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A CANONICAL FORM FOR THE GENERAL LINEAR MODEL

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The collection of variance-covariance matrices for any linear model may be represented, without altering relationships among linear unbiased estimators, as a compact convex subset of nonnegative definite matrices throughout the relative interior of which all matrices are positive definite.

Forms of the general linear model in which the variance-covariance matrix of the observations is singular have received considerable attention in the literature on linear estimation (Albert (1972), Olsen et al. (1976), Zyskind (1967) and Zyskind and Martin (1969)). The purpose of this paper is to present a canonical form for linear estimation in which the collection of variance-covariance matrices is a compact convex subset of nonnegative definite matrices having in its relative interior only positive-definite matrices.

A linear model for the random n-vector Y is customarily specified by

$$E(Y) = X\beta\,, \quad \beta \in R^q\,, \tag{1}$$

and

$$\operatorname{Cov}(Y \mid V) = V\,, \quad V \in \Theta\,, \tag{2}$$

where X is an $n \times q$ matrix of known constants and Θ is a specific set of $n \times n$ symmetric nonnegative definite matrices. The set Θ takes different forms in different applications, including: $\sigma^2 I$; $\sigma^2 V$, V, known; $\sigma_1^2 V_1 + \cdots + \sigma_k^2 V_k$, each V_i known; and other more complicated forms, such as those for autocorrelated disturbances. Zyskind (1967) discusses some, and refers to other, models in which the covariance matrix is singular.

Linear unbiased estimators $L'Y$ of linearly estimable functions $p'\beta$ ($p \in R(X')$, where $R(X')$ is the linear subspace spanned by the columns of X'), and relations among them ("as good as," "better than," "admissible," as defined in Olsen et al. (1976)) are determined by X and the risk function

$$\operatorname{Var}(L'Y \mid V) = L'VL\,, \quad V \in \Theta\,. \tag{3}$$

Relations among unbiased linear estimators are unchanged if

$$\phi_L(V) = L'(V + XX')L = \operatorname{Var}(L'Y \mid V) + p'p\,, \quad V \in \Theta\,, \tag{4}$$

is used as the risk function in place of (3).

Let

$$\Theta_X = \{V + XX' \colon V \in \Theta\}\,. \tag{5}$$

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Let Ω be the intersection of the minimal closed convex cone containing Θ_X and the set $\{A\colon A \text{ is } n \times n \text{ symmetric and } \operatorname{tr}(A) = 1\}$. Then Ω is a compact convex subset of nonnegative definite matrices. Relations among unbiased linear estimators relative to (3) are equivalent to these relative to $\phi_L(S)$, $S \in \Omega$.

Let

$$\mathscr{N} = \bigcap_{S \in \Omega} N(S)\,. \tag{6}$$

where $N(S)$ is the linear subspace of vectors $\{x\colon Sx = 0\}$. Then $\mathscr{N}$ is a linear subspace. Let F be an $n \times t$ matrix whose columns form a basis for the subspace of vectors orthogonal to $\mathscr{N}$. Then every n-vector L has the unique representation

$$L = F\alpha + l \tag{7}$$

with $l \in \mathscr{N}$. Furthermore,

$$L'SL = \alpha' F' S F \alpha \tag{8}$$

for all $S \in \Omega$ so that, if $L'Y$ is an unbiased linear estimator of $p'\beta$, $\alpha'F'Y$ is an unbiased linear estimator of $p'\beta$ and is as good as $L'Y$. That is, $\{\alpha'F'Y\colon X'F\alpha = p,\ \alpha \in R^t\}$ is an essentially complete class of unbiased linear estimators of $p'\beta$ (this result corresponds to Lemma 3.1 in Olsen et al. (1976)). Thus attention may be restricted to the linear model for $Z = F'Y$ specified by the $t \times q$ matrix $F'X$ and the risk function

$$\phi_\alpha(M) = \alpha' M \alpha\,, \qquad M \in \Omega_F = \{F'SF\colon S \in \Omega\}\,. \tag{9}$$

