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APPROXIMATE BEHAVIOR OF THE POSTERIOR DISTRIBUTION FOR A LARGE OBSERVATION

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Let X be a real valued random variable with a family of possible distributions belonging to a one parameteter exponential family with the natural parameter $\theta \in (\underline{\theta}, +\infty)$. Let g be a prior probability density for θ with unbounded support. Under some additional assumptions it is shown that for large values of x the posterior distribution of θ given $X=x$ is approximately normally distributed about its mode. If δ_g denotes the Bayes estimator for squared error loss of some function $\gamma(\theta)$ against g then the rate at which $\delta_g(x)$ approaches infinity as x approaches infinity is found. The rate is shown to depend on the behavior of the prior density $g(\theta)$ for large values of θ.

1. Introduction and summary. Let X be a normal random variable with mean θ and variance one. Consider the problem of estimating θ with squared error loss. For the prior distribution which is normal with mean 0 and variance $\sigma^2 > 0$ the corresponding Bayes estimator is $(\sigma^2/\sigma^2 + 1)X$. Since a sufficiently large observed value of x indicates that the prior was inappropriate it might be hoped that the difference between the posterior mean and x is negligible for large x; however, this difference $x - (\sigma^2/\sigma^2 + 1)x = x/(\sigma^2 + 1)$ approaches infinity as x aproaches infinity. From the modern subjective Bayesian point of view this property of the estimator for large values of x is often considered unfortunate. See Lindley (1968) for further discussion.

If δ_g denotes the Bayes estimate for θ with respect to the prior density g then Dawid (1973) showed that if g is approximately uniform for large θ then $\lim_{x\to\infty} (\delta_g(x) - x) = 0$. This proved a conjecture of Lindley (1968). In addition Dawid showed that the posterior distribution of θ for large x is approximately normal with mean x and variance 1. For example if $g(\theta) = 1/\theta^\alpha$ for θ sufficiently large where $\alpha \geqq 1$ then Dawid's results follow.

This paper studies the general problem of the relationship between the tail behavior of the density g and the behavior of the posterior distribution for large values of x. Under certain regularity conditions it is shown that the posterior distribution is approximately normally distributed about its mode. In addition the rate at which a Bayes estimator approaches infinity as x approaches infinity is found for a large class of estimators. The results are developed in the context of the one parameter exponential family.

More specifically, let μ be a σ-finite measure defined on the Borel sets of the

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real line. Assume that $\mu\{(c, \infty)\} > 0$ for each real number c. Assume that $\int \exp(\theta x)\, d\mu$ exists for $\theta \in (\underline{\theta}, +\infty)$ where $-\infty \leqq \underline{\theta} < +\infty$. (For notational convenience let $\exp(\beta(\theta)) = \int \exp(\theta x)\, d\mu$.) Let X be a real valued random variable whose family of possible densities with respect to μ is given by $\exp(\theta x - \beta(\theta))$ for $\theta \in (\underline{\theta}, \infty)$. For this family of distributions belonging to the one parameter exponential family let $g(\theta) = \exp(-\eta(\theta))$ denote a prior density (possibly improper) for θ. If $\lambda(\theta) = \beta(\theta) + \eta(\theta)$ then the posterior distribution of θ is given by

$$[\exp\{\theta x - \lambda(\theta)\}]/\textstyle\int_{\underline{\theta}}^{\infty} \exp\{tx - \lambda(t)\}\, dt\,. \tag{1}$$

Assume that $\lambda'(\theta)$ exists and is strictly increasing for θ sufficiently large. If λ is well behaved elsewhere, then for x sufficiently large, the mode of the posterior distribution will be given by $L(x) = [\lambda']^{-1}(x)$. This suggests that the posterior distribution is centered about $L(x)$ for large x.

In Section 2, Theorem 1 gives conditions under which the posterior distribution (1) is approximately normal with mean $L(x)$ and variance $1/\lambda''(L(x))$ for large x. Let $\delta(x; g)$ denote the Bayes estimator for θ against the prior density g with respect to squared error loss; i.e., $\delta(x; g)$ is the conditional expectation of θ given x. Theorem 1 also yields the approximate behavior of $\delta(x; g)$ for large x, that is

$$\lim_{x\to\infty} [\lambda''(L(x))]^{\frac{1}{2}}[\delta(x; g) - L(x)] = 0\,. \tag{2}$$

Since one of the assumptions of Theorem 1 is that $\lambda''(t)$ stays away from zero for large t it follows that

$$\lim_{x\to\infty} [\delta(x; g) - L(x)] = 0 \tag{3}$$

and equation (2) gives a bound for the rate at which the limit (3) approaches zero. Section 2 concludes with several examples illustrating Theorem 1.

