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A NONLINEAR RENEWAL THEORY WITH APPLICATIONS TO SEQUENTIAL ANALYSIS I

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Renewal theory is developed for processes of the form $Z_n = S_n + \xi_n$, where S_n is the nth partial sum of a sequence $X_1, X_2, \ldots$ of independent identically distributed random variables with finite positive mean μ and ξ_n is independent of $X_{n+1}, X_{n+2}, \ldots$ and has sample paths which are slowly changing in an appropriate sense. Applications to sequential analysis are given.

1. Introduction. Let $X_1, X_2, \cdots$ be independent identically distributed random variables with positive mean μ and finite variance σ^2, and let $S_n = X_1 + \cdots + X_n$ $(n = 1, 2, \cdots)$. This paper and its companion are concerned with renewal theory and its applications to sequential analysis for processes of the form $Z_n = S_n + \xi_n$, where ξ_n is independent of $X_{n+1}, X_{n+2}, \cdots$ and has sample paths which are slowly changing in a sense to be made precise in what follows. For $b > 0$ define

$$T = T_b = \inf\{n : Z_n > b\} \tag{1}$$

and

$$\tau = \tau_b = \inf\{n : S_n > b\}. \tag{2}$$

Our first result concerns the limiting distribution of $Z_T - b$ as $b \to \infty$, which is shown to be the same as the well-known limiting distribution of $S_\tau - b$ (cf. Feller, 1966, page 354).

THEOREM 1. *Let $\frac{1}{2} < \alpha \leqq 1$, and assume*

$$b^{-\alpha}(T_b - \mu^{-1}b) \to_P 0 \qquad b \to \infty \tag{3}$$

and that for each $\eta > 0$ there exists a $\rho > 0$ and an integer n' such that for all $n \geqq n'$

$$P\{\max_{n \leqq j \leqq n+\rho n^\alpha} |\xi_j - \xi_n| \geqq \eta\} < \eta. \tag{4}$$

If X_1 does not have a lattice distribution, then for all $x \geqq 0$

$$\lim_{b\to\infty} P\{Z_T - b \leqq x\} = (ES_{\tau_0})^{-1} \int_{(0,x]} P\{S_{\tau_0} > y\}\, dy. \tag{5}$$

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Theorem 1 is proved in Section 2. In Sections 3 and 4 it is applied to give improved approximations to the error probability of certain statistical tests. In Section 4 we compare these approximations with some Monte Carlo results. Sections 3 and 4 may be read before Section 2.

In our companion paper, we shall give a Blackwell-type renewal theorem for quantities of the form

$$\sum_1^\infty P\{b < Z_n < b + h\}$$

and asymptotic expansions up to terms which vanish as $b \to \infty$ for $E(T_b)$. Additional applications will be included.

The decomposition $Z_n = S_n + \xi_n$ with ξ_n slowly changing was implicitly used by Pollak and Siegmund (1975) in the context of open-ended statistical tests. Before proceeding, we shall indicate informally by a simpler example the kinds of processes which motivate this decomposition.

Let $x_1, x_2, \cdots$ be independent identically distributed random variables with mean $\mu_0 = E(x_k)$ and finite variance $\sigma_0^2 = E(x_k - \mu_0)^2$. Let $s_n = \sum_1^n x_k$. For a function g which is positive at μ_0 and twice continuously differentiable in a neighborhood U of μ_0 let

$$Z_n = ng(s_n/n) . \tag{6}$$

Expanding (6) by Taylor's theorem we obtain for $s_n/n \in U$

$$Z_n = ng(\mu_0) + (s_n - n\mu_0)g'(\mu_0) + (s_n - n\mu_0)^2 g''(\zeta_n)/2n ,$$

where $|\zeta_n - \mu_0| \leqq |s_n/n - \mu_0|$. Let $X_k = g(\mu_0) + (x_k - \mu_0)g'(\mu_0)$ and $\xi_k = Z_k - S_k$. Then for $s_n/n \in U$ we have $\xi_n = (s_n - n\mu_0)^2 g''(\zeta_n)/2n$ and it is not difficult using the central limit theorem, strong law of large numbers, and Kolmogorov's inequality to show that this sequence ξ_n satisfies (4). The details are omitted here but are given in greater generality in the companion to this paper.

