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GENERAL DISTRIBUTION THEORY OF THE CONCOMITANTS OF ORDER STATISTICS[1]

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Let (X_i, Y_i) $(i = 1, 2, \cdots, n)$ be n independent rv's from some bivariate distribution. If $X_{r:n}$ denotes the rth ordered X-variate, then the Y-variate $Y_{[r:n]}$ paired with $X_{r:n}$ is termed the concomitant of the rth order statistics. The exact and asymptotic distribution theory of $Y_{[r:n]}$ and of its rank are studied. The results obtained are applied to a prediction problem in a Round Robin tournament.

1. Introduction. Let (X_i, Y_i) $(i = 1, 2, \cdots, n)$ be n independent random variables from some bivariate distribution. If we arrange the X-variates in ascending order as

$$X_{1:n} \leqq X_{2:n} \leqq \cdots \leqq X_{n:n},$$

then the Y-variates paired with these order statistics are denoted by

$$Y_{[1:n]}, Y_{[2:n]}, \cdots, Y_{[n:n]},$$

and termed the *concomitants* of the order statistics. These concomitants are of interest in selection and prediction problems based on the ranks of the X's. For example, when k $(< n)$ individuals having the highest X-scores are selected, we may wish to know the behavior of the corresponding Y-scores.

Under the assumption that X_i and Y_i are linearly related apart from an independent error term, the small-sample theory of concomitants has been studied extensively by O'Connell (1974). The asymptotic distribution theory of the concomitants, in the case when the paired variates (X_i, Y_i) have a bivariate normal distribution, has been investigated by David and Galambos (1974). Their results depend heavily on the assumption of linearity between X_i and Y_i. In this paper, the general distribution theory of the concomitants and of their ranks is studied when the (X_i, Y_i) are from an arbitrary absolutely continuous bivariate distribution. The results obtained are applied to a prediction problem in a Round Robin tournament.

2. Distribution of the concomitants. For convenience, the following notation concerning the distributions of random variables will be adopted throughout this paper.

$$F_W(\omega)\text{—cdf of a random variable } W.$$
$$f_W(\omega)\text{—pdf of a random variable } W.$$

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$f(y \mid x)$—conditional pdf of Y_1 given $X_1 = x$.

$f_{r_1,\cdots,r_k:n}(x_1, \cdots, x_k)$—joint pdf of the k ordered X-variates $X_{r_1:n}$, $X_{r_2:n}, \cdots, X_{r_k:n}$ $(k \geqq 1)$ with $1 \leqq r_1 < r_2 < \cdots < r_k \leqq n$.

Let (X_i, Y_i) $(i = 1, 2, \cdots, n)$ be n independent random variables having a common bivariate cdf $F(x, y)$ and pdf $f(x, y)$. It is also assumed that $f(x, y)$ is continuous although this assumption is needed only in proving the asymptotic results in Sections 2 and 3. Since (X_i, Y_i) $(i = 1, 2, \cdots, n)$ are independent and identically distributed random variables, the conditional pdf of $Y_{[r:n]}$ given $X_{r:n} = x$ is $f_{Y[r:n]}(y \mid X_{r:n} = x) = f(y \mid x)$. Hence

$$f_{X_{r:n},Y[r:n]}(x, y) = f(y \mid x) f_{r:n}(x), \quad \text{and}$$

$$f_{Y[r:n]}(y) = \int_{-\infty}^{\infty} f(y \mid x) f_{r:n}(x)\, dx. \tag{2.1}$$

More generally, for $1 \leqq r_1 < r_2 < \cdots < r_k \leqq n$, we have

$$\begin{aligned} &f_{Y[r_1:n],\cdots,Y[r_k:n]}(y_1, \cdots, y_k) \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_2} \prod_{i=1}^{k} f(y_i \mid x_i) f_{r_1,\cdots,r_k:n}(x_1, \cdots, x_n)\, dx_1 \cdots dx_k. \end{aligned} \tag{2.2}$$

Likewise we can also show that, for $r < s$,

$$f_{X_{s:n},Y[r:n]}(x, y) = \int_{-\infty}^{x} f(y \mid t) f_{r,s:n}(t, x)\, dt. \tag{2.3}$$