In Section 3 the rates of growth of Bayes estimators as x approaches infinity is studied more generally. For example Theorem 3, under different assumptions than those in Theorem 1, states that

$$\lim_{x\to\infty} [\delta(x; g) - L(x)] = 0\,.$$

Theorem 2 gives conditions under which the cruder approximation

$$\lim_{x\to\infty} [\delta(x; g)/L(x)] = 1 \tag{4}$$

holds. Let $\delta(x; \gamma, g)$ denote the Bayes estimator of the function $\gamma(\theta)$ against the prior density g with squared error loss. In Theorem 4 the cruder approximation (4) is extended to $\delta(x; \gamma, g)$ for a wider class of γ's. That is, conditions are given so that

$$\lim_{x\to\infty} \delta(x; \gamma, g)/\gamma(L(x)) = 1\,.$$

2. Asymptotic normality. In this section the relationship between the tail behavior of the prior density g and the behavior of the posterior distribution for large values of x is studied. Actually the natural correspondence is between the posterior and the function $\lambda(\theta)$ where $\lambda(\theta) = \beta(\theta) + \eta(\theta)$. (See Section 1 for the definitions of $\beta(\theta)$ and $\eta(\theta)$.) Hence the theorems will be stated in terms of $\lambda(\theta)$. In what follows it will always be assumed that λ statisfies the following two regularity conditions.

(A) There exists a real number N_1 such that for $x \geqq N_1$

$$p_\lambda(\theta \mid x) = \exp\{\theta x - \lambda(\theta)\}/\int_{\underline{\theta}}^{+\infty} \exp\{tx - \lambda(t)\}\, dt$$

is a probability density over $(\underline{\theta}, \infty)$ whose finite expected value is denoted by

$$h_\lambda(x) = \int_{\underline{\theta}}^{\infty} \theta p_\lambda(\theta \mid x)\, d\theta \,.$$

Note that $h_\lambda(x) = \delta(x; g)$, the conditional expectation of θ given x.

(B) There exists an $N_2 > \max\{0, N_1\}$ such that for $w \geqq N_2$, $\lambda'(w)$ is continuous and is strictly increasing on $[N_2, +\infty)$ with $\lim_{w\to\infty} \lambda'(w) = \infty$.

Now for fixed, x, sufficiently large, condition B implies that $p_\lambda(\theta \mid x)$ as a function of θ on $[N_2, +\infty)$ is strictly increasing when $\lambda'(\theta) < x$ and strictly decreasing when $\lambda'(\theta) > x$. This suggests that under suitable regularity conditions $[\lambda']^{-1}(x)$ will be the mode of $p_\lambda(\theta \mid x)$ and that most of the probability under $p_\lambda(\theta \mid x)$ will be concentrated in a "small" interval about $[\lambda']^{-1}(x)$. This is the key idea behind the proofs that follow.

For notational convenience consider the reparameterization $y = [\lambda']^{-1}(x)$ or $x = \lambda'(y)$. Note that consideration of $p_\lambda(\theta \mid x)$ for large x is equivalent to consideration of

$$p_\lambda^*(\theta \mid y) = \exp\{\theta\lambda'(y) - \lambda(\theta)\}/\int_{\underline{\theta}}^{\infty} \exp\{t\lambda'(y) - \lambda(t)\}\, dt$$

for large y. In what follows it will be necessary to consider for large y the integration of $\exp\{m_y(\theta)\}$ with respect to θ where

$$m_y(\theta) = \theta\lambda'(y) - \lambda(\theta) \,.$$

Note that on the interval $[N_2, \infty)$ $m_y(\theta)$ is a concave function with $m_y'(y) = 0$. In order to facilitate the integration, $m(\theta) = m_y(\theta)$ will be approximated by straight lines that lie below $m(\theta)$ near y and lie above $m(\theta)$ away from y. In particular if $b \neq y$ then the natural line to use is the line going through $(b, m(b))$ and $(y, m(y))$; that is, in the (θ, ϕ) plane, the line is given by

