as (δ, ξ) in Lemma 6.1 where $\Delta = \operatorname{tr} \xi\xi'$. To verify Assumption 1, define

$$U \equiv U(x, y, t_2) = \operatorname{tr} (I + t_2)^{-1}[b_1 x(x'x + y'y)^{-1}x' - b_2 I] + 1 \tag{6.1}$$

where $b_1 = (m_1 + n_1)/(2m_1p_2)$ and $b_2 = 1/2m_1$, and write the test $\mathscr{K}_5$ as $\phi^*(x, y, t_2) = 1$ if $U > c_\alpha$ and $\phi^* = 0$ otherwise. Then U is positive and has a continuous distribution function for each (Δ, ξ). For $\Delta <$ some Δ_0, (5.1) implies that this distribution can be written as

$$F^U(c \mid \Delta, \xi) = 1 - E(\phi_c \mid \Delta, \xi) = 1 - \alpha(c) - B(\phi_c)\Delta + o_c(\Delta)$$

where $\phi_c(U) = 1$ if $U > c$ and $\phi_c = 0$ otherwise and $\alpha(c) = E_0(\phi_c)$. Here, writing the remainder term $o(\Delta)$ in (5.7) as $o(\Delta; x, y, t_2)$, the last term $o_c(\Delta)$ above is expressed as $\int \phi_c(U(x, y, t_2))\, o(\Delta; x, y, t_2)P(d(x, y, t_2) \mid 0, I)$. Since from Lemma 5.3 $|o(\Delta; x, y, t_2)| \leqq K\Delta$ for some $K > 0$ and Δ near zero, for $c' > c$ $|o_{c'}(\Delta) - o_c(\Delta)| \leqq K\Delta[E_0(\phi_c) - E_0(\phi_{c'})]$ for Δ near zero. Hence we obtain

$$\begin{aligned} &|F^U(c' \mid \Delta, \xi) - F^U(c \mid \Delta, \xi)| \\ &\qquad \leqq |\alpha(c) - \alpha(c')| + |B(\phi_c) - B(\phi_{c'})|\Delta + |\alpha(c) - \alpha(c')|K\Delta . \end{aligned}$$

Since $\alpha(c)$ and $B(\phi_c)$ do not depend on (Δ, ξ) and since these are uniformly continuous functions of c, the equicontinuity of $F^U(\cdot \mid \Delta, \xi)$ for $\Delta <$ some Δ_0 follows.

For Assumption 2, take $\phi_\alpha \equiv \alpha$ in (5.1). Then $\pi(\phi_\alpha, \delta) = \alpha$ implies $B(\phi_\alpha) = 0$ from (5.1). Since the test $\mathscr{K}_5$ or ϕ^* is the unique maximizer of $B(\phi)$ with respect to $\phi \in \mathscr{C}_\alpha(\mathscr{G})$, ϕ^* dominates $\phi_\alpha \equiv \alpha$ strictly for $0 < \alpha < 1$, implying $B(\phi^*) > 0$. Further from (5.1)

$$E(\phi^* \mid \Delta, \xi) = \alpha + B(\phi^*)\Delta + o(\Delta) .$$

Hence with $\Delta = \lambda$, $h(\lambda) = B(\phi^*)\lambda$ and $g(\lambda, \xi) = o(\operatorname{tr} \xi\xi') = o(\Delta)$, Assumption 2 is now satisfied.

To verify Assumption 3, we first consider the probability ratio of a maximal invariant, say $T = t(X, Y, T_2)$, *under* G_0. Then, as in Lemma 5.1, the ratio is given by

$$\begin{aligned} R_0 &\equiv [P^T(dt \mid \xi, I)/P^T(dt \mid 0, I)](t(x, y, t_2)) \\ &= K(x, y, t_2 \mid \xi, I)/K(x, y, t_2 \mid 0, I) \end{aligned} \tag{6.2}$$

where $P^T(dt \mid \xi, I)$ denotes the distribution of T under (ξ, I) and

$$K(x, y, t_2 \mid \xi, I) = \iint_{\mathscr{O}(m_1)\times G_T(p_2)} f(g(x, y, t_2) \mid \xi, I)|SS'|^{M/2}\tau_1(dQ)\tau_2(dS) \tag{6.3}$$

