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BAYESIAN SEQUENTIAL ESTIMATION[1]

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For fixed θ, let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables having density $f_\theta(x)$. Using a sequential Bayes decision theoretic approach we consider the problem of estimating any strictly monotone function $g(\theta)$ when the error incurred by a wrong estimate is measured by squared error loss and the sampling cost is c units per observation. A heuristic stopping rule is suggested. It is shown that the excess risk which results when using it is bounded above by terms of order c.

1. Introduction. What follows is an attempt to present a practical solution to a dynamic programming problem. For fixed θ, let $X_1, X_2, \cdots$, be a sequence of independent identically distributed (i.i.d.) random variables defined on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$. Let the probability function, or probability density function be of the form

$$f_\theta(x) = \exp(\theta x - K(\theta))$$

with respect to some σ-finite measure.

Suppose θ is itself a random variable having a distribution which admits density $\psi(\theta)$ with respect to Lebesgue measure. Let $g(\theta)$ be a strictly monotone real function of θ for which the expectation $Eg^2(\theta)$ is finite. It is desired to estimate $g(\theta)$ when the error incurred by a wrong estimate is squared error loss and the cost per observation is c units.

The dynamic programming solution is presented in Chow, Robbins, Siegmund (1972). Wald (1951), Anscombe (1953) and Bickel and Yahav (1965), (1969), treated this problem. However, their results were principally asymptotic ones. Our results, though in the same direction, are much stronger.

In Section 2, a lower bound on the Bayes risk using the optimal procedure is derived. In Section 4 an ad hoc procedure is suggested for each of several examples. In each case it is shown that the procedure wastes the cost of a bounded and computable number of observations. In Section 5 we prove a more general result.

2. A lower bound on the optimal risk. The main result of this section is Theorem 2 in which it is shown that under certain conditions the Bayes risk of

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the optimal procedure taking at least one observation is bounded below by $2c^{\frac{1}{2}}E\{|\sigma(\theta)g'(\theta)|\} - b^2c$ where $\sigma^{-2}(\theta)$ is the Fisher information number and b^2 is a computable constant.

The following notation is used throughout:

$\psi_n(\theta)$: the posterior density for a sample of size n when the prior is $\psi(\theta)$;
$\mathscr{F}_n$: the σ-algebra generated by $X_1, X_2, \cdots, X_n$;
$E(\cdot \mid \mathscr{F}_n)$: expectation with respect to $\psi_n(\theta)$;
$E_\theta(\cdot)$: expectation conditional on θ.
$\rho(\psi, \delta)$: total risk using the prior ψ and stopping rule δ.

Throughout we make the following assumptions:

Assumption A. $f_\theta(x) = \exp\{\theta x - K(\theta)\}$ with respect to a sigma finite measure.

Assumption B. g is a monotone increasing differentiable function for which $Eg^2(\theta) < \infty$.

Assumption C. The domain of θ is an interval $D = (a_2, a_1)$ where a_2, a_1 could be $-\infty$ and $+\infty$ respectively.

We require an adaptation of a result of Wolfowitz (1947). Let $\bar{g}_n$ be the Bayes estimate of $g(\theta)$ based on n observations.

Theorem 1. *Let $X_1, X_2, \cdots$ be a sequence of* i.i.d. *random variables with density $f_\theta(x)$. Let δ_n be an integer valued random variable for which*

$$(\delta_n = j) \in \mathscr{F}_j \qquad \textit{and} \qquad 1 \leqq \delta_n \leqq n .$$

Then under Assumptions A, B, C

$$E_\theta[\bar{g}_{\delta_n} - \mu(\theta)]^2 \geqq (\mu'(\theta)\sigma(\theta))^2/(E_\theta \delta_n)$$

where

$$\mu(\theta) = E_\theta \bar{g}_{\delta_n} .$$

Proof. We begin by noting that:

(i) $\partial f_\theta(x)/\partial\theta$ exists for all $\theta \in D$ and almost all x,

(ii) $\mu(\theta)$ can be differentiated under the integral sign with respect to θ for $\theta \in D$, and

(iii) $0 < \sigma^2(\theta) < \infty$ for all $\theta \in D$.

Setting $Y_{\delta_n} = \sum_{i=1}^{\delta_n} \{\partial \log f_\theta(x_i)/\partial\theta\}$, it follows from Wald's lemma that

$$E_\theta Y_{\delta_n} = 0 .$$

Moreover, from Wolfowitz (1947),

$$E_\theta Y_{\delta_n}^2 = (E_\theta \delta_n)\sigma^{-2}(\theta) .$$

Since $\mu'(\theta) = E_\theta(\bar{g}_{\delta_n} Y_{\delta_n})$, the desired result follows from Schwarz's inequality. □

Note that $\mu(\theta)$ is an increasing function of θ. In fact, the posterior density is a member of the family of densities defined for every t and fixed s by

$$\psi^*(t; s) = \exp(ts - K^*(t))$$

with respect to some σ-finite measure $\nu^*(t)$ and function $K(t)$. By Lemma 2, page 74 of Lehmann (1959),

$$\int g(t) \cdot \psi^*(t; s)\, d\nu^*(t)$$

is an increasing function of s. Hence $\bar{g}_n$ is an increasing function of S_n for every n where

$$S_n = \sum_{i=1}^n X_i \, .$$

A second application of this lemma allows us to deduce that $\mu(\theta)$ is an increasing function of θ.

