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A GLIVENKO-CANTELLI THEOREM AND STRONG LAWS OF LARGE NUMBERS FOR FUNCTIONS OF ORDER STATISTICS

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A strengthened version of the Glivenko-Cantelli theorem for the uniform empirical distribution function is proved. The strengthened Glivenko-Cantelli theorem is used to establish strong laws of large numbers for linear functions of order statistics.

1. Introduction. Let $\xi_1, \xi_2, \cdots$ be a sequence of independent and identically distributed uniform (0, 1) rv's with distribution function I ($I(t) = t$) on [0, 1], and let Γ_n denote the empirical df of the first n variables in the sequence:

$$\Gamma_n(t) = \frac{1}{n} \sum_{i=1}^n 1_{[0,t]}(\xi_i) \qquad \text{for} \quad 0 \leqq t \leqq 1 .$$

(Here 1_A denotes the function which is 1 for $x \in A$ and 0 otherwise.) The Glivenko-Cantelli theorem says that with probability one (w.p. 1)

$$\rho(\Gamma_n, I) \equiv \sup_{0 \leqq t \leqq 1} |\Gamma_n(t) - t| \to 0 \qquad \text{as} \quad n \to \infty .$$

Let h be a continuous nondecreasing function on [0, 1] and define

$$\rho_h(\Gamma_n, I) \equiv \rho(\Gamma_n/h, I/h) = \sup_{0 \leqq t \leqq 1} |\Gamma_n(t) - t|/h(t) .$$

In Theorem 1 below we prove that $\int_0^1 (1/h)\, dI < \infty$ is both necessary and sufficient for $\rho_h(\Gamma_n, I) \to 0$ w.p. 1 as $n \to \infty$. (Here and in the following $\int \cdot\, dI$ denotes integration with respect to Lebesgue measure.) Theorem 1 may be viewed as a (special but important) strong law of large numbers for Banach-space valued random elements; this connection will be discussed in more detail in Section 2.

Our primary motivation for this type of Glivenko-Cantelli theorem is as a tool for proving strong laws of large numbers for linear functions of order statistics. Let $\mathscr{G}$ denote the set of left continuous functions on (0, 1) that are of bounded variation on $(\theta, 1-\theta)$ for all $\theta > 0$; fix $g \in \mathscr{G}$. Let $c_{n1}, \cdots, c_{nn}$ for $n \geqq 1$ be known constants. In Section 3 we prove strong laws of large numbers for

$$T_n = n^{-1} \sum_{i=1}^n c_{ni}\, g(\xi_{ni}) \tag{1}$$

where $0 \leqq \xi_{n1} \leqq \cdots \leqq \xi_{nn} \leqq 1$ denote the order statistics of the first n ξ's (i.i.d. uniform (0, 1) rv's). Note that if $g = t(F^{-1})$ for some df F, then T_n has the same

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distribution as $S_n = n^{-1} \sum_1^n c_{ni} t(X_{ni})$ where $X_{n1} \leqq \cdots \leqq X_{nn}$ are the order statistics of a sample of size n from F.

Most limit theory for statistics like T_n has focused on central limit theorems, and several good theorems giving conditions under which T_n has a limiting normal distribution have been established: see Shorack [4], and Stigler [5] in particular. However, I know of no general strong law for T_n. Our approach to strong laws for T_n in Section 3 parallels Shorack's [4] approach to central limit theorems for T_n. By use of the strengthened Glivenko-Cantelli theorem proved in Section 2, we prove a general strong law of large numbers for T_n under weak conditions on the c_{ni}'s and the df F. Our sufficient conditions are also close to being necessary; this is illustrated by several examples.

2. A strengthened Glivenko-Cantelli theorem.

DEFINITION. Let $\mathscr{H}(\nearrow)$ denote the set of all nonnegative, nondecreasing, continuous functions on $[0, 1]$ for which $\int_0^1 (1/h)\, dI < \infty$. Let $\mathscr{H}$ denote the set of all h such that $h(t) = h(1-t) = \bar{h}(t)$ for $0 \leqq t \leqq \frac{1}{2}$ and some $\bar{h}$ in $\mathscr{H}(\nearrow)$.

