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A NOTE ON ESTIMATORS IN FINITE POPULATIONS

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Two previous results of the author, regarding the admissibility of the sample mean as estimator of the population mean in the class of all estimators for arbitrary sampling designs with the squared error as loss function, and for a wide class of loss functions, are shown to remain valid in the more general case when the population means of many variates are simultaneously under estimation.

In the previous papers [1] and [2], it has been shown that for arbitrary sampling designs, and in the entire class of all estimators, the sample mean is always admissible as an estimator of the population mean. The result is proved in [1] with the squared error as loss function and in [2] for a general class of loss functions subject only to certain very mild restrictions. In both these papers it is assumed that there is a single real number x_i associated with each population unit whose population total (or equivalently population mean value) is under estimation. But in practice, very frequently there are many population characteristics whose sample values are simultaneously observed for estimating their population totals. Hence a question arises whether the results proved in [1] and [2] remain valid in these circumstances or whether the clubbing together of different estimates has an effect like that of Stein's [3] inadmissibility result for the multivariate normal population. It is shown in this note that the results proved in [1] and [2] remain valid.

It suffices to indicate here only the modifications required in the arguments in [1] and [2].

Consider first the argument in [1]. Its adaptation is effected by treating each x_i, $i = 1, 2, \cdots, N$ as a p-dimensional vector, $(p > 1)$.

The population total $T(x)$, the estimators $e(s, x)$, $e'(s, x)$, $e^*(s, x)$, $g(s, x)$, $h(s, x)$, $h^*(s, x)$, etc. all then become p-dimensional vectors, the loss function for an estimator $e(s, x)$ being the square of the vector magnitude $|e(s, x) - T(x)|^2$. The discrete prior distribution w in equation (7) of [1] is replaced by the discrete distribution of a p-dimensional vector x with $E_w(x) = \theta(w)$, $E_w|x - \theta|^2 = \sigma^2(w)$, and in Lemma 7 of [1], $t_1, t_2, \cdots, t_k$ are taken to be the k vectors in R_p to which w assigns positive probabilities $p_1, p_2, \cdots, p_k$ with $\sum_{i=1}^k p_i = 1$. Similarly $G(x)$, $H(x)$ are treated as p-dimensional vectors, $|G(x) - H(x)|^2$ being substituted for $(G(x) - H(x))^2$. Similarly, a product such as $l_m(x)(\bar{x}_m - \theta)$ [[1], equation (6*), page 1727] is replaced by the scalar product of the p-dimensional vectors. With

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these adaptations, the whole argument in [1] is seen to hold for the case of several characteristics.

The adaptation of the argument in [2] is effected in a similar manner by treating each x_i as a p-dimensional vector. In [2], the admissibility of a ratio estimator based on a set of known positive numbers y_i, $1 = 1, 2, \cdots, N$ is proved. In adapting the argument, the y_i are continued as scalars, which means that the ratio estimates of the population totals of the p characteristic numbers are assumed to be based on the same set of numbers y_i. The estimator thus reduces to (the vector) sample mean by putting $y_i = 1/N$ for $i = 1, 2, \cdots, N$. The estimators $e(s, x)$, $e'(s, x)$, $e(s, x)$, etc. all become p-dimensional vectors, the loss function being the real-valued function $V(|e(s, x) - T(x)|)$. The prior distribution on R_N defined in the remarks preceding equation (7) of [2], is replaced by a distribution in which the p-vectors x_i, for $i = 1, 2, \cdots, N$ are distributed independently, and normally with respectively mean p-vectors θy_i and covariance matrices $y_i I$ where I is the $p \times p$ identity matrix, θ having a p-variate normal distribution with mean vector 0 and covariance matrix $\tau^2 I$. With these substitutions the entire argument in Section 3 of [2] continues to hold, thus proving the weak admissibility of $\bar{e}(s, x)$ (a p-dimensional vector estimator).