THEOREM 2. *Let N^* be the optimal procedure which takes at least one observation. If, in addition to Assumptions* A, B, C *the following conditions are satisfied,*

(1) $b^2 = \int \sigma^2[\sigma'/\sigma + \psi'/\psi]^2\psi\, d\theta < \infty$,
(2) $\lim_{y\to a_i} [g(y)\sigma(y) \cdot \psi(y)] = 0$, $\lim_{y\to a_i}[\sigma(y)\psi(y)] = 0$ *for* $i = 1, 2$,
(3) $(\sigma(y)\psi(y)/\int_y^{a_1} \sigma\psi\, d\theta) \cdot (\int_y^{a_1} \sigma^2\psi\, d\theta)^{\frac{1}{2}} = O(1)$ *as* $y \to a_1$,
(4) $(\sigma(y) \cdot \psi(y)/\int_{a_2}^y \sigma\psi\, d\theta) \cdot (\int_{a_2}^y \sigma^2\psi\, d\theta)^{\frac{1}{2}} = O(1)$ *as* $y \to a_2$,

then

$$\rho(\psi, N^*) \geqq 2c^{\frac{1}{2}}E(\sigma g') - b^2 c \, .$$

PROOF. Define $\delta_n = \min(N^*, n)$.

We first show $\rho(\psi, \delta_n) \to \rho(\psi, N^*)$ as $n \to \infty$.

Let $Z_n = E[(\bar{g}_n - g(\theta))^2 \mid \mathscr{F}_n]$ so that

$$\begin{aligned}\rho(\psi, \delta_n) &= EZ_{\delta_n} + cE\delta_n \\ &= E(Z_{N^*}; N^* \leqq n) + E(Z_n; N^* > n) + cE\delta_n \, .\end{aligned}$$

By the monotone convergence theorem,

$$E(Z_{N^*}; N^* \leqq n) + cE\delta_n \to EZ_{N^*} + cEN^* = \rho(\psi, N^*) \qquad \text{as} \quad n \to \infty \, .$$

Hence, it suffices to show $\liminf_{n\to\infty} E(Z_n; N^* > n) = 0$. If $\mathscr{F}_\infty = B(X_1, X_2, \cdots)$, then by the martingale convergence theorem $\bar{g}_n = E(g \mid \mathscr{F}_n) \to E(g \mid \mathscr{F}_\infty)$ a.s. (see Chow et al., page 7 or page 18). In order that $E(g \mid \mathscr{F}_\infty) = g$ a.s., it suffices that g be $\mathscr{F}_\infty$-measurable. However, by the strong law of large numbers, $n^{-1}\sum_1^n X_i \to E_\theta E_1$ a.s. P_θ for every θ and hence a.s. P. It follows that $E_\theta(X_1)$ is $\mathscr{F}_\infty$-measurable; and since $g(\theta)$ is a measurable function of $E_\theta X_1$, it is also $\mathscr{F}_\infty$-measurable.

In order to show $\liminf_{n\to\infty} E(Z_n; N^* > n) = 0$ note that $Z_n \to 0$ a.s. and hence it suffices to show that the sequence Z_n, $n = 1, 2, \cdots$ is uniformly integrable. But the elementary inequality $(a + b)^2 \leqq 2(a^2 + b^2)$ and the Schwarz inequality give $Z_n \leqq 2(\bar{g}_n^2 + g^2) \leqq 2(E(g^2 \mid \mathscr{F}_n) + g^2)$. Since $Eg^2 < \infty$, the uniform integrability of $\{Z_n\}$ follows from Chow et al., page 18.

Then by Theorem 1,

$$E_\theta[\bar{g}_{\delta_n} - \mu(\theta)]^2 + cE_\theta\delta_n \geqq \frac{[\mu'(\theta)\sigma(\theta)]^2}{E_\theta\delta_n} + cE_\theta\delta_n$$
$$\geqq 2c^{\frac{1}{2}}\mu'(\theta)\sigma(\theta) .$$

The last inequality follows on minimizing with respect to $E_\theta\delta_n$. Hence,

$$\rho(\psi, \delta_n) \geqq E[g(\theta) - \mu(\theta)]^2 + 2c^{\frac{1}{2}}E\mu'(\theta)\sigma(\theta)$$

$$= 2c^{\frac{1}{2}}E\sigma(\theta)g'(\theta) + E[g(\theta) - \mu(\theta)]^2 - 2c^{\frac{1}{2}} \int (\mu - g)(\sigma\psi' + \sigma'\psi)\, d\theta \tag{2.1}$$

$$\geqq 2c^{\frac{1}{2}}E\sigma g' + E[g(\theta) - \mu(\theta)]^2 - 2c^{\frac{1}{2}}\{E[g(\theta) - \mu(\theta)]^2\}^{\frac{1}{2}}b \tag{2.2}$$

$$\geqq 2c^{\frac{1}{2}}E\sigma g' - b^2c . \tag{2.3}$$

The inequality in (2.2) follows by Schwarz's inequality whereas (2.3) follows on minimizing the right side of (2.2) with respect to $\{E[g(\theta) - \mu(\theta)]^2\}^{\frac{1}{2}}$. The integration by parts in (2.1) will now be justified.

Specifically, we need only show that

$$\lim_{y\to a_i} [\mu(y)\sigma(y)\psi(y)] = 0 \qquad \text{for} \quad i = 1, 2 .$$

Treating the upper end point first, we suppose that there exists a point y for which

$$0 \leqq \mu(y) .$$

Since μ is an increasing function,

$$0 \leqq \mu(y) \leqq \frac{\int_y^{a_1} \mu\sigma\psi\, d\theta}{\int_y^{a_1} \sigma\psi\, d\theta}$$

$$\leqq \frac{[\int_y^{a_1} \mu^2\psi\, d\theta]^{\frac{1}{2}}[\int_y^{a_1} \sigma^2\psi\, d\theta]^{\frac{1}{2}}}{\int_y^{a_1} \sigma\psi\, d\theta} . \tag{2.4}$$

Since $E\mu^2 \leqq EE_\theta\bar{g}^2_{\delta_n} \leqq Eg^2(\theta) < \infty$, the first numerator factor in (2.4) approaches 0 as $y \to a_1$. By assumptions (2) and (3), our result follows.