$$\phi = \{[m(y) - m(b)]/(y - b)\}(\theta - b) + m(b) \,.$$

For technical reasons, however, the approximating line is chosen to be the line passing through $(b, m(b))$ with slope $\gamma = [\lambda'(y) - \lambda'([y + b]/2)]/2$. For the case $b < y$ it is now shown that γ lies between zero and $[m(y) - m(b)]/(y - b)$. Note that

$$\begin{aligned}
[m(y) - m(b)]/(y - b) &= [y\lambda'(y) - b\lambda'(y)]/(y - b) - [\lambda(y) - \lambda(b)]/(y - b) \\
&= \lambda'(y) - [\lambda(y) - \lambda([y + b]/2)]/(y - b) \\
&\quad - [\lambda([y + b]/2) - \lambda(b)]/(y - b) \\
&\geqq \lambda'(y) - \lambda'(y)/2 - \lambda'([y + b]/2)/2 \\
&> 0
\end{aligned}$$

by the mean value theorem and the fact that λ' is strictly increasing.

The above discussion yields the following lemma, which will be used repeatedly in what follows.

LEMMA 1. *Let $\lambda(\theta)$ satisfy condition* B. *Let $y > N_2$ be fixed and let*

$$m(\theta) = m_y(\theta) = \theta\lambda'(y) - \lambda(\theta) .$$

Then

$$m(\theta) \leqq m(b) + \gamma(\theta - b) \quad \text{for} \quad N_2 < \theta < b < y \quad \text{and for} \quad y < b < \theta$$

and

$$m(\theta) \geqq m(b) + \gamma(\theta - b) \quad \text{for} \quad N_2 < b < \theta < y \quad \text{and for} \quad y < \theta < b$$

where

$$\gamma = [\lambda'(y) - \lambda'([y + b]/2)]/2 .$$

The following theorem gives sufficient conditions for the approximate normality of the posterior distribution for a large observation.

THEOREM 1. *Let λ satisfy conditions* A *and* B. *Suppose $\lambda''(y)$ exists for every real number $y > \underline{\theta}$ and $\inf_{y>N_2} \lambda''(y) > 0$. For $y > N_2$ let*

$$p_\lambda^*(\theta \mid y) = \exp\{\theta\lambda'(y) - \lambda(\theta)\}/\int_{\underline{\theta}}^{\infty} \exp\{t\lambda'(y) - \lambda(t)\}\, dt$$

denote the probability density of a random variable θ_y^* on $(\underline{\theta}, \infty)$. Let f_y denote the probability density of the random variable $Z_y = [\lambda''(y)]^{\frac{1}{2}}(\theta_y^* - y)$. If for each $c > 0$

$$\lim_{y\to\infty} \{\lambda''(y + \alpha(y)[\lambda''(y)]^{-\frac{1}{2}})/\lambda''(y)\} = 1 \tag{5}$$

where $|\alpha(y)| < c$ for all y then

$$\lim_{y\to\infty} f_y(z) = (2\pi)^{-\frac{1}{2}} \exp(-z^2/2) \tag{6}$$

for each real number z. In addition,

$$\lim_{y\to\infty} [\lambda''(y)]^{\frac{1}{2}}[\int_{\underline{\theta}}^{\infty} \theta p_\lambda^*(\theta/y)\, d\theta - y] = 0 . \tag{7}$$

PROOF. Without loss of generality assume $N_2 = 0$. If the density of θ_y^* is given by $p_\lambda^*(\theta \mid y)$, then the density of Z_y is given by

$$f_y(z) = \frac{\exp\{z\lambda'(y)[\lambda''(y)]^{-\frac{1}{2}} + y\lambda'(y) - \lambda(z[\lambda''(y)]^{-\frac{1}{2}} + y)\}}{[\lambda''(y)]^{\frac{1}{2}} \int_{\underline{\theta}}^{\infty} \exp\{t\lambda'(y) - \lambda(t)\}\, dt} \tag{8}$$

for $z \in ([\lambda''(y)]^{\frac{1}{2}}(\underline{\theta} - y), \infty)$. By making the change of variable $r = [\lambda''(y)]^{\frac{1}{2}}(t - y)$ in the integral in the denominator and cancelling and adding factors depending only on y one finds that

$$f_y(z) = \frac{\exp\{z\lambda'(y)[\lambda''(y)]^{-\frac{1}{2}} + \lambda(y) - \lambda(z[\lambda''(y)]^{-\frac{1}{2}} + y)\}}{\int \exp\{r\lambda'(y)[\lambda''(y)]^{-\frac{1}{2}} + \lambda(y) - \lambda(r[\lambda''(y)]^{-\frac{1}{2}} + y\}\, dr} . \tag{9}$$