Special cases of (6) have appeared elsewhere. For example, Woodroofe (1976a) studies the stopping rule $T_c^* = \inf\{n : s_n > cn^\gamma\}(0 < \gamma < 1)$, which with $g(x) = (x^+)^{1/(1-\gamma)}$ and $b = c^{1/(1-\gamma)}$ becomes $T_c^* = \inf\{n : Z_n > b\}$, where Z_n is given by (6). Similarly the stopping rule $T = \inf\{n : (|s_n| + n/2)^2/2n \geqq c\}$ considered by Chernoff (1972, page 80) as well as those of some sequential tests obtained by Wald's method of weight functions may be rewritten in the form considered in this paper with Z_n defined by (6). A more complicated application arises in studying invariant sequential tests, where the log likelihood ratio may frequently be expressed asymptotically in a form similar to (6) (cf. Wijsman, 1971), although in these cases s_n may be a vector-valued random walk.

2. Proof of Theorem 1. Let α, η and ρ be as in the conditions of the theorem. Let $n_0 = \mu^{-1}b$, $n_1 = [n_0 - \rho n_0^\alpha/4]$, $n_2 = [n_0 + \rho n_0^\alpha/4]$, so that for all large b

$$n_1 + \rho n_1^\alpha > n_2 . \tag{7}$$

Let $A_b = \{\max_{1 \leqq n \leqq n_1} (S_n + \xi_n) < b - b^{\alpha/2}\}$. It follows from (3) that

$$P(\Omega - A_b) + P\{T > n_2\} \to 0 \tag{8}$$

as $b \to \infty$. Let $\mathscr{F}_n = \mathscr{B}((X_k, \xi_k),\ 1 \leqq k \leqq n)$ and for $-\infty < \beta < \infty$ define $t = t_b(\beta) = \inf\{n: S_n + \xi_{n_1} > b + \beta\}$. By the renewal theorem (cf. Feller (1966), page 354), for all b sufficiently large and all $x > 0$

$$(9) \qquad |P\{S_{t(\beta)} + \xi_{n_1} - (b + \beta) \leqq x \mid \mathscr{F}_{n_1}\} - G(x)| < \eta \qquad \text{on} \quad A_b$$

where $G(x)$ denotes the right-hand side of (5).

The remainder of the proof consists of showing that under the condition (4) for suitable β the conditional probability in (9) is essentially equal to the unconditional probability in (5). Let $x > 2\eta$. On the event $A_b{}^* = A_b \cap \{T < n_2, \max_{n_1 \leqq k \leqq n_2} |\xi_k - \xi_{n_1}| < \eta\}$

$$(10) \qquad \{S_T + \xi_T - b > x\} \subset \{t(\eta) = T,\ S_{t(\eta)} + \xi_{n_1} - (b + \eta) > x - 2\eta\}$$

and

$$(11) \qquad \{S_{t(-\eta)} + \xi_{n_1} - (b - \eta) > x + 2\eta\} \subset \{t(-\eta) = T,\ S_T + \xi_T - b > x\}\,.$$

For example, if $n_1 < k \leqq n_2$ and $T = k$, $S_k + \xi_k - b > x$, then on $A_b{}^*$ we have $S_k + \xi_{n_1} - (b + \eta) > x - 2\eta > 0$, so $t(\eta) \leqq k$. But if for some $n_1 < j < k$, $t(\eta) = j$ then $S_j + \xi_{n_1} > b + \eta$; and hence on $A_b{}^*$ we have $S_j + \xi_j > b$, so $T \leqq j$ in contradiction to the hypothesis $T = k$. This proves (10) and a similar argument proves (11). Hence by (11)

$$\begin{aligned} P\{S_T + \xi_T - b > x\} &\geqq P(A_b{}^*\{S_{t(-\eta)} + \xi_{n_1} - (b - \eta) > x + 2\eta\}) \\ &\geqq \int_{A_b} P\{S_{t(-\eta)} + \xi_{n_1} - (b - \eta) > x + 2\eta \mid \mathscr{F}_{n_1}\}\, dP \\ &\qquad - P\{T \geqq n_2\} - P\{\max_{n_1 < k \leqq n_2} |\xi_k - \xi_{n_1}| \geqq \eta\}\,. \end{aligned}$$

Letting first $b \to \infty$, then $\eta \to 0$ we obtain from (4), (8) and (9)

$$\liminf_{b \to \infty} P\{S_T + \xi_T - b > x\} \geqq 1 - G(x) \qquad x > 0\,.$$

A similar argument using (10) shows the reverse inequality, which completes the proof.