Suppose now $\mu(\theta) < 0$ for all $\theta \in D$. Then there exists a point z and a constant C for which

$$|\mu(y)|\sigma(y) \cdot \psi(y) \leqq C \cdot \sigma(y)\psi(y) \qquad \text{for} \quad y > z .$$

Letting $y \to a_1$, our result follows once again by (2). An analogous argument shows

$$\lim_{y\to a_2} \mu(y)\sigma(y) \cdot \psi(y) = 0 . \qquad \square$$

3. Preliminaries. In this section we develop two tools and then use them in some examples to obtain upper bounds on the excess risk.

First note that the Bayes estimate $\bar{g}_n$ is a martingale with respect to $\mathscr{F}_n$. Moreover, for all stopping rules N for which $P(N < \infty) = 1$, $E\bar{g}_N$ exists and $E\bar{g}_N = Eg(\theta)$. (See Breiman (1968), page 98.)

Our next result is derivable from Frame (1949).

LEMMA 1. *Let*

$$h(p) = \frac{p^{\frac{1}{2}}\Gamma(p)}{\Gamma(p+\frac{1}{2})} \qquad \textit{for} \quad p > 0\,.$$

Then,

$$1 \leqq h(p) \leqq 1 + \frac{1}{4p}\,.$$

PROOF. From Frame (1949), we quote the result that for $n > 0$

$$\frac{\Gamma\left(n+\frac{1+u}{2}\right)}{\Gamma\left(n+\frac{1-u}{2}\right)} = \left(n^2 + \frac{1-u^2}{12}\right)^{u/2} e^{-E_n(u)}$$

where $|u| \leqq 1$, and

$$0 \leqq E_n(u) < \frac{|u| \cdot (1-u^2)(4-u^2)}{6!\, n^4}\,.$$

Now substituting first $u = \frac{1}{2}$ and then $n = p + \frac{1}{4}$ we get

$$\begin{aligned} h(p) &= \left(1 + \frac{1}{2p} + \frac{1}{8p^2}\right)^{\frac{1}{4}} e^{-E_p(u)} \\ &\leqq \left[\left(1 + \frac{1}{4p}\right)^4 - \frac{1}{2p} - \frac{1}{16p^2} - \frac{1}{16p^3} - \frac{1}{64p^4}\right]^{\frac{1}{4}} \\ &\leqq 1 + \frac{1}{4p}\,. \end{aligned}$$

Moreover, $h(p)$ is a decreasing function of p converging to 1. □

4. Examples. Four examples are presented illustrating the use of the main result of Section 2. First, a lower bound on the optimal risk is computed using Theorem 2. Then using a heuristic argument formulated below, an ad hoc stopping rule is suggested and an upper bound on its risk is computed. In all cases, one can take advantage of the particular form of the distribution and the upper bound is easily obtained.

The choice of stopping rule used is based on the following argument. For a fixed sample of size n and for known θ, the risk is usually of the form $(k^2(\theta)/(l + n)) + cn$, where $k(\theta)$ is some function of θ and l is a positive constant. Treating n as a continuous variable, the value of n which minimizes the risk satisfies the equation

$$l + n = |k(\theta)|/c^{\frac{1}{2}}\,.$$

In the same way, the rule proposed when the Bayes posterior risk is of the form

$$(k_n^2/(l+n)) + cn$$

is to stop sampling at the Nth observation where $N =$ least integer $n \geqq 1$ such that $l + n \geqq |k_n|/c^{\frac{1}{2}}$.

EXAMPLE 1. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables from a negative exponential distribution with mean θ^{-1}. Let $g(\theta) = \theta^{-1}$ and assume the prior density $\psi(\theta)$ on θ has the form

$$\begin{aligned}\psi(\theta) &= \beta^\alpha/\Gamma(\alpha)\cdot\theta^{\alpha-1}e^{-\theta\beta}, \qquad \theta>0, \quad \alpha>2, \quad \beta>0\\ &= 0 \qquad\qquad \text{otherwise.}\end{aligned}$$

Theorem 2 holds with $b^2 = \alpha$. The posterior distribution of θ given $x_1, x_2, \cdots, x_n$ have been observed is again of the same form as $\psi(\theta)$ but with parameters α_n, β_n where $\alpha_n = \alpha + n$ and $\beta_n = \beta + (x_1 + \cdots + x_n)$. The posterior risk denoted by $\rho_0(\psi_n)$ is given by

$$\rho_0(\psi_n) = (\bar{g}_n{}^2/(\alpha_n - 2)) + cn ,$$

where $\bar{g}_n = \beta_n/(\alpha_n - 1)$ is the mean of the posterior distribution. In view of the heuristic argument presented above, it follows that the ad hoc procedure is to estimate θ by $\bar{g}_{N_1}$ where the stopping rule N_1 is given by

$$N_1 = \text{least integer} \quad n \geqq 1 \quad \text{such that} \quad \alpha_n - 2 \geqq \bar{g}_n/c^{\frac{1}{2}} .$$