By (2.2) and (2.3), we can easily show the following:

$$\begin{aligned} &E(Y_{[r:n]}) = E[E(Y_1 \mid X_1 = X_{r:n})]; \\ &\mathrm{Var}\,(Y_{[r:n]}) = E[\mathrm{Var}\,(Y_1 \mid X_1 = X_{r:n})] + \mathrm{Var}\,[E(Y_1 \mid X_1 = X_{r:n})]; \\ &\mathrm{Cov}\,(Y_{[r:n]}, Y_{[s:n]}) = \mathrm{Cov}\,[E(Y_1 \mid X_1 = X_{r:n}), E(Y_1 \mid X_1 = X_{s:n})] \quad (r \neq s); \\ &\mathrm{Cov}\,(X_{s:n}, Y_{[r:n]}) = \mathrm{Cov}\,[X_{s:n}, E(Y_1 \mid X_1 = X_{r:n})]. \end{aligned} \tag{2.4}$$

The asymptotic distribution of the concomitants can also be easily obtained. For convenience, instead of working with variables $X_1, \cdots, X_n$, we shall work with uniform variables $F_X(X_1), \cdots, F_X(X_n)$. Thus, without loss of generality, at this point and throughout the next section, the X's are assumed to be uniformly distributed on $[0, 1]$.

THEOREM 2.1. *Let* $1 \leqq r_1 < r_2 < \cdots < r_k \leqq n$ *be sequences of integers such that, as* $n \to \infty$, $r_i/n \to \lambda_i$ *with* $0 < \lambda_i < 1$ $(i = 1, 2, \cdots, k)$. *Then*

$$\lim_{n\to\infty} \Pr\,(Y_{[r_1:n]} \leqq y_1, \cdots, Y_{[r_k:n]} \leqq y_k) = \prod_{i=1}^{k} \Pr\,(Y_i \leqq y_i \mid X_i = \lambda_i).$$

PROOF. Suppose $r_i/n \to \lambda_i$ $(i = 1, 2, \cdots, k)$ as $n \to \infty$ with $0 < \lambda_i < 1$. Then $(X_{r_1:n}, \cdots, X_{r_k:n})'$ converges in probability to $(\lambda_1, \cdots, \lambda_n)'$ as $n \to \infty$. Since $\prod_{i=1}^{k} \Pr\,(Y_i \leqq y_i \mid X_i = x_i)$ is a bounded continuous function of $(x_1, \cdots, x_n)$, it follows from (2.2) that

$$\lim_{n\to\infty} \Pr\,(Y_{[r_1:n]} \leqq y_1, \cdots, Y_{[r_k:n]} \leqq y_k) = \prod_{i=1}^{k} \Pr\,(Y_i \leqq y_i \mid X_i = \lambda_i).$$

3. Distribution of the rank of the concomitants. Let $R_{[r:n]}$ denote the rank of $Y_{[r:n]}$. Let

$$I(x) = 1 \quad \text{if } x \geqq 0\,, \qquad = 0 \quad \text{if } x < 0\,.$$

Then

$$R_{[r:n]} = \sum_{i=1}^{n} I(Y_{[r:n]} - Y_i)\,. \tag{3.1}$$

The distribution of $R_{[r:n]}$ and the expected value of $R_{[r:n]}$ are obtained by David, O'Connell and Yang (1976). For completeness and easier reference, we shall state the results here.

$$\begin{aligned} \Pr(R_{[r:n]} = s) &= n \sum_{k=1}^{s-1} \binom{n-1}{s-1}\binom{s-1}{k}\binom{n-s}{r-1-k} \\ &\times \Pr\begin{pmatrix} Y_i \leqq Y_n,\, X_i \leqq X_n,\, i = 1, 2, \cdots, k\,; \\ Y_i \leqq Y_n,\, X_i > X_n,\, i = k+1, \cdots, s-1\,; \\ Y_i > Y_n,\, X_i \leqq X_n,\, i = s, \cdots, s-1+(r-k-1)\,; \\ Y_i > Y_n,\, X_i > X_n,\, i = s+(r-k-1), \cdots, n-1 \end{pmatrix}. \end{aligned} \tag{3.2}$$

Note that (3.2) continues to hold if $(X_1, Y_1), \cdots, (X_n, Y_n)$ form a set of exchangeable pairs of random variables. This fact will be used in Section 4.