with f given by (5.3). Here τ_1 and τ_2 are left invariant measures on $\mathscr{O}(m_1)$ and $G_T(p_2)$ respectively with $\tau_1(\mathscr{O}(m_1)) = 1$. This is because G_0 is a subgroup of G and hence the space of (x, y, t_2) is a Cartan G_0-space with the relative topology (see Palais (1961) and Wijsman (1967)). The ratio R_0 is the density of T with respect to $P^T(dt \mid 0, I)$. In the same way as in the proof of Lemma 5.3 (see (5.11) through (5.13)), the ratio R_0 is evaluated as

$$R_0 = 1 - (2m_1)^{-1}\Delta \operatorname{tr}(I + t_2)^{-1} + I(v, \xi) + o(\Delta) , \tag{6.4}$$

where $o(\Delta)$ is uniform in (x, y, t_2) and with $D_0 = \int_{G_T(p_2)} |SS'|^{M/2} \operatorname{etr}(-\frac{1}{2}SS')\tau_2(dS)$,

$$I(v, \xi) = D_0^{-1} \int_{G_T(p_2)} (2m_1)^{-1} \operatorname{tr} vS'\xi'\xi Sv'|SS'|^{M/2} \operatorname{etr}(-\tfrac{1}{2}SS')\tau_2(dS) . \tag{6.5}$$

In Section 5, the integral in (6.5) was over $Gl(p_2)$ instead of $G_T(p_2)$ and hence resulted in $I(v, \xi) = (2m_1p_2)^{-1}(m_1 + n_1)\Delta \operatorname{tr} v'v$ for arbitrary ξ. This is not true now, but if we can show that there is a special value of ξ for which the same result holds, then for $\lambda <$ some λ_0,

$$R_0 = 1 + \lambda[U(x, y, t_2) - 1] + o(\Delta) \tag{6.6}$$

follows from (6.4) and the definition of $U(x, y, t_2)$. Therefore putting the probability measure η on that particular ξ, Assumption 3 is verifield with $h(\lambda) = B(\phi^*)\lambda$, $g(\lambda) = -B(\phi^*)^{-1}$, $r(\lambda) = B(\phi^*)^{-1}$ and $B(x, \lambda)$ being the last term in (6.6). This observation and the next proof for the existence of such a special value of ξ are due to a referee. Kariya (1975), using an argument similar to that in Schwartz (1967), found a measure η for which the same result holds.

Now what is left is to show the existence of ξ such that

$$I(v, \xi) = (2m_1 p_2)^{-1} M\Delta \operatorname{tr} v'v$$

for any v where $\Delta = \operatorname{tr} \xi\xi'$. For any $p_2 \times p_2$ matrices A and Γ, define

$$h(A, \Gamma) = D_0^{-1} \int_{G_T(p_2)} (\operatorname{tr} \Gamma S A S')|SS'|^{M/2} \operatorname{etr}(-\tfrac{1}{2}SS')\tau_2(dS) . \tag{6.7}$$

Let $\Gamma = (\gamma_{ij})$, $A = (a_{ij})$ and $S = (s_{ij})$. Then $\operatorname{tr} \Gamma S A S' = \sum_{ijkl} \gamma_{ki} s_{ij} a_{jl} s_{kl}$ but the integration of $s_{ij} s_{kl}$ gives 0 unless $i = k$ and $j = l$. Thus only $\sum_{i \geqq j} \gamma_{ii} a_{jj} s_{ij}^2$ remains in the integration. Define for $i \geqq j$,

$$c_{ij} = D_0^{-1} \int s_{ij}^2 |SS'|^{M/2} \operatorname{etr}(-\tfrac{1}{2}SS')\tau_2(dS) \quad (> 0) .$$

Let C be the lower triangular matrix with elements c_{ij} and let $u' = (u_1, \cdots, u_{p_2})$ with $u_i = \gamma_{ii}$. Then

$$h(A, \Gamma) = \sum_{i \geqq j} u_i a_{jj} c_{ij} = \sum_j (\sum_{i \geqq j} u_i c_{ij}) a_{jj} .$$