In order to derive an upper bound on the total risk using N_1, write

$$\rho_0(\psi_n) = 2c^{\frac{1}{2}}\bar{g}_n + (2-\alpha)c + (\bar{g}_n - c^{\frac{1}{2}}\alpha_{n-2})^2/(\alpha_n - 2) .$$

For $n \geqq 1$, it is clear that $\bar{g}_{n-1} \leqq \bar{g}_n(1 + (\alpha_n - 2)^{-1})$. Since at time $N_1 - 1$, $\alpha_{N_1-1} - 1 < \bar{g}_{N_1-1}/c^{\frac{1}{2}}$, it follows that,

$$(\bar{g}_{N_1} - c^{\frac{1}{2}}\alpha_{N_1-2}) \leqq \alpha c^{\frac{1}{2}} .$$

Moreover, by Jensen's inequality,

$$(\alpha_{N_1} - 2)^{-1} \leqq c^{\frac{1}{2}}\bar{g}_{N_1}^{-1} \leqq c^{\frac{1}{2}}E\{(g(\theta))^{-1} \mid \mathscr{F}_{N_1}\} .$$

Hence, $\rho(\psi, N_1) \leqq 2c^{\frac{1}{2}}Eg(\theta) + (2-\alpha)c + (\alpha^3/\beta)c^{\frac{3}{2}}$. Consequently, the excess risk is bounded above by

$$2c + (\alpha^3/\beta)c^{\frac{3}{2}} .$$

The remaining examples are done in much less detail.

EXAMPLE 2. Let

$$\begin{aligned}f_\theta(x) &= e^{-\theta}\theta^x/x! , \qquad \theta>0 \quad x = 0, 1, 2, \cdots\\ &= 0 \qquad\qquad \text{otherwise.}\end{aligned}$$

Set $g(\theta) = \theta$ and assume $\psi(\theta)$ is a gamma density given by

$$\begin{aligned}\psi(\theta) &= (\beta^\alpha/\Gamma(\alpha))\theta^{\alpha-1}e^{-\theta\beta}, \qquad \theta>0, \quad \alpha>1, \quad \beta>0\\ &= 0 \qquad\qquad \text{otherwise.}\end{aligned}$$

Theorem 2 holds with $b^2 = \beta/(4(\alpha-1)) + \beta$. The posterior distribution of θ given $x_1, x_2, \cdots, x_n$ is given by a gamma density $\psi_n(\theta)$ with parameters $\alpha_n = \alpha + (x_1 + \cdots + x_n)$ and $\beta_n = \beta + n$. Since the posterior mean and posterior risk are given respectively by $\bar{g}_n = \alpha_n/\beta_n$ and

$$\rho_0(\psi_n) = (\bar{g}_n/\beta_n) + cn ,$$

it follows that the ad hoc procedure is to estimate θ by $\bar{g}_{N_2}$ where $N_2 =$ least integer $n \geqq 1$ such that $\beta_n \geqq \bar{g}_n^{\frac{1}{2}}/c^{\frac{1}{2}}$. Write

$$\rho_0(\psi_n) = 2c^{\frac{1}{2}}\bar{g}_n^{\frac{1}{2}} - \beta c + (\bar{g}_n^{\frac{1}{2}} - \beta_n c^{\frac{1}{2}})^2/\beta_n .$$

First note for $n \geqq 1$,

$$\bar{g}_{n-1} \leqq \bar{g}_n\left(\frac{\beta_n}{\beta_{n-1}}\right).$$

Next recalling Lemma 1, and the definition of the function h, we get

$$\begin{aligned} E\{\bar{g}_{N_2}^{\frac{1}{2}} - g^{\frac{1}{2}}(\theta)\} &= E\left\{\bar{g}_{N_2}^{\frac{1}{2}}\left(1 - \frac{1}{h(\alpha_{N_2})}\right)\right\} \\ &\leqq \tfrac{1}{4}E\left\{\frac{\bar{g}_{N_2}^{\frac{1}{2}}}{\alpha_{N_2}}\right\} \\ &\leqq \frac{c^{\frac{1}{2}}}{4}E\left\{\frac{\beta_{N_2}}{\alpha_{N_2}}\right\} \\ &\leqq \frac{c^{\frac{1}{2}}}{4}EE\{\theta^{-1} \mid \mathscr{F}_{N_2}\} = \frac{c^{\frac{1}{2}}}{4}\left(\frac{\beta}{\alpha - 1}\right). \end{aligned}$$

It is then easy to see that for $N_2 > 1$,

$$\begin{aligned} 0 \leqq c^{\frac{1}{2}}\beta_{N_2} - \bar{g}_{N_2}^{\frac{1}{2}} &\leqq (\bar{g}_{N_2-1}^{\frac{1}{2}} - \bar{g}_{N_2}^{\frac{1}{2}}) + c^{\frac{1}{2}} \\ &\leqq \frac{1}{2}\left(\frac{\beta_{N_2}}{\beta_{N_2} - 1}\right)c^{\frac{1}{2}} + c^{\frac{1}{2}} \\ &\leqq \left(3 + \frac{1}{\beta}\right)\frac{c^{\frac{1}{2}}}{2} \end{aligned}$$

whereas for $N_2 = 1$,

$$0 \leqq c^{\frac{1}{2}}\beta_{N_2} - \bar{g}_{N_2}^{\frac{1}{2}} \leqq (\beta + 1)c^{\frac{1}{2}} .$$

Therefore $$\rho(\psi, N_2) \leqq 2c^{\frac{1}{2}}E\{g^{\frac{1}{2}}(\theta)\} + \left(\frac{\beta}{2(\alpha - 1)} - \beta\right)c + \left(\frac{\beta}{\alpha - 1}\right)^{\frac{1}{2}}\frac{1}{h(\alpha - 1)}\left(\tfrac{3}{2} + \beta + \frac{1}{2\beta}\right)^2 c^{\frac{3}{2}} .$$

If we neglect the term of order $c^{\frac{3}{2}}$, the excess risk is then no more than $\{3\beta/(4(\alpha - 1))\}c$.