We shall obtain the asymptotic distribution of $R_{[r:n]}$ by first determining the asymptotic moments of $R_{[r:n]}/n$. From (3.1), as in David and Galambos (1974),

$$\begin{aligned} R_{[r:n]}^k &= [\textstyle\sum_{i=1}^{n} I(Y_{[r:n]} - Y_i)]^k \\ &= \textstyle\sum^* I(Y_{[r:n]} - Y_{i_1}) \cdots I(Y_{[r:n]} - Y_{i_k}) + O(n^{k-1})\,, \end{aligned}$$

where $\sum^*$ denotes the summation over all $(i_1, \cdots, i_k)$ with distinct components and $Y_{i_l} \neq Y_{[r:n]}$ for $l = 1, \cdots, k$. Therefore,

$$E\left[\left(\frac{R_{[r:n]}}{n}\right)^k\right] = \frac{1}{n^k} \sum^* \Pr(Y_{i_1} \leqq {}_{[r:n]}, Y_{i_2} \leqq Y_{[r:n]}, \cdots, Y_{i_k} \leqq Y_{[r:n]}) + o\left(\frac{1}{n}\right). \tag{3.3}$$

Since (X_i, Y_i) $(i = 1, 2, \cdots, n)$ are i.i.d. random vectors, we may write

$$\begin{aligned} &\textstyle\sum^* \Pr(Y_{i_1} \leqq Y_{[r:n]}, Y_{i_2} \leqq Y_{[r:n]}, \cdots, Y_{i_k} \leqq Y_{[r:n]}) \\ &\quad = \textstyle\sum_{l=1}^{n} \sum_{i_1 \neq \cdots \neq i_k \neq l} \Pr(Y_{i_1} \leqq Y_l, Y_{i_2} \leqq Y_l, \cdots, Y_{i_k} \leqq Y_l, \\ &\qquad\qquad \operatorname{rank}(X_l) = r) \\ &\quad = n(n-1) \cdots (n-k) \Pr(Y_1 \leqq Y_n, Y_2 \leqq Y_n, \cdots, Y_k \leqq Y_n, \\ &\qquad\qquad \operatorname{rank}(X_n) = r) \\ &\quad = n(n-1) \cdots (n-k) \Pr(Y_1 \leqq Y_n, Y_2 \leqq Y_n, \cdots, Y_k \leqq Y_n, \\ &\qquad\qquad (r-1)X_i's < X_n, (n-r)X_i's > X_n) \\ &\quad = n(n-1) \cdots (n-k) \textstyle\sum_{l=0}^{k} \binom{k}{l} \Pr(X_i < X_n, Y_i \leqq Y_n, \\ &\qquad\qquad i = 1, \cdots, l;\, X_i > X_n, Y_i \leqq Y_n, i = l+1, \cdots, k; \\ &\qquad\qquad \text{and exactly } r-1-l \text{ of } X_{k+1}, \cdots, X_{n-1} \text{ are } < X_n)\,. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$E\left[\left(\frac{R_{[n:r]}}{n}\right)^k\right] = \frac{n(n-1)\cdots(n-k+1)}{n^k} \sum_{l=0}^{k} \binom{k}{l} \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} [F(x,y)]^l \times [F_Y(y) - F(x,y)]^{k-l} f(y\,|\,x)\,dy\} f_{r-l:n-k}(x)\,dx + O\left(\frac{1}{n}\right). \tag{3.5}$$

As $n, r \to \infty$ with $r/n \to \lambda$ and $0 < \lambda < 1$, for any fixed k and $0 \leqq l \leqq k$, $X_{r-l:n-k}$ converges in probability to λ. Clearly, for $l \leqq k$ the inner integral in (3.5) is a bounded continuous function of x. Hence

$$\begin{aligned}\lim_{n\to\infty} E\left[\left(\frac{R_{[r:n]}}{n}\right)^k\right] &= \sum_{l=0}^{k} \binom{k}{l} \int_{-\infty}^{\infty} [F(\lambda,y)]^l [F_Y(y) - F(\lambda,y)]^{k-l} f(y\,|\,x=\lambda)\,dy \\ &= \int_{-\infty}^{\infty} [F_Y(y)]^k f(y\,|\,x=\lambda)\,dy\,.\end{aligned} \tag{3.6}$$

(3.6) leads immediately to the following theorem.