Upon expanding $\lambda(z[\lambda''(y)]^{-\frac{1}{2}} + y)$ in a Taylor series about $z = 0$ one finds that the numerator of $f_y(z)$ is given by

$$\exp\{-\lambda''(y + \xi(z, y)[\lambda''(y)]^{-\frac{1}{2}})[\lambda''(y)]^{-1}(z^2/2)\} \tag{10}$$

where $\xi(z, y)$ is between 0 and z. Note that for a fixed z expression (10) converges to $\exp\{-z^2/2\}$ as y approaches infinity by assumption (5). From this it follows that (6) will be proved if it can be shown that $\lim_{K\to\infty} \liminf_{y\to\infty} \int_{-K}^{K} f_y(z)\, dz = 1$ or alternatively

$$\lim_{K\to\infty} \limsup_{y\to\infty} \int_{\{|z|>K\}} f_y(z)\, dz = 0\,. \tag{11}$$

To this end note that by an appropriate change of variable in (8) one finds that

$$\int_K^\infty f_y(z)\, dz < \frac{\int_{b(y)}^\infty \exp\{t\lambda'(y) - \lambda(t)\}\, dt}{\int_y^{b(y)} \exp\{t\lambda'(y) - \lambda(t)\}\, dt} \tag{12}$$

where $b(y) = y + K[\lambda''(y)]^{-\frac{1}{2}}$. The next step is to use Lemma 1 to find suitable approximations for the above integrals. Note that for $t > b(y) > y$ it follows from the lemma that

$$\begin{aligned} t\lambda'(y) - \lambda(t) &\leqq m(b(y)) - [\lambda'(y + K[\lambda''(y)]^{-\frac{1}{2}}/2) - \lambda'(y)](t - b(y))/2 \\ &= m(b(y)) - K[\lambda''(y)]^{-\frac{1}{2}}\lambda''(\xi(y))(t - b(y))/4 \end{aligned}$$

where by the mean value theorem $y < \xi(y) < b(y)$. Hence by assumption (5) it follows that for $t > b(y)$

$$t\lambda'(y) - \lambda(t) \leqq m(b(y)) - (K/8)[\lambda''(y)]^{\frac{1}{2}}(t - b(y)) \tag{13}$$

when y is sufficiently large.

Turning now to the denominator of (12) one has from the lemma for the case $y < t < b(y)$ that

$$\begin{aligned} t\lambda'(y) - \lambda(t) &\geqq m(b(y)) - [\lambda'(y + K[\lambda''(y]^{-\frac{1}{2}}/2) - \lambda'(y)](t - b(y))/2 \\ &= m(b(y)) - K[\lambda''(y)]^{-\frac{1}{2}}\lambda''(\xi(y))(t - b(y))/4 \end{aligned}$$

where by the mean value theorem $y < \xi(y) < b(y)$. Hence by assumption (5) it follows that for $y < t < b(y)$.

$$t\lambda'(y) - \lambda(t) \geqq m(b(y)) - (K/8)[\lambda''(y)]^{\frac{1}{2}}(t - b(y)) \tag{14}$$

when y is sufficiently large.

Upon substituting equations (13) and (14) into equation (12) it follows that

$$\begin{aligned} \int_K^\infty f_y(z)\, dz &\leqq \frac{\int_{b(y)}^\infty \exp\{-(K/8)[\lambda''(y)]^{\frac{1}{2}}t\}\, dt}{\int_y^{b(y)} \exp\{-(K/8)[\lambda''(y)]^{\frac{1}{2}}t\}\, dt} \\ &\leqq [\exp\{K^2/8\} - 1]^{-1}\,. \end{aligned}$$

From this it follows that $\lim_{K\to\infty} \limsup_{y\to\infty} \int_K^\infty f_y(z)\, dz = 0$.

The expression for $\int_{\{z<-K\}} f_y(z)\, dz$ may be split up into two parts corresponding to the ranges of integration $0 < t < y - K[\lambda''(y)]^{-\frac{1}{2}}$ and $\underline{\theta} < t < 0$. The first of these is handled in a way entirely analogous to the above argument where the integral in the denominator is over the interval $(y - K[\lambda''(y)]^{-\frac{1}{2}}, y)$; the second of these will be considered now.