It may be easily seen that

$$\mathscr{N}_F = \bigcap_{M \in \Omega_F} N(M) = \{0\}\,. \tag{10}$$

As an example illustrating the preceding development, suppose $\Theta = \{\sigma^2 A\colon \sigma^2 \geqq 0\}$ where A is a known $n \times n$ nonnegative definite matrix. Then $\Theta_X = \{\sigma^2 A + XX'\colon \sigma^2 \geqq 0\}$ and $\Omega = \{\rho_1 A + \rho_2 XX'\colon \rho_1 \geqq 0,\ \rho_2 \geqq 0,\ \rho_1 \operatorname{tr}(V) + \rho_2 \operatorname{tr}(XX') = 1\}$. By noting that $N(\rho_1 A + \rho_2 XX') = N(A) \cap N(XX')$ if ρ_1 and ρ_2 are both positive it may be seen that $\mathscr{N} = N(A) \cap N(XX') = N(A + XX')$. Thus the columns of F form a basis of $R(A + XX')$, and every matrix $F'(\rho_1 A + \rho_2 XX')F$ in Ω_F with ρ_1 and ρ_2 positive (that is, in the relative interior of Ω_F) is positive definite. To show in general that Ω_F contains only positive definite matrices in its relative interior the following two lemmas may be used.

LEMMA 1. *If Ω is a convex set of nonnegative definite $n \times n$ symmetric matrices, then there exists a matrix $M_* \in \Omega$ such that*

$$N(M_*) = \bigcap_{M \in \Omega} N(M)\,. \tag{11}$$

PROOF. Let $M_* \in \Omega$ be such that $\operatorname{rank}(M_*) = \max\{\operatorname{rank}(M)\colon M \in \Omega\}$. Suppose M_* does not satisfy (11). Then there exists a matrix $M_0 \in \Omega$ such that $N(M_*)$ contains a vector not in $N(M_0)$. Since Ω is convex, $M_1 = \frac{1}{2}M_0 + \frac{1}{2}M_* \in \Omega$. But $N(M_1) = N(M_0) \cap N(M_*)$ which is a proper subset of $N(M_*)$ and hence has lesser dimension than $N(M_*)$. This implies that the dimension of $R(M_1)$, and hence the rank of M_1, is greater than the rank of M_*, contrary to the fact that M_* has maximal rank in Ω. □

LEMMA 2. *If Ω is a convex subset of nonnegative definite $n \times n$ symmetric matrices and $m = \max\{\text{rank}(M) : M \in \Omega\}$, then every matrix in the relative interior of Ω has rank m.*

PROOF. Let M_0 be in the relative interior of Ω. Let M_* be a matrix in Ω having maximal rank. Then there exists a matrix M_1 "on the other side of M_0 from M_*" on the line through M_0 and M_*: that is, there exists a $\lambda > 1$ such that $M_1 = \lambda M_0 + (1 - \lambda)M_* \in \Omega$. Thus $M_0 = 1/\lambda M_1 + (\lambda - 1)/\lambda M_*$. Therefore the rank of M_0 is equal to the rank of M_*. □

By Lemma 1 and (10) there exists an $M_* \in \Omega_F$ which is positive definite; and by Lemma 2 every matrix in the relative interior of Ω_F is positive definite. Thus linear unbiased estimation with squared-error loss in the model (1), (2) is equivalent to linear unbiased estimation in the model specified by the matrix $F'X$ and the risk function ϕ_α.

Other representations equivalent with respect to linear unbiased estimation to (1), (2) are clearly possible. One such has Ω_F represented as

$$M = \textstyle\sum_{i=1}^{k} \gamma_i V_i \,, \quad \gamma \in \Gamma \,, \tag{12}$$

where $\gamma = (\gamma_1, \cdots, \gamma_k)'$, each V_i is a known $t \times t$ symmetric nonnegative definite matrix and Γ is a compact convex subset of R^k.

The net result of the preceding results is that Θ in the model (1), (2) may, with no loss of generality for purposes of unbiased linear estimation, be regarded as a compact convex set of nonnegative definite matrices, throughout the relative interior of which all matrices are nonsingular.

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