THEOREM 1. (A) *If* $h \in \mathscr{H}(\nearrow)$ *then*

$$\lim_{n\to\infty} \rho_h(\Gamma_n, I) = 0 \quad \text{w.p. } 1. \tag{2}$$

(B) *Furthermore, if* h *is increasing on* $[0, 1]$ *and* $\int_0^1 (1/h)\, dI = +\infty$ *then*

$$\limsup_{n\to\infty} \rho_h(\Gamma_n, 0) = +\infty \quad \text{w.p. } 1.$$

PROOF. We begin with (B). Suppose that h is increasing and $\int_0^1 (1/h)\, dI = +\infty$. Since

$$\rho_I(\Gamma_n, 0) = \sup_{0\leqq t\leqq 1} (\Gamma_n(t)/t) \geqq \Gamma_n(\xi_{n1})/\xi_{n1} = (n\xi_{n1})^{-1},$$

(i) of Theorem 1 of Robbins and Siegmund [3] implies that $\limsup_{n\to\infty} \rho_I(\Gamma_n, 0) = +\infty$ w.p. 1. Hence, if $h \leqq aI$ for some $a > 0$,

$$\limsup_{n\to\infty} \rho_h(\Gamma_n, 0) \geqq a^{-1} \limsup_{n\to\infty} \rho_I(\Gamma_n, 0) = +\infty \quad \text{w.p. } 1,$$

and therefore we may assume that $h \geqq aI$ for some $a > 0$. (If $h \leqq aI$ for some $a > 0$ does not hold, then, for every $a > 0$, $h(t) > at$ for some $t \in [0, 1]$; by monotonicity of h this implies that $h \geqq aI$ for some $a > 0$.) Let $Q_i(t) = 1_{[0,t]}(\xi_i)$ so that

$$\Gamma_n(t) = n^{-1} \textstyle\sum_1^n Q_i(t).$$

Let $M > 0$ and define events B_n and D_n by

$$B_n \equiv \{\rho_h(\Gamma_n, 0) > M\} = \{\rho_h(\textstyle\sum_1^n Q_i, 0) > nM\}$$

and

$$D_n \equiv \{\rho_h(Q_n, 0) > nM\}.$$

Then, since $\sum_1^n Q_i \geqq Q_n$, $\rho_h(\sum_1^n Q_i, 0) \geqq \rho_h(Q_n, 0)$, and hence $\{D_n \text{ i.o.}\} \subset \{B_n \text{ i.o.}\}$. But the events D_n are independent and therefore, by Borel-Cantelli,

$$P(D_n \text{ i.o.}) = 0 \quad \text{or} \quad 1 \quad \text{according as} \quad \textstyle\sum_1^\infty P(D_n) < \infty \quad \text{or} \quad = \infty. \tag{3}$$

Now we compute $P(D_n)$. Since the Q_i's are independent and identically distributed we may drop the subscript n; hence for n sufficiently large

$$\begin{aligned} P(D_n) &= P(\rho_h(Q, 0) > nM) \\ &= P(1/h(\xi) > nM) \\ &= P(\xi < h^{\sim}(n^{-1}M^{-1})) \\ &= h^{\sim}(n^{-1}M^{-1}) \end{aligned}$$

where $h^{\sim}$ denotes the inverse of h. Therefore the series in (3) is $\sum_1^\infty h^{\sim}(n^{-1}M^{-1})$ and this converges or diverges, by monotonicity, with

$$\int^\infty h^{\sim}(t^{-1}M^{-1})\, dt = M^{-1} \int_0 s^{-2} h^{\sim}(s)\, ds \, .$$

Integration by parts together with $h \geqq aI$ shows that the latter integral converges and diverges with $\int_0^1 (1/h)\, dI$. Hence $\int_0^1 (1/h)\, dI = +\infty$ implies, by the divergence half of (3), that $P(D_n \text{ i.o.}) = 1$ and therefore $P(B_n \text{ i.o.}) = 1$ for all $M > 0$. Since M is arbitrary, this completes the proof of (B).

Note that we have also proved that $\int_0^1 (1/h)\, dI < \infty$ implies $P(\rho_h(Q_n, 0) > n\varepsilon \text{ i.o.}) = 0$ for all $\varepsilon > 0$.