Hence to complete the proof of (11) and hence of (6) it remains to show that

$$\limsup_{y\to\infty} \frac{\int_{\underline{\theta}}^{0} \exp\{t\lambda'(y) - \lambda(t)\}\,dt}{\int_{\underline{\theta}}^{\infty} \exp\{t\lambda'(y) - \lambda(t)\}\,dt} = 0\,. \tag{15}$$

Note that for $y > a > 0$, where $\lambda'(a) > 0$ it follows that

$$\begin{aligned}\int_{\underline{\theta}}^{0} \exp\{t\lambda'(y) - \lambda(t)\}\,dt &= \int_{\underline{\theta}}^{0} \exp\{t\lambda'(y) - t\lambda'(a) + t\lambda'(a) - \lambda(t)\}\,dt \\ &\leqq \int_{\underline{\theta}}^{0} \exp\{t\lambda'(a) - \lambda(t)\}\,dt \\ &< \infty\end{aligned}$$

by assumption A. On the other hand for $y > 0$ the denominator of (15) is greater than $y\exp\{-\lambda(0)\}$ and hence (15) follows.

Assertion (7) of the theorem follows from $\lim_{K\to\infty} \limsup_{y\to\infty} \int_{\{|z|>K\}} |z| f_y(z)\,dz = 0$. This can be verified in a fashion analogous to the proof of (11) and will be omitted. This completes the proof.

The assumptions on λ in Theorem 1 loosely interpreted imply that for large y, λ is a reasonably behaved convex function which grows at least as fast as y^2 but not too fast. For example, if for large y, λ is a polynomial of degree $\geqq 2$ whose leading coefficient is positive or if $\lambda(y) = e^y$ for large y, then the assumptions are satisfied. The assumptions are not satisfied when for large y, $\lambda(y) = e^{y^2}$. It is not known to the authors if the results of Theorem 1 hold when $\lambda(y) = e^{y^2}$ for large y.

In Theorem 1 let $x = \lambda'(y)$, $L(x) = (\lambda')^{-1}(x)$ and $h_\lambda(x) = \int_{-\infty}^{+\infty} \theta p_\lambda^*(\theta\,|\,y = L(x))\,d\theta$, Then equation (7) can be rewritten as

$$\lim_{x\to\infty} [\lambda''(L(x))]^{\frac{1}{2}}(h_\lambda(x) - L(x)) = 0\,.$$

This section is concluded with several examples illustrating Theorem 1.

EXAMPLES. Let X be Normal $(\theta, 1)$. Then $\lambda(\theta) = \theta^2/2 + \eta(\theta)$ when $g(\theta) = \exp(-\eta(\theta))$ is the prior density. In what follows, $\eta(\theta)$ will only be specified for larger values of θ. We assume, however, it is defined in such a way that the regularity assumptions of Theorem 1 are satisfied.

Suppose $g(\theta) = 1/\theta^C = \exp(-C\ln\theta)$ for $\theta > K > 0$. Then for large $x, L(x) = (x + (x^2 - 4C)^{\frac{1}{2}})/2$ which is approximately x and $\lambda''(L(x))$ is approximately one. Hence for large x the posterior distribution of θ given x is approximately Normal $(x, 1)$. For $C > 0$ this result was given by Dawid (1973). For a similar multivariate result see Hill (1974).

Suppose now $g(\theta) = \exp(-C\theta)$ for $\theta > K > 0$ where $C > 0$. Then for large x, $L(x) = x - C$ and the posterior distribution of θ given x is approximately Normal $(x - C, 1)$.

Suppose now $g(\theta) = \exp(-C\theta^{\frac{3}{2}})$ for $\theta > K > 0$ where $C > 0$. In this case it follows for large x that $L(x) = x - 3C[4x + 9C^2/4]^{\frac{1}{2}}/4 + 9C^2/8$ and the posterior distribution of θ given x is approximately Normal $(L(x), 1/\lambda''(L(x)))$. Note that the variance $1/\lambda''(L(x))$ is approximately one for large x.