It remains to show how the argument in Section 4 of [2], showing that weak admissibility implies strict admissibility, is adapted. The argument here essentially consists in showing that if the subset of the parametric space R_N defined by $|h(s, x)| = |e'(s, x) - \bar{x}_s| \neq 0$ is not empty then it contains at least one point from which one can successively build up larger and larger subsets of R_N until a contradiction with Theorem 3.1 or 4.1 is reached. Thus it suffices to indicate how the set Q^a_{m-n} in equations (69) to (71) of [2] is defined in the case under consideration.

Let $a = (a_1, a_2, \cdots, a_N)$, where each a_i is a p-dimensional vector, denote the point of R_{Np} and s_0 a sample with positive probability such that $|h(s_0, a)| = |e'(s_0, a) - \bar{x}_{s_0}(a)| > 0$. As in [2], s_0 is taken without loss of generality as the sample consisting of units $u_1, u_2, \cdots, u_m$. The $(N - m)$ p-dimensional hyperplane P^a_{N-m} is defined by

$$x_i = a_i \,, \qquad i = 1, 2, \cdots, m \,. \tag{1}$$

Let $(g \cdot h)$ denote the inner product of the p-dimensional vectors g and h. Then in place of (69) in [2], the set $Q^a_{N-m} \subset P^a_{N-m}$ is defined by

$$x \in Q^a_{N-m} \quad \text{if and only if} \quad x \in P^a_{N-m} \,, \tag{2}$$

and

$$(h(s_0, x) \cdot [\bar{x}_{s_0} - \bar{X}_N]) \geqq 0 \,.$$

Since by equation (3) in [2],

$$\bar{x}_{s_0} = (y(s_0))^{-1} \textstyle\sum_{i=1}^m a_i \,,$$
$$\bar{X}_N = Y^{-1}\{\textstyle\sum_{i=1}^m a_i + \sum_{i=m+1}^N x_i\}$$

and by the definition of an estimator $h(s_0, x) = h(s_0, a)$ for $x \in P^a_{N-m}$, it follows from (2) that

$$\left[\frac{1}{y(s_0)} - \frac{1}{Y}\right] \sum_{i=1}^{m} (a_i \cdot h(s_0, a)) \geqq Y^{-1} \sum_{i=m+1}^{N} (x_i \cdot h(s_0, a)) . \tag{3}$$

Then in place of equation (70) of [2] we obtain that, for $x \in Q^a_{N-m}$,

$$|e'(s_0, x) - \bar{X}_N|^2 \geqq |x_{s_0} - \bar{X}_N|^2 + h_0^2 \tag{4}$$

where

$$|h_0| = |h(s_0, a)| > 0 \quad \text{by assumption.}$$

In Lemma 3.2 of [2], it is shown that for any positive constant h, the set of values of t for which $V(t + h) > V(t)$ has positive measure. By a slight modification of the argument the result can be shown to hold also for the set of values of t for which $V((t^2 + h^2)^{\frac{1}{2}}) > V(t)$, the required modification being to define the sequence of values $T_1, T_2, \cdots$ by $T_1 = \max((T^2 - h^2)^{\frac{1}{2}}, 0)$, $T_2 = \max((T_1^2 - h^2)^{\frac{1}{2}}, 0) = \max((T^2 - 2h^2)^{\frac{1}{2}}, 0)$, etc., instead of by $T_1 = \max(T - h, 0)$, $T_2 = \max(T_1 - h_1, 0) = \max(T - 2h, 0)$ etc. The rest of the argument remains unchanged. With this modification in the lemma, (4) implies that there exists a subset $Q^{*a}_{N-m} \subset Q^a_{N-m}$ with positive $(N - m)$ p-dimensional measure such that for $u \in Q^{*a}_{N-m}$

$$V(|e'(s_0, x) - \bar{X}_N|) > V(|\bar{x}_{s_0} - \bar{X}_N|)$$

which corresponds to equation (71) in [2]. The argument is now continued as in [2].

Thus the results proved in [1] and [2] remain valid in the case when population totals of several characteristics are simultaneously estimated.

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