We now prove (A). Suppose $h \in \mathscr{H}(\nearrow)$. Let $\varepsilon > 0$ and choose θ so small that $\int_0^\theta (1/h)\, dI < \varepsilon/2$. Then

$$\rho_h(\Gamma_n, I) \leqq \sup_{0<t\leqq\theta} (\Gamma_n(t)/h(t)) + \sup_{0<t\leqq\theta} (t/h(t)) + \sup_{\theta\leqq t\leqq 1} |\Gamma_n(t) - t|/h(\theta)\, , \tag{4}$$

and

$$\begin{aligned} \sup_{0<t\leqq\theta} (\Gamma_n(t)/h(t)) &= \sup_{0<t\leqq\theta} n^{-1} \textstyle\sum_1^n 1_{[0,t]}(\xi_i)/h(t) \\ &\leqq n^{-1} \textstyle\sum_1^n 1_{[0,\theta]}(\xi_i)/h(\xi_i) \\ &\to \textstyle\int_0^\theta (1/h)\, dI \quad \text{w.p. } 1 \end{aligned}$$

by the ordinary strong law of large numbers. Since $t/h(t) \leqq \int_0^t (1/h)\, dI$ implies $\sup_{0<t\leqq\theta} (t/h(t)) \leqq \int_0^\theta (1/h)\, dI$, the first two terms in (4) have a limit superior on n which is less than ε w.p. 1, and the third term converges to zero w.p. 1 by the Glivenko–Cantelli theorem. Therefore

$$\limsup_{n\to\infty} \rho_h(\Gamma_n, I) < \varepsilon$$

w.p. 1 for any $\varepsilon > 0$, and (A) is proved. □

REMARK 1. Note that (A) of the theorem may be extended, using symmetry, to give the conclusion

$$\lim_{n\to\infty} \rho_h(\Gamma_n, I) = \lim_{n\to\infty} \rho_h(\Gamma_n - I, 0) = 0 \quad \text{w.p. } 1$$

for $h \in \mathscr{H}$. Also note, however, that (A) implies

$$\lim_{n\to\infty} \rho_h(\Gamma_n, 0) = \rho_h(I, 0) \quad \text{w.p. } 1$$

for $h \in \mathscr{H}(\nearrow)$, but that the latter is an empty statement for $h \in \mathscr{H}$ (both sides being $+\infty$). The point is that the functions $h \in \mathscr{H}(\nearrow)$ are appropriate for either of the processes $Q(t) = 1_{[0,t]}(\xi)$ or $Q(t) - t = 1_{[0,t]}(\xi) - t$ whereas the functions

$h \in \mathscr{H}$ are appropriate only for processes which are zero at both 0 and 1 such as $Q(t) - t$. For $h \in \mathscr{H}$, $\rho_h(Q, 0) = +\infty$ w.p. 1.

REMARK 2. For $h \in \mathscr{H}(\nearrow)$, define processes X_i on [0, 1] by $X_i(t) = Q_i(t)/h(t) = 1_{[0,t]}(\xi_i)/h(t)$, and write $\|f\| = \rho(f, 0)$ for $f \in D[0, 1] \equiv D$ where $D[0, 1]$ is the set of right continuous functions on [0, 1] with left limits. Then $(D, \|\cdot\|)$ is an (inseparable) Banach space, and (A) of Theorem 1 is a strong law of large numbers for Banach space valued random elements: $E(X_1) = I/h$, $\|X_1\| = \rho_h(Q_1, 0) = 1/h(\xi_1)$, and hence (A) asserts that if

$$E\|X_1\| = \int_0^1 (1/h)\, dI < \infty ,$$

then

$$\lim_{n\to\infty} \|n^{-1} \textstyle\sum_1^n X_i - E(X_1)\| = 0 \quad \text{w.p. 1.}$$

REMARK 3. The convergence half of Theorem 1 has also been established by Lai [1] (page 81). For other strong laws for Banach spaces see [2].

COROLLARY 1. *If* $h \in \mathscr{H}(\nearrow)$ *then for all* $\tau > 1$

$$P(\Gamma_n(t) > \tau\rho_h(I, 0)h(t) \text{ for some } 0 < t \leqq 1 \text{ i.o.}) = 0 .$$

PROOF. (A) implies that $\rho_h(\Gamma_n, 0) \to \rho_h(I, 0)$ w.p. 1 as $n \to \infty$. Hence, for any $\tau > 1$, $P(\rho_h(\Gamma_n, 0) > \tau\rho_h(I, 0) \text{ i.o.}) = 0$. □

For example, when $h(t) = t(\log(e/t))^\gamma$ with $\gamma > 1$, then $h \in \mathscr{H}(\nearrow)$, $\rho_h(I, 0) = 1$, (A) of the theorem holds, and for every $\tau > 1$, $\Gamma_n \leqq \tau h$ for $n \geqq N_{\omega,\tau}$ w.p. 1. On the other hand, if $\gamma \leqq 1$, $\int_0^1 (1/h)\, dI = +\infty$ and by (B) of the theorem, $\limsup_{n\to\infty} \rho_h(\Gamma_n, 0) = +\infty$ w.p. 1.