Now suppose $g(\theta) = \exp[(-C\theta^{\alpha}/\alpha) + (\theta^2/2)]$ for $\theta > K > 0$ where $C > 0$ and $\alpha > 2$. We have for large x that $L(x) = (x/C)^{1/(\alpha-1)}$ and the posterior distribution of θ given x is approximately Normal $((x/C)^{1/(\alpha-1)}, 1/(\alpha - 1)C(x/C)^{(\alpha-2)/(\alpha-1)})$. Finally

$$\lim_{x\to\infty} x^{(\alpha-2)/\alpha(\alpha-1)}[\delta(x; g) - L(x)] = 0 .$$

Suppose now $g(\theta) = \exp\{-e^{C\theta} + \theta^2/2\}$ for $\theta > K > 0$ where $C > 0$. A simple calculation shows that for large x, $L(x) = (1/C) \ln (x/C)$ and

$$\lim_{x\to\infty} (Cx)^{\frac{1}{2}}[\delta(x; g) - (1/C) \ln (x/C)] = 0 .$$

Now consider the case where X is Poisson (λ) where $\lambda \in (0, +\infty)$. Letting $\theta = \ln \lambda$ one finds that $f_\theta(x) = \exp\{-\exp(\theta) + x\theta\}(1/x!)$ for $x = 0, 1, 2, \cdots$ where $\theta \in (-\infty, +\infty)$. Letting $g(\theta) = \exp(-\eta(\theta))$ denote the prior density it follows that $\lambda(\theta) = \eta(\theta) - e^{\theta}$. If the prior density is uniform over the real numbers, then the posterior is identical to the posterior in the previous example when $C = 1$.

3. Rates of growth for estimators of θ and $\gamma(\theta)$. In this section the single question of the rate of growth of Bayes estimators as $x\to\infty$ is considered. Note that no assumptions about $\lambda''(y)$ are needed for these theorems.

The first theorem gives sufficient conditions for the ratio $h_\lambda(x)/(\lambda')^{-1}(x)$ to approach one as x approaches infinity.

THEOREM 2. *Let λ satisfy conditions* A *and* B *and let $L(x) = (\lambda')^{-1}(x)$ for $x \geqq \lambda'(N_2)$. If for all $\delta > 0$ there is a $K = K(\delta)$ and an $N_3 = N_3(\delta)$ such that $\lambda'[(1 + \delta)y] - \lambda'(y) \geqq K(\delta) > 0$ for $y \geqq N_3$, then* $\lim_{x\to\infty} h_\lambda(x)/L(x) = 1$.

PROOF. Let $x = \lambda'(y)$ and consider

$$\frac{h_\lambda[\lambda'(y)]}{y} = \frac{\int_{\underline{\theta}}^{\infty} \theta \exp[\theta\lambda'(y) - \lambda(\theta)]\, d\theta}{y \int_{\underline{\theta}}^{\infty} \exp[\theta\lambda'(y) - \lambda(\theta)]\, d\theta} \quad \text{for} \quad y > N_2 . \tag{16}$$

(Note that as in Section 1 $\underline{\theta} \geqq -\infty$.) Let $\varepsilon > 0$ be given and rewrite (16) as a sum of four terms as follows where $m_y(\theta) = m(\theta) = \theta\lambda'(y) - \lambda(\theta)$ and $D(y) = \int_{\underline{\theta}}^{\infty} \exp[m(\theta)]\, d\theta$. (As in Theorem 1 we take $N_2 = 0$.)

$$\begin{array}{lll} & \text{(i)} & \int_{\underline{\theta}}^{0} \theta \exp[m(\theta)]\, d\theta/yD(y) + \\ (17) & \text{(ii)} & \int_{0}^{y(1-\varepsilon)} \theta \exp[m(\theta)]\, d\theta/yD(y) + \\ & \text{(iii)} & \int_{y(1-\varepsilon)}^{y(1+\varepsilon)} \theta \exp[m(\theta)]\, d\theta/yD(y) + \\ & \text{(iv)} & \int_{y(1+\varepsilon)}^{\infty} \theta \exp[m(\theta)]\, d\theta/yD(y) . \end{array}$$

First it is shown that parts (i), (ii), and (iv) go to zero as $y \to \infty$. For part (iv) let $b = (1 + \varepsilon)y$ so $y < b < \theta$. By hypothesis $\gamma = \gamma(y, \varepsilon/2) = \frac{1}{2}[\lambda'(y) - \lambda'(y + \varepsilon/2)] \leqq K < 0$ for $y \geqq N_3$. Hence by using Lemma 1 it follows that

$$\begin{aligned}\text{(iv)} &\leqq \frac{\int_{b}^{\infty} \theta \exp[m(b) + \gamma(\theta - b)]\, d\theta}{y \int_{b-[\varepsilon b/(1+\varepsilon)]}^{b} \exp[m(b) + \gamma(\theta - b)]\, d\theta} \\ &= \frac{-b\gamma e^{b\gamma} + e^{b\gamma}}{y\gamma e^{b\gamma}[1 - e^{-b\gamma\varepsilon/(1+\varepsilon)}]} \to 0\end{aligned}$$

as $b \to \infty$ since $\gamma \leqq K < 0$ for $y \geqq N_3$. For part (ii) let $b = y(1-\varepsilon)$ so $\gamma \geqq K > 0$ for $y \geqq N_3$.