EXAMPLE 3. Consider a Bernoulli distribution with mean θ. Let $g(\theta) = \theta$ and assume

$$\begin{aligned} \psi(\theta) &= \{\Gamma(\alpha + \beta)/(\Gamma(\alpha)\Gamma(\beta))\}\theta^{\alpha-1}(1 - \theta)^{\beta-1} , & 0 < \theta < 1 , \quad \alpha > 1 , \quad \beta > 1 \\ &= 0 & \text{otherwise.} \end{aligned}$$

Here Theorem 2 holds with

$$b^2 = (\beta/(\alpha-1))(\alpha-\tfrac{1}{2})^2 + (\alpha/(\beta-1))(\beta-\tfrac{1}{2})^2 - 2(\alpha-\tfrac{1}{2})(\beta-\tfrac{1}{2}) .$$

Define

$$\begin{aligned}
\gamma &= \alpha + \beta , \\
\gamma_n &= \gamma + n ; \\
\alpha_n &= \alpha + (x_1 + \cdots + x_n) , \\
\beta_n &= \beta + n - (x_1 + \cdots + x_n) , \\
\bar{\sigma}_n &= E\{(\theta(1-\theta))^{\frac{1}{2}} \mid \mathscr{F}_n\} , \\
\bar{g}_n &= \alpha_n/\beta_n .
\end{aligned}$$

The posterior risk $\rho_0(\psi_n)$ is given by

$$\rho_0(\psi_n) = (\bar{\sigma}_n{}^2/(\gamma_n + 1))h^2(\alpha_n)h^2(\beta_n) + cn .$$

The stopping rule suggested by the heuristic argument enunciated above is technically difficult to work with. Instead consider $N_3 =$ least integer $n \geqq 1$ such that $\gamma_n + 1 \geqq \bar{\sigma}_n/c^{\frac{1}{2}}$. Write

$$\begin{aligned}
\rho_0(\psi_n) = 2c^{\frac{1}{2}}\bar{\sigma}_n &+ 2c^{\frac{1}{2}}\bar{\sigma}_n(h(\alpha_n)h(\beta_n) - 1) \\
&+ (\bar{\sigma}_n h(\alpha_n)h(\beta_n) - c^{\frac{1}{2}}\gamma_{n+1})^2/\gamma_{n+1} - (\gamma+1)c .
\end{aligned}$$

Since for $n \geqq 1$, $\bar{\sigma}_{n-1} \leqq \bar{\sigma}_n + \bar{\sigma}_n/(\gamma_n - 1)$ a.s. and

$$\begin{aligned}
E(\bar{\sigma}_{N_3}/\alpha_{N_3}) &\leqq c^{\frac{1}{2}}((\gamma+1)/(\alpha-1)) , \\
E(\bar{\sigma}_{N_3}/\beta_{N_3}) &\leqq c^{\frac{1}{2}}((\gamma+1)/(\beta-1)) ,
\end{aligned}$$

it can be shown, neglecting terms of order $c^{\frac{3}{2}}$, that the excess cost incurred is no more than

$$\{(5\beta+8)/4(\alpha-1) + (5\alpha+8)/4(\beta-1) + \tfrac{1}{2}\}c .$$

EXAMPLE 4. Suppose the X's have a normal density with mean M and precision R, both unknown. It is desired to estimate M. Let the prior ψ be such that the conditional distribution of M given $R = r$ is normal with mean μ and precision τr, $\tau > 0$ and the marginal distribution of R is given by

$$\frac{(\beta/2)^{\alpha/2}}{\Gamma(\alpha/2)} \cdot \exp(-R(\beta/2)) \cdot R^{\alpha/2-1} , \qquad R > 0 , \quad \alpha > 2 , \quad \beta > 0 .$$

It follows from Theorem 2 that with this prior the optimal risk is bounded below by $2c^{\frac{1}{2}}ER^{-\frac{1}{2}} - \tau \cdot c$.

Define $\mu_n = (\tau\mu + (x_1 + \cdots + x_n))/(\tau + n)$,

$$\begin{aligned}
\alpha_n &= \alpha + n , \\
\beta_n &= \beta + \textstyle\sum_{i=1}^n (x_i - \bar{x}_n)^2 + \tau n(\bar{x}_n - \mu)^2/(\tau + n) , \\
\bar{\sigma}_n{}^{-2} &= E\{R^{-1} \mid \mathscr{F}_n\} .
\end{aligned}$$

It follows that the posterior mean and posterior risk are respectively given by μ_n and

$$(\bar{\sigma}_n^2/(\tau + n)) + cn .$$

It can be shown that with the stopping rule N_4 defined by

$$N_4 = \text{least integer} \quad n \geqq 1 \quad \text{such} \quad n \geqq \bar{\sigma}_n/c^{\frac{1}{2}} ,$$

the excess cost neglecting terms of order $c^{\frac{3}{2}}$ is no more than $c/2$.