In [6] (Theorem 1) we established almost sure "nearly linear" bounds for Γ_n and Γ_n^{-1}, the left continuous inverse of Γ_n. Those nearly linear bounds are crucial to our proofs in the following section and therefore we restate them here as Theorem 2. The inequalities (5) and (8) below, especially the upper bound half of (5), are strongly related to Corollary 1 above. In fact, Corollary 1 together with (B) of Theorem 1 give a strong (integral test) version, and a completely different proof, of the upper bound half of (5). Thus (5) and (8) below are easy further corollaries of Corollary 1; for the proofs of (6), (7), (9) and (10) see [6]. It would be interesting to know the strong form corresponding to (6).

THEOREM 2. *Let* $\tau_1, \tau_2 > 1$ *be fixed. Then there exists* $0 < \beta = \beta(\tau_1, \tau_2) < \frac{1}{2}$ *and a set* $A \subset \Omega$ *with* $P(A) = 1$ *having the following properties: for all* $\omega \in A$ *there is an* $N \equiv N(\omega, \tau_1, \tau_2)$ *for which* $n > N$ *implies*

$$1 - \left(\frac{1-t}{\beta}\right)^{1/\tau_2} \leqq \Gamma_n(t) \leqq (t/\beta)^{1/\tau_1} \quad \textit{for} \quad 0 \leqq t \leqq 1 , \tag{5}$$

$$\beta t^{\tau_1} \leqq \Gamma_n(t) \quad \textit{for all} \quad t \quad \textit{such that} \quad 0 < \Gamma_n(t) , \tag{6}$$

$$\Gamma_n(t) \leqq 1 - \beta(1-t)^{\tau_2} \quad \textit{for all} \quad t \quad \textit{such that} \quad \Gamma_n(t) < 1 , \tag{7}$$

$$\beta t^{\tau_1} \leqq \Gamma_n^{-1}(t) \leqq 1 - \beta(1-t)^{\tau_2} \quad \textit{for} \quad 0 \leqq t \leqq 1 , \tag{8}$$

(9) $$\Gamma_n^{-1}(t) \leqq (t/\beta)^{1/\tau_1} \quad \textit{for} \quad t \geqq \frac{1}{n} \qquad \textit{and}$$

(10) $$1 - \left(\frac{1-t}{\beta}\right)^{1/\tau_2} \leqq \Gamma_n^{-1}(t) \quad \textit{for} \quad t \leqq 1 - \frac{1}{n}.$$

3. A strong law for T_n. We now consider the statistic T_n given in (1). For $n \geqq 1$ define functions J_n on $[0, 1]$ by $J_n(t) = c_{ni}$ for $(i-1)/n < t \leqq i/n$ and $1 \leqq i \leqq n$ and $J_n(0) = c_{n1}$. Set

(11) $$\mu_n = \int_0^1 J_n g \, dI .$$

The main theorem of this section, Theorem 3, gives sufficient conditions for $(T_n - \mu_n) \to 0$ w.p. 1 as $n \to \infty$; that these conditions are almost necessary may be seen from Examples 2, 3 and 4. Theorem 3 takes care of the "random" part of the strong law for T_n; Assumption 1 below suffices for its proof. No assumption concerning the convergence of the J_n's is needed for Theorem 3.

Let J denote a fixed measurable function on (0, 1) and set

(12) $$\mu = \int_0^1 Jg \, dI ;$$

note that if J and g satisfy Assumption 1 below then $|\mu| < \infty$. In Theorem 4 we give conditions on the J_n's which imply $\mu_n \to \mu$. This is an easy deterministic problem as opposed to the more difficult random problem which is handled by Theorem 3. For Theorem 4, convergence of the J_n's is the essential additional requirement; it is not necessary that J be continuous a.e. $|g|$ as in the central limit theorem for T_n (confer [4], page 413 and Example 3, page 418).