$$\begin{aligned}
\text{(ii)} &\leqq \frac{\int_0^b \theta \exp[m(b) + \gamma(\theta - b)]\, d\theta}{y \int_b^{b+\varepsilon b/(1-\varepsilon)} \exp[m(b) + \gamma(\theta - b)]\, d\theta} \\
&= \frac{\gamma b e^{b\gamma} - e^{b\gamma} + 1}{y\gamma e^{\gamma b} e^{\varepsilon b\gamma/(1-\varepsilon)} - y\gamma e^{b\gamma}} \\
&\leqq (\gamma b + 1)/[y\gamma e^{\varepsilon b\gamma/(1-\varepsilon)} - y\gamma] \to 0 \\
&\qquad \text{as} \quad b \to \infty \quad \text{since} \quad \gamma \geqq K > 0 \quad \text{for} \quad y \geqq N_3 .
\end{aligned}$$

For part (i) let $\varepsilon_1 = \lambda'(a+1) - \lambda'(a)$. From the fact that $|\theta \exp[\varepsilon_1 \theta]| \leqq B_1 < \infty$ for $\theta \leqq 0$ and assumption A, it follows that for $y \geqq a + 1$ where a is such that $\lambda'(a) > 0$,

$$\begin{aligned}
\int_{\underline{\theta}}^0 |\theta| \exp[\theta\lambda'(y) - \lambda(\theta)]\, d\theta &= \int_{\underline{\theta}}^0 |\theta| \exp[\theta\lambda'(y) - \theta\lambda'(a) + \theta\lambda'(a) - \lambda(\theta)]\, d\theta \\
&\leqq B_1 \int_{\underline{\theta}}^0 \exp[\theta\lambda'(a) - \lambda(\theta)]\, d\theta < \infty .
\end{aligned}$$

In the proof of Theorem 1 it was shown that $D(y) \to \infty$ as $y \to \infty$ so (i) $\to 0$ as $y \to \infty$. The above arguments can also be used to show that (i), (ii) and (iv) $\to 0$ as $y \to \infty$ without the term θ/y so it follows that

$$\lim_{y\to\infty} \frac{\int_{y(1-\varepsilon)}^{y(1+\varepsilon)} \exp[\theta\lambda'(y) - \lambda(\theta)]\, d\theta}{D(y)} = 1 .$$

Hence

$$1 - \varepsilon \leqq \liminf \text{(iii)} \leqq \limsup \text{(iii)} \leqq 1 + \varepsilon .$$

But since ε is arbitrary the proof is complete.

In Section 2 sufficient conditions for $\lim_{x\to\infty} [h_\lambda(x) - [\lambda']^{-1}(x)] = 0$ were given as a consequence of proving asymptotic normality. Now it is shown that $\lim_{x\to\infty} [h_\lambda(x) - [\lambda']^{-1}(x)] = 0$ under a different set of conditions. These new conditions put no upper bound on how fast λ can grow, which was the case in Theorem 1.

THEOREM 3. *Let λ satisfy conditions* A *and* B *and let $L(x) = (\lambda')^{-1}(x)$ for $x \geqq \lambda'(N_2)$. If for every $\varepsilon > 0$ $\lim_{y\to\infty}[\lambda'(y) - \lambda'(y-\varepsilon)] = \infty$, then*

$$\lim_{x\to\infty} [h_\lambda(x) - L(x)] = 0 .$$

PROOF. The proof is similar to the proof of Theorem 2 except that in this case the integral in the numerator of $h_\lambda(\lambda'(y))$ is considered over the intervals $(\underline{\theta}, 0]$, $(0, y-\varepsilon]$, $(y-\varepsilon, y+\varepsilon]$, $(y+\varepsilon, +\infty)$ separately where $\varepsilon > 0$.