A similar argument gives the joint asymptotic behavior of T and $z_T - b$.

THEOREM 2. *In addition to conditions* (3) *and* (4) *of Theorem* 1, *assume that* $n^{-\frac{1}{2}} \xi_n \to_P 0$. *Let* $n_0 = \mu^{-1}b + t\sigma\mu^{-\frac{3}{2}}b^{\frac{1}{2}}$. *If* X_1 *does not have a lattice distribution, then for all* $-\infty < t < \infty$ *and* $x \geqq 0$ *as* $b \to \infty$

$$P\{T \leqq n_0,\ Z_T - b \leqq x\} \to \Phi(t)G(x)\,,$$

where Φ *is the standard normal distribution function and* G *denotes the right-hand side of* (5). *If, in fact,* $P\{n^{-\frac{1}{2}}\xi_n \to 0\} = 1$, *then* (3) *is a consequence and need not be assumed.*

In the case $\xi_n \equiv 0$, so $Z_n = S_n$, Theorem 2 is a special case of (for example) Theorem 1 of Siegmund (1975). A proof of Theorem 2 may be obtained by combining the argument of Siegmund (1975) with the preceding proof of Theorem 1 and hence is omitted. In some special cases Theorem 2 is equivalent to Theorem 4.3 of Woodroofe (1976a).

3. Application to the "optional stopping" problem. Suppose that under P_θ the random variables $x_1, x_2, \cdots$ are independent and normally distributed with mean θ and variance 1. Let $s_n = x_1 + \cdots + x_n$, and for $a, r > 0$ define

$$T = \inf\{n: |s_n| \geqq (2a(r+n))^{\frac{1}{2}}\}. \tag{12}$$

Interest in evaluating $P_0\{T \leqq m\}$ has been expressed by Robbins (1952), Armitage (1967), and others. As an application of Theorem 1 we shall prove that as $a \to \infty$

$$P_0\{T \leqq at\} \sim 2(a/\pi)^{\frac{1}{2}} e^{-a} \int_{(2/t)^{\frac{1}{2}}}^{\infty} \{\nu(x) \exp(-rx^2/2)/x\}\, dx, \tag{13}$$

where

$$\nu(x) = 2\exp(-2 \textstyle\sum_1^\infty \Phi(-xn^{\frac{1}{2}}/2)/n)/x^2 \tag{14}$$

and

$$\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2), \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(u)\, du. \tag{15}$$

The function ν in (14) and the integral on the right-hand side of (13) may be evaluated numerically.

Motivated by a different application than those authors cited above, Woodroofe (1976b) obtained a result which is equivalent to (13) by a different method. Although the applications in the next section give new results, the approximation (13) involves essentially the same ideas with fewer technical details and hence is given first.

To prove (13) define a probability Q by

$$Q(A) = \int_{-\infty}^{\infty} P_\theta(A) r^{\frac{1}{2}} \varphi(r^{\frac{1}{2}}\theta)\, d\theta. \tag{16}$$

If $Q^{(n)}(P_0^{(n)})$ denotes the restriction of $Q(P_0)$ to the space of $x_1, \cdots, x_n$, then it is easy to verify that

$$dQ^{(n)} = L_n\, dP_0^{(n)}, \tag{17}$$

where

$$L_n = \int_{-\infty}^{\infty} \exp(ys_n - ny^2/2) r^{\frac{1}{2}} \varphi(r^{\frac{1}{2}}y)\, dy = (r/(r+n))^{\frac{1}{2}} \exp(s_n^2/2(r+n)). \tag{18}$$

Then for each $m = 1, 2, \cdots$

$$P_0\{T \leqq m\} = \textstyle\sum_{n=1}^{m} \int_{\{T=n\}} (dP_0^{(n)}/dQ^{(n)})\, dQ = \int_{\{T \leqq m\}} (1/L_T)\, dQ. \tag{19}$$

From (16) and (18) we obtain

$$P_0\{T \leqq m\} = \int_{-\infty}^{\infty} [\int_{\{T \leqq m\}} (r+T)^{\frac{1}{2}} \exp(-s_T^2/2(r+T))\, dP_\theta] \varphi(r^{\frac{1}{2}}\theta)\, d\theta,$$

which for $m = at$ becomes

$$P_0\{T \leqq at\} = a^{\frac{1}{2}} e^{-a} \int_{-\infty}^{\infty} [\int_{\{T \leqq at\}} ((r+T)/a)^{\frac{1}{2}} \exp\{-(s_T^2/2(r+T) - a)\}\, dP_\theta] \times \varphi(r^{\frac{1}{2}}\theta)\, d\theta. \tag{20}$$