For fixed b_1, b_2 and M define a "scores bounding function" B by

$$B(t) = Mt^{-b_1}(1-t)^{-b_2}, \qquad 0 < t < 1 .$$

For $\delta > 0$ define

$$D(t) = Mt^{-1+b_1+\delta}(1-t)^{-1+b_2+\delta}, \qquad 0 < t < 1,$$
$$h(t) = [t(1-t)]^{1-\delta/2}, \qquad 0 < t < 1,$$
$$h^*(t) = [t(1-t)]^{1-\delta/4}, \qquad 0 < t < 1 .$$

Let g be a fixed function in $\mathscr{G}$ (see Section 1).

ASSUMPTION 1 (Boundedness). Let $|g| \leqq D$, all $|J_n| \leqq B$, and $|J| \leqq B$ on (0, 1). Suppose that $\int_0^1 Bh \, d|g| < \infty$.

THEOREM 3. *If Assumption* 1 *holds, then*

$$\lim_{n\to\infty} (T_n - \mu_n) = 0 \quad \text{w.p. } 1.$$

PROOF. From Shorack [4] (modifying the notation there by the factor $n^{\frac{1}{2}}$), integration by parts yields

$$T_n - \mu_n = -(S_n + \gamma_{n1} + \gamma_{n2} + \gamma_{n3})$$

where

$$S_n = \int_{\xi_{n1}}^{\xi_{nn}} A_n(\Gamma_n - I)\, dg = \int_0^1 A_n^*(\Gamma_n - I)\, dg ,$$
$$A_n = [\psi_n(\Gamma_n) - \psi_n]/(\Gamma_n - I),$$

with A_n^* equal to A_n on $[\xi_{n1}, \xi_{nn})$ and 0 otherwise, and where $\psi_n(t) = -\int_t^1 J_n\, dI$ for $0 \leqq t \leqq 1$. Here

$$\gamma_{n1} = g(\xi_{n1})[\psi_n(0) - \psi_n(\xi_{n1})]\,,$$
$$\gamma_{n2} = g(\xi_{nn})\psi_n(\xi_{nn})\,,$$

and

$$\gamma_{n3} = \int_{[\xi_{n1},\xi_{nn}]^c} J_n g\, dI$$

are terms which will be shown to be negligible. By Assumption 1, when b_1, $b_2 > 0$,

$$|A_n| = |\int_I^{\Gamma_n} J_n\, dI/(\Gamma_n - I)| \leqq \int_I^{\Gamma_n} B\, dI/(\Gamma_n - I) \leqq B \vee B(\Gamma_n)\,,$$

and Theorem 2 may be used to bound A_n in terms of B. Choose τ_1, τ_2 in Theorem 2 so that $b_1\tau_1 = b_1 + \delta/4$, $b_2\tau_2 = b_2 + \delta/4$, and fix $\omega \in A$. Then, $n \geqq N_\omega$, (6) and (7) imply that

$$|A_n^*| \leqq M_{1,2} M I^{-b_1\tau_1}(1 - I)^{-b_2\tau_2} = M_{1,2} B[I(1 - I)]^{-\delta/4} \tag{13}$$

for some constant $M_{1,2}$ depending on β of Theorem 2. Clearly (13) also holds if either b_1 or b_2 equals zero. In the case b_1 or $b_2 < 0$, use of (5) of Theorem 2 and an argument similar to that given for b_1, $b_2 > 0$ also yields (13).

Hence, w.p. 1, for $n \geqq N_\omega$

$$\begin{aligned}|S_n| &\leqq \int_0^1 |A_n^*|\,|\Gamma_n - I|\, d|g| \\ &\leqq M_{1,2} \int_0^1 B[I(1 - I)]^{-\delta/4}(|\Gamma_n - I|/h^*)h^*\, d|g| \\ &\leqq M_{1,2}\rho_{h^*}(\Gamma_n - I, 0) \int_0^1 Bh\, d|g|\end{aligned}$$

using (13) and $h^*[I(1 - I)]^{-\delta/4} = h$. But $h^* \in \mathscr{H}$, so Theorem 1 implies $\rho_{h^*}(\Gamma_n - I, 0) \to 0$ w.p. 1 as $n \to \infty$; also, $\int_0^1 Bh\, d|g| < \infty$ by Assumption 1. Hence $S_n \to 0$ w.p. 1 as $n \to \infty$.