THEOREM 3.1. *Let $\{r\}$ be a sequence of integers such that, as $n \to \infty$, $r/n \to \lambda$ with $0 < \lambda < 1$. Then for $0 \leqq a \leqq 1$,*

$$\lim_{n\to\infty} \Pr(R_{[r:n]} \leqq na) = \Pr(Y \leqq F_Y^{-1}(a)\,|\,X = \lambda)\,. \tag{3.7}$$

PROOF. Since the limit obtained in (3.6) is the kth moment of a random variable $F_Y(W)$, where W is the conditional random variable $Y\,|\,X = \lambda$, the boundness of $F_Y(W)$ implies that its moments uniquely determine its distribution. Equation (3.7) now follows from (3.6).

Theorem 3.1 leads to the following theorem.

THEOREM 3.2. *If $\{r\}$ and $\{s\}$ are two sequences of integers such that r/n and s/n converge to λ_r and λ_s respectively as $n \to \infty$, with $0 < \lambda_r < \lambda_s < 1$, then for $0 \leqq a \leqq 1$ and any choice i ($1 \leqq i \leqq n$),*

$$\lim_{n\to\infty} \Pr(\text{rank}(Y_i) \leqq na\,|\,r \leqq \text{rank}(X_i) \leqq s) = \frac{1}{\lambda_s - \lambda_r} \int_{\lambda_r}^{\lambda_s} \Pr(Y \leqq F_Y^{-1}(a)\,|\,X = \lambda)\,d\lambda\,. \tag{3.8}$$

PROOF.

$$\begin{aligned}&\Pr(\text{rank}(Y_i) \leqq na\,|\,r \leqq \text{rank}(X_i) \leqq s) \\ &= \frac{n}{s-r+1} \sum_{j=r}^{s} \Pr(\text{rank}(X_i)=j) \Pr(\text{rank}(Y_i) \leqq na\,|\,\text{rank}(X_i)=j) \\ &= \frac{n}{s-r+1} \sum_{j=r}^{s} \frac{1}{n} \Pr(R_{[j:n]} \leqq na)\,.\end{aligned} \tag{3.9}$$

Now, if we let $s/n \to \lambda_s$ and $r/n \to \lambda_r$ as $n \to \infty$, then by (3.7), for large n,

$$\sum_{j=r}^{s} \Pr(R_{[j:n]} \leqq na) \frac{1}{n} \sim \sum_{j=r}^{s} \Pr\left(Y \leqq F_Y^{-1}(a)\,|\,X = \frac{j}{n}\right) \frac{1}{n}\,. \tag{3.10}$$

Recognizing that the right-hand side of (3.10) is a Riemann sum to the integral

$$\int_{\lambda_r}^{\lambda_s} \Pr(Y \leqq F_Y^{-1}(a) \mid X = \lambda)\, d\lambda ,$$

and letting $n \to \infty$, (3.9) and (3.10) yield (3.8). This completes the proof.

4. Application to a prediction problem in a Round Robin tournament. Suppose we have a Round Robin tournament among q teams, $A_1, A_2, \cdots, A_q$, with the tournament to be replicated n times. In each tournament, every team A_i $(i = 1, 2, \cdots, q)$ plays every other team once, making a total of $\frac{1}{2}q(q-1)n$ matches. It is also assumed that ties are forbidden.

Let $\delta_{ij\alpha}$ be a characteristic random variable corresponding to the outcome of the match between A_i and A_j in the αth replication. That is,

$$\begin{aligned} \delta_{ij\alpha} &= 1 \quad \text{if} \quad A_i \to A_j \qquad i, j = 1, 2, \cdots, q;\ i \neq j ,\\ &= 0 \quad \text{if} \quad A_j \to A_i \qquad i, j = 1, 2, \cdots, q;\ i \neq j , \end{aligned}$$

where $A_i \to A_j$ denotes A_i defeating A_j. We assume that there is no replication effect, and that all $\frac{1}{2}nq(q-1)$ matches are independent. The probability π_{ij} of A_i defeating A_j is $\Pr(\delta_{ij\alpha} = 1) = \pi_{ij}$ and of A_i being defeated by A_j is $\Pr(\delta_{ij\alpha} = 0) = 1 - \pi_{ij} = \pi_{ji}$. The total scores a_i of team A_i after n replications is given by $a_i = \sum_{\alpha=1}^{n} a_{i\alpha} = \sum_{\alpha=1, j\neq i}^{n} \delta_{ij\alpha}$, where $a_{i\alpha}$ denotes the score of A_i in the αth replication.