Denote the inner integral in (20) by $J_a(\theta)$. By Lemma 1 below for each $\theta \neq 0$

$$P_\theta\{\lim_{a\to\infty} a^{-1}T = 2/\theta^2\} = 1 \tag{21}$$

and $a^{-1}T$ is uniformly integrable. Hence for each $\theta \neq 0$ with $t < 2/\theta^2$,

$$\lim_{a\to\infty} J_a(\theta) = 0 . \tag{22}$$

By algebra

$$s_n^2/2(r+n) = \theta(s_n - n\theta/2) + (s_n - n\theta)^2/2(r+n) - r\theta(s_n - n\theta/2)/(r+n) .$$

If we set $S_n = \theta(s_n - n\theta/2)$ and $\xi_n = (s_n - n\theta)^2/2(r+n) - r\theta(s_n - n\theta/2)/(r+n)$, then from (21), Lemma 2 below, and the strong law of large numbers, it follows that conditions (3) and (4) hold with $\alpha = 1$. Hence by Theorem 1, (21), and the uniform integrability of $a^{-1}T$, for each θ with $t > 2/\theta^2$

$$\begin{aligned}\lim_{a\to\infty} 2^{-\frac{1}{2}}|\theta|J_a(\theta) &= (E_\theta S_{\tau_0})^{-1} \textstyle\int_0^\infty e^{-x} P_\theta\{S_{\tau_0} > x\}\, dx \\ &= (E_\theta \tau_0 E_\theta X_1)^{-1}(1 - E_\theta \exp(-S_{\tau_0})) .\end{aligned}$$

This last quantity may be evaluated according to results of Spitzer (cf. Feller, 1966, Chapter 18 or Chung, 1968, Chapter 8) to give

$$\lim_{a\to\infty} J_a(\theta) = 2^{\frac{1}{2}}\nu(|\theta|)/|\theta| , \qquad |\theta| > (2/t)^{\frac{1}{2}} , \tag{23}$$

where ν is defined in (14). Moreover, $J_a(\theta) \leqq (r/a + t)^{\frac{1}{2}}$ for all θ. Hence (13) follows from (20), (22), (23), and the dominated convergence theorem.

LEMMA 1. *For T defined by* (12), *for each $\theta \neq 0$* (21) *holds and $E_\theta(T) \sim 2a/\theta^2$ as $a \to \infty$. Hence $\{a^{-1}T, a \geqq 1\}$ is uniformly integrable.*

PROOF. The proof involves standard arguments. For more general results along these lines cf. Siegmund (1967).

LEMMA 2. *For arbitrary $\eta > 0$ there exists $\rho > 0$ such that*

$$P_\theta\{\max_{n\leqq k\leqq n+\rho n} |(s_k - k\theta)^2/(r+k) - (s_n - n\theta)^2/(r+n)| > \eta\} < \eta . \tag{24}$$

PROOF. It suffices to consider the case $\theta = 0$. By simple algebra for $n \leqq k \leqq n + \rho n$

$$|s_k^2/(r+k) - s_n^2/(r+n)| \leqq \rho s_n^2/n + (s_k - s_n)^2/n + 2|s_n(s_k - s_n)|/n . \tag{25}$$

Let $\varepsilon, \lambda > 0$ and define $A = \{|s_n|/n^{\frac{1}{2}} < \lambda\}$, $B = \{\max_{n\leqq k\leqq n+\rho n} |s_k - s_n| < \varepsilon n^{\frac{1}{2}}\}$. On $A \cap B$ the right-hand side of (25) is majorized by $\rho\lambda^2 + \varepsilon^2 + 2\lambda\varepsilon$, which for any fixed λ can be made less than η by taking ρ and ε sufficiently small. Then by Kolmogorov's and Chebyshev's inequalities

$$\begin{aligned}&P_0\{\max_{n\leqq k\leqq n+\rho n} |s_k^2/(r+k) - s_n^2/(r+n)| > \eta\} \\ &\qquad \leqq P_0(A^c) + P_0(B^c) \leqq \lambda^{-2} + \rho\varepsilon^{-2} .\end{aligned}$$

Thus the lemma follows, if we first fix λ so large that $\lambda^{-2} < \eta/2$ and then choose ρ and ε so small that $\rho\lambda^2 + \varepsilon^2 + 2\lambda\varepsilon < \eta$ and $\rho\varepsilon^{-2} < \eta/2$.