It remains only to show that $(\gamma_{n1} + \gamma_{n2} + \gamma_{n3}) \to 0$ w.p. 1 as $n \to \infty$; but each γ_{ni} is easily shown, using Assumption 1, to be of the order $\xi_{n1}^\delta \to 0$ w.p. 1. □

COROLLARY 2. *If* $\lim_{n\to\infty} \mu_n = \mu_\infty$ *exists* (*with* $|\mu_\infty| < \infty$) *and Assumption* 1 *holds, then*

$$\lim_{n\to\infty} T_n = \mu_\infty \quad \text{w.p. } 1.$$

PROOF. $|T_n - \mu_\infty| \leqq |T_n - \mu_n| + |\mu_n - \mu_\infty|$. □

ASSUMPTION 2 (Convergence). $J(t) = \lim_{n\to\infty} J_n(t)$ exists for every $t \in (0, 1)$.

THEOREM 4. *If Assumptions* 1 *and* 2 *hold, then*

$$\lim_{n\to\infty} T_n = \mu \quad \text{w.p. } 1$$

with μ *of* (12) *finite.*

PROOF. If we show that $\lim_{n\to\infty} \mu_n = \mu$, then Corollary 2 with $\mu_\infty = \mu$ is in force and the proof is complete. But, by Assumption 1,

$$|J_n g| \leqq M^2[I(1 - I)]^{-1+\delta}$$

which is in $L^1(I)$, and, by Assumption 2, $J_n(t)g(t) \to J(t)g(t)$ for all $t \in (0, 1)$. Hence, by the dominated convergence theorem, $\mu_n = \int_0^1 J_n g\, dI \to \int_0^1 Jg\, dI = \mu$. □

Now we give several Examples; the first two parallel Examples 1 and 1a of [4].

EXAMPLE 1. Let $X_1, \cdots, X_n$ be a random sample from an arbitrary df F for which $E|X|^r < \infty$ for some $r > 0$. Let

$$T_n = n^{-1} \sum_1^n J(t_{ni})X_{ni}$$

where $\max_{1 \leqq i \leqq n} |t_{ni} - i/n| \to 0$ as $n \to \infty$ and suppose that for some $a > 0$

$$a\left[\frac{i}{n} \wedge \left(1 - \frac{i}{n}\right)\right] \leqq t_{ni} \leqq 1 - a\left[\frac{i}{n} \wedge \left(1 - \frac{i}{n}\right)\right]$$

for $1 \leqq i \leqq n$. Suppose that

$$|J(t)| \leqq M[t(1-t)]^{-1+1/r+\delta}, \qquad 0 < t < 1$$

for some $\delta > 0$ where J is continuous with the exception of a finite number of points. Then

$$\lim_{n\to\infty} T_n = \int_0^1 JF^{-1}\, dI \quad \text{w.p. } 1.$$

PROOF. We use Theorem 4 with $g = F^{-1}$ and $b_1 = b_2 \equiv 1 - 1/r - \delta$. Since $E|X|^r < \infty$ we have

$$t|F^{-1}(t)|^r \leqq \int_0^t |F^{-1}(s)|^r\, ds \leqq \int_{-\infty}^{F^{-1}(t)} |s|^r\, dF(s) \to 0 \qquad \text{as} \quad t \to 0\,.$$

using $F^{-1} \circ F(t) \leqq t$ for $-\infty < t < \infty$. Thus $|g| \leqq D$ with the choice $-1 + b + \delta = -1/r$. The "a-condition" on the t_{ni}'s implies that $|J_n| \leqq B \equiv M_a[t(1-t)]^{-1+1/r+\delta}$ for some constant M_a. The above inequalities and an integration by parts also show that $\int_0^1 Bh\, d|g| < \infty$. Thus Assumption 1 holds. The "max condition" on the t_{ni}'s and the continuity of J implies Assumption 2. Hence the result follows from Theorem 4. □

EXAMPLE 2. Let $X_1, \cdots, X_n$ be a random sample from a df having $E|X|^r < \infty$ for some $r > 1$. Then

$$\lim_{n\to\infty} \bar{X} = \int_0^1 F^{-1}\, dI = \int x\, dF(x) \quad \text{w.p. } 1.$$

This example shows that the ordinary law of large numbers "just fails" to be a corollary to Theorem 4.