The conclusion of Theorem 3 is the desired conclusion yet Theorem 2 has its value. In the first place the hypotheses on λ' are weaker for Theorem 2. Secondly from a practical standpoint Theorem 2 is easier to use, as the subsequent remarks illustrate. Let

$$h_\lambda(x) = \int_{-\infty}^{\infty} \theta \exp[-(\theta - x)^2/2] g(\theta)\, d\theta / \int_{-\infty}^{\infty} \exp[-(\theta - x)^2/2] g(\theta)\, d\theta$$

where $g(\theta) = e^{-|\theta|^3/3+\theta^2/2}$ is the prior distribution. In this case $\lambda(\theta) = +|\theta|^3/3$ so $\lambda'(\theta) = \theta^2$ for $\theta > 0$. Hence $L(y) = y^{\frac{1}{2}}$ so $\lim_{x\to\infty} h_\lambda(x)x^{-\frac{1}{2}} = 1$ and $\lim_{x\to\infty} h_\lambda(x) - x^{\frac{1}{2}} = 0$. This example was made easy by the fact that the inverse function for $\lambda'(\theta)$ was easy to calculate. If in fact for $\theta > 0$, $g(\theta) = \exp[-|\theta|^3/3 + \theta^2/2 + \log\theta]$ then $\lambda'(\theta) = \theta^2 + 1/\theta$; the calculation of $L(y)$ becomes much more difficult. Fortunately the smaller order terms of $\lambda(\theta)$ can often be ignored when applying Theorem 2, as the following lemma shows.

LEMMA 2. *Let λ satisfy the conditions of Theorem* 2 *and assume that* $\lim_{\theta\to\infty} \lambda'(\theta)/\theta^\beta = K$ *for some* $\beta > 0$ *and* $0 < K < \infty$. *Then* $\lim_{x\to\infty} h_\lambda(x)/(x/K)^{1/\beta} = 1$.

PROOF. Using the fact that $\lim_{\theta\to\infty} \lambda'(\theta)/\theta^\beta = K$ it is possible to show that $\lim_{\theta\to\infty} [\lambda']^{-1}(\theta)/(\theta/K)^{1/\beta} = 1$. Hence $\lim_{x\to\infty} h_\lambda(x)/(x/K)^{1/\beta} = \lim_{x\to\infty} [h_\lambda(x)/[\lambda']^{-1}(x)]$ $\lim_{x\to\infty} [\lambda']^{-1}(x)/(x/K)^{1/\beta} = 1$.

REMARK. The condition $\lim_{\theta\to\infty} \lambda'(\theta)/\theta^\beta = K$ in Lemma 2 can be replaced by the condition $\lim_{\theta\to\infty} \lambda'(\theta)/e^{p(\theta)} = K$ where $p(\theta)$ is a polynomial in θ and the result still holds. However, $\lambda'(\theta) = \log\theta$ and $f'(\theta) = \log[\theta\log\theta]$ shows that it does not always follow that $[\lambda']^{-1}(\theta)/[f']^{-1}(\theta) \to 1$ when $\lambda'(\theta)/f'(\theta) \to 1$.

REMARK. Lemma 2 can be used to simplify the problem of finding the asymptotic growth of $h_\lambda(x)$. In particular if $\lambda'(\theta) = \theta^2 - 3\theta + \log\theta$ it is nontrivial to find $[\lambda']^{-1}(\theta)$. However, it is sufficient to find the inverse of $f'(\theta) = \theta^2$. Unfortunately the smaller order terms cannot be ignored when using Theorem 3.

Theorems 2 and 3 yield the growth rate of the expected value of θ given x for large x. Next the growth rate of the expected value of $\gamma(\theta)$ given x for large x for some function γ is considered. A partial result is given in the following theorem.

THEOREM 4. *Let $\lambda(\theta)$ be a real valued function satisfying the conditions of Theorem* 2. *Assume that $\gamma(\theta)$ is a polynomial in θ such that* $\int_{\underline{\theta}}^\infty \gamma(\theta)\exp[\theta x - \lambda(\theta)]\,d\theta$ *exists for large x. Then*

$$\lim_{x\to\infty} \frac{1}{\gamma[L(x)]} \frac{\int_{\underline{\theta}}^\infty \gamma(\theta)\exp[\theta x - \lambda(\theta)]\,d\theta}{\int_{\underline{\theta}}^\infty \exp[\theta x - \lambda(\theta)]\,d\theta} = 1\,.$$

PROOF. The proof is similar to that of Theorem 2.

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