4. Application of Theorem 1 to open-ended tests and confidence sequences. Let $x_1, x_2 \cdots$ be independent and identically distributed random variables with probability distribution of the form

$$P_\theta\{x_k \in dx\} = \exp(\theta x - \Psi(\theta))H(dx) \tag{26}$$

for some $\theta \in J$, J an open interval containing 0. Assume also that H is a nonlattice probability distribution which gives positive measure to both $(-\infty, 0)$ and $(0, \infty)$, and without loss of generality that

$$\Psi(0) = \Psi'(0) = 0\,. \tag{27}$$

It is easily verified that $\Psi'(\theta) = E_\theta(x_1)$ and $\Psi''(\theta) = \operatorname{Var}_\theta(x_1) > 0$, so that by (27)

$$\operatorname{sgn} E_\theta(x_1) = \operatorname{sgn}\theta\,. \tag{28}$$

For a probability distribution G on J, in analogy with the preceding section define a probability Q by

$$Q(A) = \int_J P_\theta(A)G(d\theta)\,. \tag{29}$$

With $Q^{(n)}$ and $P_0^{(n)}$ also analogously defined as the restrictions of Q and P_0 to the space of $x_1, \cdots, x_n$, it follows from (26), (27), and (29) that (17) holds but now with

$$L_n = \int_J \exp(ys_n - n\Psi(y))G(dy)\,. \tag{30}$$

For any stopping rule T and $m = 1, 2, \cdots$ we have (19), which as $m \to \infty$ becomes

$$P_0\{T < \infty\} = \int_{\{T<\infty\}} (1/L_T)\, dQ\,. \tag{31}$$

For the particular choice

$$T = \inf\{n : L_n > c\} \tag{32}$$

we obtain at once from (31)

$$P\{L_n > c \text{ for some } n \geqq 1\} \leqq c^{-1}\,. \tag{33}$$

This inequality forms the basis of the theory of open-ended tests and confidence sequences as given, for example, by Robbins (1970). For these purpose it is desirable to have a more accurate approximation to the left-hand side of (33). To study the effect of truncation on these tests it would be useful to have an approximation to $P_0\{T < m\}$ for finite values of m. Interest in such approximations has also been expressed by Armitage (1967).

By (29) we may rewrite (19) as

$$P_0\{T < m\} = e^{-a} \int_J \left(\int_{\{T<m\}} \exp[-(\log L_T - a)]\, dP_\theta\right)G(d\theta)\,, \tag{34}$$

where we have set $c = e^a$. Equation (34) also holds for $m = \infty$. If we set

$$Z_n = \log L_n = \theta s_n - n\Psi(\theta) + \log \int_J \exp[(y-\theta)s_n - n(\Psi(y) - \Psi(\theta))]G(dy)$$

and make the identifications $S_n = \theta s_n - n\Psi(\theta)$, $\xi_n = Z_n - S_n$, then with the assumption that G' exists in a neighborhood of $\theta \neq 0$ and is positive and continuous at θ, it may be shown that Theorem 1 applies under P_θ to the stopping rule T defined by (33). The technical aspects of this proof have been developed by Pollak and Siegmund (1975) in a related context and will not be repeated here. If G' exists and is positive and continuous on an open set of G-measure 1, then by arguing as in the preceding section we obtain for each $0 < t \leqq \infty$

$$P_0\{T < at\} \sim e^{-a} \int_{\{\theta:\theta\Psi'(\theta)-\Psi(\theta)>1/t\}} \nu^*(\theta)G(d\theta)\,, \tag{35}$$

where

$$\nu^*(\theta) = (\theta\Psi'(\theta) - \Psi(\theta))^{-1} \exp[-\textstyle\sum_1^\infty n^{-1}(P_\theta\{S_n \leqq 0\} + \int_{(S_n>0)} \exp(-S_n)\, dP_\theta)]\,. \tag{36}$$

For the important special case of normal random variables, $\Psi(\theta) = \theta^2/2$ and ν^* is just the function ν defined in (14).