EXAMPLE 3. Let $g = I$ and let $c_{ni} = J(i/n)$ where $J = B = I^{-2+\delta}$ with $\delta > 0$, i.e., $b_1 = 2 - \delta$, $b_2 = 0$. Since $\int_0^1 Bh\, dI = \int_0^1 I^{-1+\delta/2}(1-I)^{1-\delta/2}\, dI < \infty$, Assumption 1 holds. Assumption 2 holds easily and Theorem 4 yields

$$T_n = n^{-1} \sum_1^n \left(\frac{i}{n}\right)^{-2+\delta} \xi_{ni} \to \int_0^1 I^{-1+\delta}\, dI = \mu \quad \text{w.p. } 1$$

as $n \to \infty$. Note that $\mu = \int_0^1 I^{-1+\delta}\, dI = \delta^{-1}$.

EXAMPLE 4. Now let $g = I$ and let $c_{ni} = J(i/n)$ where $J = B = I^{-2}$; i.e., $b_1 = 2$, $b_2 = 0$. Then Assumption 1 fails since $\int_0^1 Bh\, dI = \int_0^1 I^{-1-\delta/2}(1-I)^{1-\delta/2}\, dI = +\infty$.

Note that

$$T_n = n^{-1} \sum_1^n \left(\frac{i}{n}\right)^{-2} \xi_{ni} \geqq n\xi_{n1} \sum_1^n i^{-2}$$

and hence

$$\limsup_{n\to\infty} T_n \geqq (\limsup_{n\to\infty} n\xi_{n1}) \sum_1^\infty i^{-2} = +\infty$$

w.p. 1 by Theorem 1(ii) of Robbins and Siegmund [3]. Taking a sequence of δ's converging to zero in Example 3 shows, in fact, that

$$\lim_{n\to\infty} T_n = +\infty \quad \text{w.p. } 1.$$

EXAMPLE 5. Let $X_1, \cdots, X_n$ be a random sample from a df F for which $E|X|^r < \infty$ for some $r > 1$. Let $J_n \equiv c_1$ for n odd and $J_n \equiv c_2 \neq c_1$ for n even. Then, by the same reasoning used in Example 1, Assumption 1 holds, but clearly Assumption 2 fails to hold. Hence, by Theorem 3,

$$\lim_{n\to\infty} (T_n - \mu_n) = 0 \quad \text{w.p. } 1;$$

but μ_n oscillates between $c_1 \int_0^1 F^{-1}\, dI$ and $c_2 \int_0^1 F^{-1}\, dI$. Thus Theorem 3 may hold while Theorem 4 fails.

EXAMPLE 6. Let $X_1, \cdots, X_n$ be independent Bernoulli ($\frac{1}{2}$) rv's. Let $g = F^{-1}$. Thus $g(t) = -\infty, 0, 1$ for $t = 0, 0 < t \leqq \frac{1}{2}, \frac{1}{2} < t \leqq 1$. Let $J(t)$ equal 0, 1 for $0 \leqq t < \frac{1}{2}, \frac{1}{2} \leqq t \leqq 1$, and let $c_{ni} = J(i/n)$. Then T_n equals $\frac{1}{2}$ if more than $\frac{1}{2}$ of the X_i's are positive, while T_n equals the proportion of positive X_i's if less than $\frac{1}{2}$ of the X_i's are positive. Thus $\mu = \int_0^1 1_{(1/2,1]}\, dI = \frac{1}{2}$, Assumptions 1 and 2 are satisfied, and $T_n \to \frac{1}{2}$ w.p. 1 as $n \to \infty$ by Theorem 4.

This example illustrates that J need *not* be continuous a.e. $|g|$ for T_n to obey a strong law (confer Example 3 of [4], page 416).

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REFERENCES

[1] LAI, T. L. (1974). Convergence rates in the strong law of large numbers for random variables taking values in Banach spaces. *Bull. Inst. Math. Acad. Sinica* **2** 67–85.

[2] PADGETT, W. J. and TAYLOR, R. L. (1973). Laws of large numbers for normed linear spaces and certain Frechet spaces. In *Springer-Verlag Lecture Notes in Mathematics No.* 360. Springer-Verlag, New York.

[3] ROBBINS, H. and SIEGMUND, D. (1971). On the law of the iterated logarithm for maxima and minima. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **3** 51–70, Univ. of California Press.

[4] SHORACK, G. R. (1972). Functions of order statistics. *Ann. Math. Statist.* **43** 412–427.

[5] STIGLER, S. M. (1974). Linear functions of statistics with smooth weight functions. *Ann. Statist.* **2** 676–693.

[6] WELLNER, J. A. (1977). A law of the iterated logarithm for functions of order statistics. *Ann. Statist.* **5** 481–494.

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