5. Asymptotic upper bound. In this section, a general stopping rule is proposed for the estimation of the parameter θ in the family of densities defined by

$$f(x \mid \theta) = \exp(\theta x - K(\theta))$$

with respect to some σ-finite measure. Under the assumption that the prior density on θ has compact support and possesses 5 continuous derivatives we will obtain an asymptotic expansion on the total Bayes risk using a proposed stopping variable. The expansion consists of a term of order $c^{\frac{1}{2}}$ comparable to the one obtained in Section 2 and an error term of order c. The remainder term will be of order $c^{\frac{5}{4}}$.

Let $\mathbf{x} = (x_1, \cdots, x_n)$ be a set of n independent observations on X. Let the posterior expectation of θ for given $\mathbf{x}$ be denoted $E_n \theta$. For any function $w(\theta)$ of θ, let $\hat{w}$, $\bar{w}$ denote respectively the maximum likelihood estimator and the Bayes estimator of $w(\theta)$ based on $\mathbf{x}$. Sometimes the index n will appear on the estimators in order to emphasize the dependence on the sample size. Let $\sigma^{-2}(\theta)$ denote the Fisher information number which is known to be nonzero and finite for the exponential family of densities. It can be shown that $K'(\hat{\theta}) = \bar{X}_n$, where $\bar{X}_n = (\sum_{i=1}^n X_i/n)$. The following result is due to Johnson (1970).

THEOREM 3. *Define $\phi = (\theta - \hat{\theta})\hat{\sigma}^{-1}$. Let M be a fixed positive integer. Suppose that the prior density of θ, $\psi(\theta)$, has $M + 1$ continuous derivatives in a neighborhood of θ_0 with $\psi(\theta_0) > 0$. If, for $k < M$, $E|\theta|^k < \infty$, then there exist continuous functions $\{\lambda_{i,j}\}$ defined on D, a constant C depending on θ_0, and for all $\mathbf{x}$ outside a P_{θ_0}-null set, an $N_{\mathbf{x}}$ depending on $\mathbf{x}$, such that for all $n > N_{\mathbf{x}}$,*

$$E_n(\phi^k) = \sum_{j=k}^M \hat{\lambda}_{k,j} n^{-j/2} + r_{k,n} n^{-(M+1)/2} ,$$

where $\sup_n |r_{i,n}| \leqq C$ *and* $\hat{\lambda}_{i,j} = 0$ *for odd* j.

Information about the $\lambda_{i,j}$ may be obtained from Johnson (1970). In fact if we omit the argument θ and use primes to denote derivatives, we have,

$$\lambda_{1,2} = \sigma(\sigma'/\sigma + \psi'/\psi)$$
$$\lambda_{2,2} = 1$$
$$\lambda_{2,4} = 2\sigma^2(\psi''/\psi) + 2(\sigma')^2 + \sigma\sigma'' + 4\sigma\sigma'(\psi'/\psi) .$$

A readaptation of the proof in Johnson (1970) yields the following.

COROLLARY 1. *If the support of ψ is a compact interval D_0 of D, then the constant C is a universal constant independent of θ_0.*

It follows that any continuous real-valued function defined on D is uniformly continuous and bounded on D_0. Some information about $N_{\mathbf{x}}$ may be obtained when ψ has compact support.

COROLLARY 2. *If the support of ψ is a compact interval D_0 of D, then the random variable $N_{\mathbf{x}}$ has moments of all orders.*

PROOF. From Johnson (1970), there exists a $\delta > 0$ such that

$$|\hat{\theta}_n - \theta| \leqq \delta$$

whenever $n > N_{\mathbf{x}}$. Since K' is strictly increasing, the inequality

$$\theta - \delta < \hat{\theta}_k < \theta + \delta$$

implies

$$K'(\theta - \delta) < K'(\hat{\theta}_k) < K'(\theta + \delta) .$$

By the mean value theorem, it follows that there exist θ_1, θ_2 with $\theta - \delta < \theta_2 < \theta < \theta_1 < \theta + \delta$ for which

$$K'(\theta + \delta) - K'(\theta) = K''(\theta_1)\delta$$

and

$$K'(\theta - \delta) - K'(\theta) = -K''(\theta_2)\delta .$$

Hence, since $K''(\cdot) < C$, it follows from Chernoff (1952),

$$\begin{aligned} P_\theta[N_{\mathbf{x}} > n] &\leqq P_\theta[|\hat{\theta}_k - \theta| > \delta \ \text{ for some } \ k > n] \\ &= P_\theta[|K'(\hat{\theta}_k) - K'(\theta)| > \delta C \ \text{ for some } \ k > n] \\ &= P_\theta[|S_k - k\mu| > k\delta c \ \text{ for some } \ k > n] \\ &\leqq \textstyle\sum_{k=n+1}^{\infty} P[|S_k - k\mu| > k\delta c] \\ &\leqq Le^{-tn} \qquad \text{for some } \quad t > 0 \quad \text{and constant} \quad L > 0 . \qquad \square \end{aligned}$$

THEOREM 4. *Let the support of ψ be a compact interval D_0 of D and assume that ψ possesses 5 continuous derivatives. Define the stopping variable N by:*

$$N = \textit{least integer} \quad n \geqq 1 \quad \textit{such that} \quad n \geqq \bar{\sigma}_n / c^{\frac{1}{2}} .$$

Then $\rho(\psi, N) = 2c^{\frac{1}{2}} E\sigma(\theta) + cEl(\theta) + O(c^{\frac{5}{4}})$ *where*

$$l(\theta) = (\lambda_{2,4} - \lambda_{1,2}^2 - 2\sigma'\lambda_{1,2} - \sigma''\sigma) .$$