Suppose further that the q teams $A_1, A_2, \cdots, A_q$, are of similar caliber. We are interested in the following problem: If after the first m $(m < n)$ replications of the tournament, team A_i has rank r, then what is the probability that it will have rank s after n replications? In other words, we wish to find, using obvious notation,

$$\Pr(\text{rank}(\textstyle\sum_{\alpha=1}^{n} a_{i\alpha}) = s \mid \text{rank}(\sum_{\alpha=1}^{m} a_{i\alpha}) = r) \tag{4.1}$$

for $1 \leqq i, r, s \leqq q$.

Since the q teams are of similar caliber, $\pi_{ij} = \frac{1}{2}$ $(i, j = 1, 2, \cdots, q;\ i \neq j)$, and for each α $(\alpha = 1, 2, \cdots)$, $(a_{1\alpha}, a_{2\alpha}, \cdots, a_{q\alpha})$ form a set of exchangeable variates (cf. Trawinski and David (1963)). Further, clearly for fixed i $(i = 1, 2, \cdots, q)$ $a_{i\alpha}$ $(\alpha = 1, 2, \cdots, n)$ are n independent binomial $B(\frac{1}{2}, q-1)$ variates, $E(a_{i\alpha}) = \frac{1}{2}(q-1)$, $\text{Var}(a_{i\alpha}) = \frac{1}{4}(q-1)$, and the common correlation between $a_{i\alpha}$ and $a_{j\alpha}$ $(i \neq j)$ is $-1/(q-1)$. Let for $i = 1, 2, \cdots, q$,

$$\begin{aligned} U_i &= \textstyle\sum_{\alpha=1}^{m} 2(a_{i\alpha} - \frac{1}{2}(q-1))/(m(q-1))^{\frac{1}{2}} ,\\ V_i &= \textstyle\sum_{\alpha=m+1}^{n} 2(a_{i\alpha} - \frac{1}{2}(q-1))/(m(q-1))^{\frac{1}{2}} . \end{aligned}$$

Then, clearly, $(U_1, U_2, \cdots, U_q)$ and $(V_1, V_2, \cdots, V_q)$ are two independent sets of exchangeable random variables. Also let

$$T_i = U_i + V_i , \qquad i = 1, 2, \cdots, q . \tag{4.2}$$

Then (4.1) can be written as

$$\Pr(\text{rank}(T_i) = s \mid \text{rank}(U_i) = r) .$$

By (3.2), and the exchangeability of $(T_1, U_1), \cdots, (T_q, U_q)$ we now have

$$\Pr(\text{rank}(T_i) = s \mid \text{rank}(U_i) = r) = q \sum_{k=1}^{s-1} \binom{q-1}{s-1}\binom{s-1}{k}\binom{q-s}{r-1-k} \cdot$$

$$(4.3) \quad \Pr\begin{bmatrix} U_i + V_i \leqq U_q + V_q, U_i \leqq U_q, i = 1, \cdots, k; \\ U_i + V_i \leqq U_q + V_q, U_i > U_q, i = k+1, \cdots, s-1; \\ U_i + V_i > U_q + V_q, U_i \leqq U_q, i = s, \cdots, s-1+(r-k-1); \\ U_i + V_i > U_q + V_q, U_i > U_q, i = s+(r-k-1), \cdots, q-1 \end{bmatrix}$$

$$= q \sum_{k=1}^{s-1} \binom{q-1}{s-1}\binom{s-1}{k}\binom{q-s}{r-1-k} P_n^*(s, k) \quad \text{(say)}.$$

It follows from the independence of replications and the multivariate central limit theorem that as $m, n \to \infty$ and $m/n \to \lambda$ $(0 < \lambda < 1)$, $(U_1, \cdots, U_q, V_1, \cdots, V_q)$, converges in distribution to $(X_1', \cdots, X_q', Z_1', \cdots, Z_q')'$ which has a multivariate

$$N_{2q}\left(\mathbf{0}, \begin{bmatrix} \begin{pmatrix} 1 & & \rho^* \\ & \ddots & \\ \rho^* & & 1 \end{pmatrix} & 0 \\ 0 & \tau\begin{pmatrix} 1 & & \rho^* \\ & \ddots & \\ \rho^* & & 1 \end{pmatrix} \end{bmatrix}\right)$$