In order that this be proportional to $\operatorname{tr} A$, $\sum_{i \geqq j} u_i c_{ij} = \text{const.} = c_0$, say, is necessary and sufficient. That is, $u'C = c_0 1'$ where $1' = (1, \cdots, 1)$, and so $u' = c_0 1' C^{-1}$ since $|C| \neq 0$. With this choice for u, $h(A, \Gamma) = c_0 \operatorname{tr} A$ and in particular $h(I, \Gamma) = c_0 \operatorname{tr} I = c_0 p_2$. But as in (5.12) $h(I, \Gamma) = M \operatorname{tr} \Gamma$. Therefore $c_0 = M p_2^{-1} \operatorname{tr} \Gamma$ and finally $h(A, \Gamma) = M p_2^{-1} \operatorname{tr} \Gamma \operatorname{tr} A$ provided the ith diagonal element of Γ is proportional to the ith element of $1'C^{-1}$ for all $i = 1, \cdots, p_2$. Taking $\Gamma = \xi'\xi$ and $A = v'v$ proves our claim.

Thus we have verified all the assumptions required in Lemma 6.1 under the action of the group G_0. By virtue of the Hunt–Stein theorem we obtain

THEOREM 6.1. *For* $0 < \alpha < 1$, *the test* $\mathscr{K}_5$ *is locally minimax with respect to the contour* $\{(\Theta_{12}, \Sigma_{22.3}) \mid \operatorname{tr} \Theta_{12} \Sigma_{22.3}^{-1} \Theta_{12}' = \lambda\}$ *as* $\lambda \to 0$.

We just proved the local minimaxity of the test $\mathscr{K}_5$ on the contour $\operatorname{tr} \Theta_{12} \Sigma_{22.3}^{-1} \Theta_{12}' = \lambda$. In the MANOVA problem Schwartz (1967) defined a local family of contours containing such a contour as we used. His argument is applicable to our case to prove the local minimaxity of the test $\mathscr{K}_5$ for a wider class of contours. Secondly it is remarked that if any fully invariant ($\mathscr{G}$-invariant) test is to be locally minimax, it must be the one which is locally best invariant. Hence Pillai's test, which is locally minimax in the MANOVA problem, cannot be locally minimax in the GMANOVA problem. The above result shows that the test $\mathscr{K}_5$ is an alternative test to the noninvariant Bayes test proposed by Kiefer and Schwartz (1965).

7. Remarks on admissibility. It has been shown that the test $\mathscr{K}_5$ is admissible in the class of all $\mathscr{G}$-invariant tests $\mathscr{D}(\mathscr{G})$. Here we briefly discuss the admissibility of the test $\mathscr{K}_i$'s ($i = 1, \cdots, 4$) in $\mathscr{D}(\mathscr{G})$. The class of $\mathscr{G}$-invariant tests based on T_1 alone does not form an essentially complete class. Hence from the viewpoint of the power of a test, there is no reason to restrict our attention to that class. As has been remarked, conditionally on $T_2 = t_2$, our problem is exactly the same as the usual MANOVA problem. From Schwartz (1967a), the tests $\mathscr{K}_i$'s ($i = 1, \cdots, 4$) are conditionally admissible. However, without completeness the conditional admissibility does not imply the unconditional admissibility in general. (See Matthes and Truax (1967) for certain special cases.) For the Bayes method, the problem is the same. That is, it is very difficult to find the prior distributions that yield the test $\mathscr{K}_i$'s. Of course, conditionally the argument in Kiefer and Schwartz (1965) holds. Thus, the usual methods, which are powerful in many problems, turned out to be not so powerful in our problem. The admissibility of these tests is an open question.

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REFERENCES

ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

COCHRAN, W. G. and BLISS, C. I. (1948). Discriminant functions with covariance. *Ann. Math. Statist.* **19** 151-176.

CONSTANTINE, A. G. (1963). Some noncentral distribution problems in multivariate analysis. *Ann. Math. Statist.* **34** 1270-1284.

EATON, M. L. (1972). *Multivariate Statistical Analysis.* Univ. of Copenhagen.

FUJIKOSHI, Y. (1973). Monotonicity of the power functions of some tests in general MANOVA models. *Ann. Statist.* **1** 338-391.