PROOF. From Theorem 3, it follows that for $n > N_{\mathbf{x}}$,

$$\bar{\sigma} = \hat{\sigma} + (\hat{\sigma}'\hat{\sigma}\lambda_{1,2} + \hat{\sigma}''\hat{\sigma}^2/2)n^{-1} + V_{1,n} n^{-\frac{3}{2}} , \tag{5.1}$$

and the posterior variance of θ given $\mathbf{x}$ is given by

$$\begin{aligned} \operatorname{Var}_n \theta &= E_n(\theta - \hat{\theta})^2 - (\bar{\theta} - \hat{\theta})^2 \\ &= \hat{\sigma}^2 n^{-1} + (\hat{\lambda}_{2,4} - \hat{\lambda}_{1,2})\hat{\sigma}^2 n^{-2} + V_{2n} n^{-\frac{5}{2}} \\ &= \bar{\sigma}^2 n^{-1} + \hat{l}\hat{\sigma}^2 n^{-2} + V_{3,n} n^{-\frac{5}{2}} \end{aligned} \tag{5.2}$$

where $\sup_n |V_{i,n}| < \infty$, $i = 1, 2, 3$. Equation (5.1) follows upon expanding σ in a third order Taylor series about $\hat{\theta}$ and then taking posterior expectation. In view of Corollary 1, $|V_{i,N}|$ for $i = 1, 2, 3$ have moments of all orders.

Similarly, by expanding l in a one-order Taylor series about $\hat{\theta}$ it follows that for $n > N_{\mathbf{x}}$,

$$\text{(5.3)} \qquad \operatorname{Var}_n \theta = \bar{\sigma}^2 n^{-1} + \bar{l}\hat{\sigma}^2 n^{-2} + V_{4,n} n^{-\frac{5}{2}}$$

where $\sup_n |V_{4,n}| < \infty$ and $|V_{4,N}|$ has moments of all orders.

$$\begin{aligned} \text{Set} \quad V_n &= V_{4,n} + n^{\frac{3}{2}}(\bar{\sigma} - nc^{\frac{1}{2}})^2 + n^{\frac{5}{2}}\bar{l}(\hat{\sigma}^2 n^{-2} - c) \qquad &\text{for} \quad n > N_{\mathbf{x}} \\ &= n^{\frac{5}{2}}\{\rho_0(\psi_n) - 2c^{\frac{1}{2}}\bar{\sigma} - \bar{l}c\} \qquad &\text{for} \quad n \leqq N_{\mathbf{x}} . \end{aligned}$$

Consequently, the posterior risk given $\mathbf{x}$ can be expressed as

$$\begin{aligned} \rho_0(\psi_n) &= \operatorname{Var}_n \theta + nc \\ &= 2c^{\frac{1}{2}}\bar{\sigma} + \bar{l}c + V_n n^{-\frac{5}{2}} . \end{aligned}$$

Due to the martingale nature of $\bar{\sigma}_n$ and $\bar{l}_n$, it follows that

$$E\bar{\sigma}_N = E\sigma(\theta) \qquad \text{and} \qquad E\bar{l}_N = El(\theta) .$$

In order to prove the theorem we need only show that

$$EV_N^2 = O(1) .$$

It will then follow that

$$\begin{aligned} EV_N N^{-\frac{5}{2}} &\leqq (EV_N^2)^{\frac{1}{2}}(EN^{-5})^{\frac{1}{2}} \\ &\leqq (EV_N^2)^{\frac{1}{2}}(E\bar{\sigma}_N^{-5})^{\frac{1}{2}} c^{\frac{5}{4}} \\ &\leqq (EV_N^2)^{\frac{1}{2}}(E\sigma^{-5})^{\frac{1}{2}} c^{\frac{5}{4}} \\ &= O(c^{\frac{5}{4}}) . \end{aligned}$$

Now note that on $N \leqq N_{\mathbf{x}}$,

$$V_N^2 \leqq 5N_{\mathbf{x}}^2\{E_N \theta^2 + c\bar{\sigma}_N^2 + c^2\bar{l}_N^2\} .$$

By Schwarz's inequality and Corollary 2 it follows that

$$EV_N^2 I_{[N \leqq N_{\mathbf{x}}]} = O(1) .$$

We will show:

$$\text{(5.4)} \qquad EN^3(\bar{\sigma}_N - Nc^{\frac{1}{2}})^4 I_{[N > N_{\mathbf{x}}]} = O(1) ,$$

and

$$\text{(5.5)} \qquad EN^5 \bar{l}_N^2(\hat{\sigma}_N^2 N^{-2} - c)^2 I_{[N > N_{\mathbf{x}}]} = O(1) .$$

Both (5.4) and (5.5) will imply $EV_N^2 I_{[N > N_{\mathbf{x}}]} = O(1)$. To prove (5.4) note that from (5.1)

$$\text{(5.6)} \qquad \begin{aligned} 0 \leqq c^{\frac{1}{2}}N - \bar{\sigma}_N &= c^{\frac{1}{2}}(N - 1) - \bar{\sigma}_N + c^{\frac{1}{2}} \\ &\leqq (\bar{\sigma}_{N-1} - \bar{\sigma}_N) + c^{\frac{1}{2}} \\ &= (\hat{\sigma}_{N-1} - \hat{\sigma}_N) + W_{1,N} \cdot N^{-1} + W_{2,N} N^{-\frac{3}{2}} + c^{\frac{1}{2}} \end{aligned}$$

where $|W_{1,N}|$ and $|W_{2,N}|$ have moments of all orders. Since $\hat{\sigma}$ is a differentiable function of $\bar{X}_n$,