distribution, where $\rho^* = -1/(q-1)$ and $\tau = (1-\lambda)/\lambda$. Furthermore, since the X_i' are equicorrelated normal variates, X_i' may be generated as follows (e.g., Gupta (1963)):

$$X_i' = (-\rho^*)^{\frac{1}{2}} X_0 + (1-\rho^*)^{\frac{1}{2}} X_i, \qquad i = 1, 2, \cdots, q,$$

where $X_1, X_2, \cdots, X_q$ are i.i.d. $N(0, 1)$ variates, X_0 is also $N(0, 1)$, and $E(X_i X_0) = -(-\rho^*)^{\frac{1}{2}}/(1-\rho^*)^{\frac{1}{2}}$. Likewise, Z_i' may be written as follows:

$$Z_i' = \tau^{\frac{1}{2}}((-\rho^*)^{\frac{1}{2}} Z_0 + (1-\rho^*)^{\frac{1}{2}} Z_i), \qquad i = 1, 2, \cdots, q.$$

Hence as $m, n \to \infty$ with $m/n \to \lambda$ $(0 < \lambda < 1)$, we have

$$\lim_{n,m\to\infty} P_n^*(s, k) = \Pr\begin{bmatrix} Y_i \leqq Y_q, X_i \leqq X_q, i = 1, \cdots, k; \\ Y_i \leqq Y_q, X_i > X_q, i = k+1, \cdots, s-1; \\ Y_i > Y_q, X_i \leqq X_q, i = s, \cdots, s-1+(r-k-1); \\ Y_i > Y_q, X_i > X_q, i = s+(r-k-1), \cdots, q-1 \end{bmatrix}$$

where $Y_i = X_i + \tau^{\frac{1}{2}} Z_i$. Clearly, the (X_i, Y_i) are independent and have a bivariate normal distribution with correlation $\lambda^{\frac{1}{2}}$. Hence the limiting values of (4.1) as $m, n \to \infty$ with $m/n \to \lambda$ $(0 < \lambda < 1)$ is given by (3.2) with $n = q$ and $F(x, y)$, $f(x, y)$ being respectively the cdf and pdf of bivariate normal variates (X_i, Y_i). Therefore, Tables 1 and 2 constructed in David, O'Connell and Yang (1976) can be used here. For example, for $\lambda = 0.5$ and $q = 9$, from the column for $\rho = 0.7$ of Table 1, we have, for sufficiently large m and n with $m/n = 0.5$,

$$\Pr(\text{rank}(\textstyle\sum_{\alpha=1}^{n} a_{i\alpha}) = 9 \mid \text{rank}(\textstyle\sum_{\alpha=1}^{m} a_{i\alpha}) = 9) \sim 0.4404.$$

The case when one of the team is superior and the others continue to be equal is also considered. Supposed team A_q is the superior team, then

$$\pi_{qj} = \pi \quad (> \tfrac{1}{2}) \qquad j = 1, 2, \cdots, q-1\,,$$
$$\pi_{ij} = \tfrac{1}{2} \qquad i, j = 1, 2, \cdots, q-1;\ i \neq j\,.$$

In this case, only the problem of predicting the rank of the superior team A_q is studied. The development is similar to that of the null case considered above. Interested readers are referred to Yang (1976). However, for illustration, the approximate values of $P_q = \Pr(\text{rank}(\sum_{\alpha=1}^{n} a_{q\alpha}) = q \mid \text{rank}(\sum_{\alpha=1}^{m} a_{q\alpha}) = q)$ for $q = 3(2)9$, $n = 10$, $m/n = 0.5$ and $\pi = 0.5(0.05)0.8$ are included here. The following are the computed P_q values:

π	0.5	0.55	0.60	0.65	0.70	0.75	0.80
P_3	0.644	0.760	0.854	0.923	0.967	0.989	0.997
P_5	0.539	0.710	0.847	0.935	0.980	0.996	0.999
P_7	0.483	0.694	0.858	0.952	0.989	0.998	0.999
P_9	0.446	0.690	0.873	0.966	0.995	0.999	0.999

It is interesting to note that P_q is a decreasing function of q for $\pi = 0.5$, 0.55 and increasing function of q for $\pi = 0.65(0.05)0.8$.

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