A systematic numerical comparison of a number of sequential tests using the results of this paper will be presented elsewhere. As a brief indication of the accuracy of our approximations we consider the special case of normal random variables, for which $\Psi(\theta) = \theta^2/2$, and take $G(dy) = \varphi(y)dy$, so that T defined by (33) becomes

$$T = \inf\{n\colon |s_n| \geqq [(n+1)(\log(n+1) + 2a)]^{\frac{1}{2}}\}\,.$$

Table 1 compares the right hand of (35) with Monte Carlo estimates of $P_0\{T < at\}$ obtained by averaging 400 Q-realizations of $I_{\{T<at\}}(1/L_T)$. By (19) this is an unbiased estimator of $P_0\{T < at\}$. It is easy to see by the preceding methods that it has variance of order e^{-2a} as $a \to \infty$ and hence is much more accurate than direct simulation, which yields a variance of order e^{-a}. Moreover, on the average it requires fewer random variables $x_1, x_2, \cdots$ to obtain a single Q-realization of $I_{\{T<at\}}(1/L_T)$ than a single P_0-realization of $I_{\{T<at\}}$. A more systematic discussion of the use of such "importance sampling" in sequential analysis is given by Siegmund (1976).

The $\pm$ value appearing in the Monte Carlo column of Table 1 is one standard error. The values of a correspond to values of $c = e^a$ of 10, 20, and 100. In all cases the theoretical approximation appears to be slightly too large, but the largest discrepancy obtained is only $5\frac{1}{2}\%$ of the Monte Carlo value.

TABLE 1

	Asymptotic theory from (35)	Monte Carlo
$a = 2.3$, $t = 25$	.0461	.0454 $\pm$ .0017
$t = 50$	.0517	.0500 $\pm$ .0017
$a = 3$, $t = 25$	.0229	.0217 $\pm$ .0009
$t = 50$	.0257	.0246 $\pm$ .0008
$a = 4.6$, $t = 25$	.00462	.00450 $\pm$ .00018
$t = 50$	.00518	.00512 $\pm$ .00018

For $t = \infty$ the integral on the right-hand side of (35) is .67. However, effective estimation of $P_0\{T < \infty\}$ by Monte Carlo methods is rather difficult. Obviously direct simulation is impossible. Generating Q-realizations of $I_{\{T<\infty\}}(1/L_T)$ requires very large sample sizes, for it may be shown that $E_\theta(T) \sim 2P_0(T = \infty) \log \theta^{-1}/\theta^2$ as $\theta \to 0$ (e.g., Robbins and Siegmund, 1973) and hence $\int T\, dQ = \int_{-\infty}^{\infty} E_\theta(T)G(d\theta) = \infty$. A different Monte Carlo estimator proposed by Darling and Robbins has been shown to have no moments of order greater than 1 (cf. Berk, 1969).

An alternative essentially Monte Carlo approximation to $P_0\{T < \infty\}$ may be obtained as follows. In (34) with $m = \infty$, $e^a P_0\{T < \infty\}$ is expressed as an integral of

$$E_\theta\{\exp[-(\log L_T - a)]\} = E_\theta(c/L_T) \tag{37}$$

with respect to a distribution on θ. Except for $\theta = 0$ the expectation (37) may be estimated by Monte Carlo methods. The first column of Table 2 contains such Monte Carlo estimates based on 400 repetition experiments for $a = 3$ and various values of θ. For comparison the limiting value as $a \to \infty$ of the expectation (37), namely $\nu(\theta)$, is given in the second column of Table 2. The agreement between asymptotic theory and Monte Carlo is sufficiently close that one expects the asymptotic approximation to $P_0\{T < \infty\}$ to be quite good. It may also be shown that

$$\lim_{\theta\to 0} E_\theta(c/L_T) = \int_{\{T<\infty\}} (c/L_T)\, dP_0 + P_0\{T = \infty\},$$

which is about .97 for $a = 3$. Inclusion of any value between .95 and 1 as the $\theta = 0$ entry for the first column of Table 2 and integration of this column by Simpson's rule yields the estimate .65 for $e^a P_0\{T < \infty\}$, in good agreement with the asymptotic theory.

TABLE 2
$E_\theta(c/L_T)$

θ	Monte Carlo for $a = 3$ ($c = 20$)	Asymptotic value $a \to \infty$
0	—	1
.2	.86	.89
.4	.76	.79
.6	.69	.71
.8	.61	.63
1.0	.57	.56
1.2	.51	.50
1.4	.46	.45
1.6	.45	.40
1.8	.40	.36
2.0	.37	.32
2.2	.34	.29
2.4	.33	.26

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