$$\hat{\sigma}_{N-1} - \hat{\sigma}_N = W_{3,N} \cdot (X_N - \bar{X}_{N-1})N^{-1}, \tag{5.7}$$

where $|W_{3,N}|$ has moments of all orders. It now follows from (5.6), (5.7) and the fact that $\sup_n (X_N{}^2 n^{-\frac{1}{2}}) < \infty$ that $N^{\frac{3}{2}}(\bar{\sigma}_N - Nc^{\frac{1}{2}})$ has moments of all orders on $[N > N_x]$. Hence (5.4) holds. In order to prove (5.5) note from (5.1), we get

$$0 \leqq c^{\frac{1}{2}}N \leqq \hat{\sigma}_N + W_{4,N} \cdot N^{-1} \tag{5.8}$$

where $|W_{4,N}|$ has moments of all orders. Hence

$$cN^2 - \hat{\sigma}_N{}^2 = W_{5,N}N^{-1}$$

where $|W_{5,N}|$ has moments of all orders. Now (5.5) follows. □

COROLLARY. *Let $g(\theta)$ be a monotone function of θ having 4 continuous derivatives. Then, for the rule $N^* =$ least integer $n \geqq 1$ such that $n \geqq E_n|\sigma g'|/c^{\frac{1}{2}}$,*

$$E(g - \bar{g}_{N^*})^2 + EN^*c = 2c^{\frac{1}{2}}E|\sigma g'| + El^*(\theta) \cdot c + O(c^{\frac{5}{4}})$$

where

$$l^* = (v_2 - v_1^2 - 2v_3k)k^{-2}$$

and

$$v_2 = (k^2\lambda_{2,4} + g'g''\sigma^3\lambda_{3,4}) + 3\{\tfrac{1}{4}(g'')^2 + \tfrac{1}{3}g'g'''\}\sigma^4$$
$$v_1 = (\lambda_{1,2}\sigma g' + \tfrac{1}{2}g''\sigma^2)$$
$$v_3 = (k'\sigma\lambda_{1,2} + k''\sigma^2/2)$$
$$k = |\sigma g'| \quad \text{and} \quad \lambda_{3,4} = 5\sigma' + 3\sigma\psi'/\psi\,.$$

We omit the details of the proof since they are quite similar to those of the theorem.

In order to compare with the examples of Section 2 we now compute the coefficient of the term of order c for some of the examples of Section 4.

EXAMPLE 1. (Exponential). The rule $N_1^* =$ least integer $n \geqq 1$ such that $n \geqq \bar{g}_n/c^{\frac{1}{2}}$ and the rule $N_1 =$ least integer $n \geqq 1$ such that $\alpha_n - 2 \geqq \bar{g}_n/c^{\frac{1}{2}}$ where $\bar{g}_n = E_n\theta^{-1}$ are asymptotically the same except for the constant $\alpha - 2$. We have

$$El^*(\theta) = 4\,.$$

The excess cost is 2 using N_1 and one stops sooner with N_1 than with N_1^*.

EXAMPLE 2. (Poisson). Here,

$$l^*(\theta) = \theta\left(\frac{\alpha - 1}{\theta} - \beta\right)^2 - \tfrac{5}{4}(\alpha - 1)\theta^{-1} + \frac{\theta^{-1}}{8} - \tfrac{3}{4}\beta$$

and

$$El^*(\theta) = \beta\left[\frac{1}{8(\alpha - 1)} - 1\right].$$

Hence as $c \to 0$, the excess risk has an order c term equal to

$$\beta\left[\frac{1}{8(\alpha-1)} - 1\right] - \left[-\beta - \frac{\beta}{4(\alpha-1)}\right] = \frac{3\beta}{8(\alpha-1)}.$$

Note

$$N_2^* = \text{least integer} \quad n \geqq 1 \quad \text{such that} \quad n \geqq \frac{\Gamma(\alpha_n + \frac{1}{2})}{\alpha_n^{\frac{1}{2}}\Gamma(\alpha_n)} \cdot \bar{\theta}_n^{\frac{1}{2}}/c^{\frac{1}{2}}.$$

The ad hoc rule N_2 = least integer $n \geqq 1$ such that $n + \beta \geqq \bar{\theta}_n^{\frac{1}{2}}c^{\frac{1}{2}}$ yields an excess risk with an order c term equal to $3\beta/(4(\alpha - 1))$.

Asymptotically the two rules are almost the same except for the constant β.

EXAMPLE 3. (Normal). Here N_4^* = least integer $n \geqq 1$, such that $n \geqq E_n R^{-\frac{1}{2}}/c^{\frac{1}{2}}$ where

$$E_n R^{-\frac{1}{2}} = (\beta_n/2)^{\frac{1}{2}} \frac{\Gamma((\alpha_n - 1)/2)}{\Gamma(\alpha_n/2)},$$

whereas

$$N_4 = \text{least integer} \quad n \geqq 1 \quad \text{such that} \quad n \geqq E_n^{\frac{1}{2}}(R^{-1})/c^{\frac{1}{2}}$$

where

$$E_n^{\frac{1}{2}} R^{-1} = (\beta_n/2)^{\frac{1}{2}} \left\{\frac{\Gamma(\alpha_n/2 - 1)}{\Gamma(\alpha_n/2)}\right\}^{\frac{1}{2}}.$$

In view of the definition of the h function the two rules are asymptotically the same. Keeping the precision fixed,

$$l(\theta) = \lambda_{2,4} - \lambda_{1,2}^2$$

and

$$El(\theta) = 0.$$

Hence the excess risk has a term of order c whose coefficient is τ. Compared to the rule of the example we do not necessarily do better.

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