GIRI, N. (1961). On tests with likelihood ratio criteria in some problems of multivariate analysis. Ph. D. Thesis, Stanford Univ.

GIRI, N. (1962). On a multivariate testing problem. *Calcutta Statist. Assoc. Bull.* **11** 55-60.

GIRI, N. (1964). On the likelihood ratio test of a normal multivariate testing problem. *Ann. Math. Statist.* **35** 181-190; correction, *ibid.* **35** 1388.

GIRI, N. (1965). On the likelihood ratio test of a normal multivariate testing problem II. *Ann. Math. Statist.* **36** 1061-1065.

GIRI, N. (1968). Locally and asymptotically minimax tests of a multivariate problem. *Ann. Math. Statist.* **39** 171-178.

GIRI, N. and KIEFER, J. (1964). Local and asymptotic minimax properties of a normal multivariate testing problem. *Ann. Math. Statist.* **35** 21-35.

GLESER, L. J. and OLKIN, I. (1966). A k-sample regression model with covariance. *Multivariate Analysis* (P. R. Krishnaiah, ed.). 59-72. Academic Press, New York.

GLESER, L. J. and OLKIN, I. (1970). Linear models in multivariate analysis. *Essays in Probability and Statistics.* 267-292. Wiley, New York.

HALL, W. J., WIJSMAN, R. A. and GHOSH, J. K. (1965). The relationship between sufficiency and invariance with applications in sequential analysis. *Ann. Math. Statist.* **36** 575–614.

JAMES, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

JOHN, S. (1971). Some optimal multivariate tests. *Biometrika* **58** 123–127.

JOHNSON, N. L. and KOTZ, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions.* Wiley, New York.

KARIYA, T. (1975). Hypothesis testing problems in the general MANOVA model. Ph. D. Thesis, Univ. of Minnesota.

KHATRI, C. G. (1966). A note on a MANOVA model applied to problems in growth curves. *Ann. Inst. Statist. Math.* **18** 75–86.

KIEFER, J. and SCHWARTZ, R. (1965). Admissible Bayes character of T^2-, R^2-, and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.* **36** 747–770.

LEHMANN, E. L. (1959). *Testing Statistical Hypotheses.* Wiley, New York.

MATTHES, T. K. and Truax, D. R. (1967). Tests of composite hypotheses for the multivariate exponential family. *Ann. Math. Statist.* **38** 681–697.

NACHBIN, L. (1965). *The Haar Integral.* Van Nostrand, New York.

OLKIN, I. and SHRIKHANDE, S. (1954). On a modified T^2-problem. *Ann. Math. Statist.* **25** 808.

PALAIS, R. S. (1961). On the existence of slices for actions of noncompact Lie groups. *Ann. of Math.* **73** 295–323.

PONTRYAGIN, L. (1966). *Topological Groups.* 2nd ed., translated from the Russian. Gordon and Breach, New York.

POTTHOFF, R. F. and ROY, S. N. (1964). A generalized multivariate analysis of variance model useful especially for growth curve problems. *Biometrika* **51** 313–326.

RAO, C. R. (1949). On some problems out of discrimination with multiple character. *Sankhya* **9** 343–366.

RAO, C. R. (1965). The theory of least squares when the parameters are stochastic and its application to the analysis of growth curves. *Biometrika* **52** 447–458.

RAO, C. R. (1967). Least squares theory using an estimated dispersion matrix and its application to measurement of signals. *Fifth Berk. Symp. Math. Statist. Prob.* **1** 355–372. Univ. of California Press.

SCHWARTZ, R. (1967). Local minimax tests. *Ann. Math. Statist.* **38** 340–360.

SCHWARTZ, R. (1967a). Admissible tests in multivariate analysis of variance. *Ann. Math. Statist.* **38** 698–710.

STEIN, C. (1969). *Multivariate Analysis I.* Technical Report 42. Lecture notes recorded by M. L. Eaton. Stanford Univ.

SUGIURA, N. (1972). Locally best invariant test for sphericity and the limiting distributions. *Ann. Math. Statist.* **43** 1312–1316.

WIJSMAN, R. A. (1967). Cross-sections of orbits and their application to densities of maximal invariants. *Fifth Berk. Symp. Math. Statist. Prob.* **1** 389–400. Univ. of California Press.

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