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A MODIFIED ROBBINS-MONRO PROCEDURE APPROXIMATING THE ZERO OF A REGRESSION FUNCTION FROM BELOW

BY DAN ANBAR[1]

Tel Aviv University

A Robbins-Monro type procedure for estimating the zero of a regression function is discussed. The procedure is a modification of the Robbins-Monro procedure which is designed to approximate the zero from below. An almost sure convergence is proved and it is shown that one can guarantee that the procedure overestimate the zero only finitely many times with probability one.

1. Introduction. Let $\{Y(x): -\infty < x < \infty\}$ be a family of random variables defined on some probability space. Assume $M(x) = EY(x)$ and $\sigma^2(x) = \text{Var } Y(x)$ to be Borel measurable functions. Denote by θ the solution of the equation $M(x) = 0$ which is assumed to exist and be unique. Let X_1 be a random variable and let $\{X_n\}$ be defined recursively by

$$X_{n+1} = X_n - a_n Y_n \qquad n = 1, 2, \cdots \tag{1}$$

where Y_n is a random variable which has conditional distribution given $\mathscr{F}_n = \mathscr{F}(X_1, \cdots, X_n)$ equal to that of $Y(X_n)$ and a_n $(n = 1, 2, \cdots)$ is a sequence of real numbers.

The process (1) is known as the Robbins-Monro (R-M) process and is designed to approximate θ.

It is known that under some general conditions $X_n \to \theta$ a.s. This result was first proved by Blum (1954). A very elegant proof was provided more recently by Robbins and Siegmund (1971). It is also known that under fairly general conditions X_n suitably normalized converges in law to a normal random variable. (Sacks, (1958)).

During the last decade procedures related to the R-M process have become widely used in many fields of application. The simplicity of the iterative relationships, the distribution free nature of the processes and other desirable properties made them attractive for use in various system control situations.

There are, however, cases in which it is advantageous to use a process which converges to θ from below (in some sense). For example, θ may be the optimal level of operating a system where the costs of operating at a level above θ may be considerably greater than the costs of operating at a level below θ. This situation arises quite frequently in medical and biological applications when both the desirable effects and the potentially harmful side effects increase with

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[1] Now at the Department of Mathematics and Statistics, Case Western Reserve University.

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the dose. Eichhorn and Zacks (1973) considered this situation. They defined the optimal dose to be the highest dose for which the toxicity does not exceed a preassigned value. They constructed a search procedure for the optimal dose with the property that at each step the probability of overestimating is bounded by a given number. Eichhorn and Zacks' results hold provided some fairly strong conditions are satisfied (e.g., linear dose-response curve, normally and independently distributed errors).

In this paper it is proposed to modify the R–M procedure by setting

$$X_{n+1} = X_n - a_n(Y_n + b_n) \qquad n = 1, 2, \cdots \tag{2}$$

where the b_n's and a_n's are measurable functions of $(X_1, \cdots, X_n)$ and X_1 is a random variable which is chosen by the statistician. It is proved that if the b_n's are small enough then the a.s. convergence of the process is preserved. Furthermore, an iterated logarithm result due to Heyde (1974) makes it possible to choose the b_n's such that with probability one X_n exceeds θ only finitely many times.

2. Convergence of the modified R–M process. In this section, an a.s. convergence of the modified R–M process is proved. The method of proof is a modification of Robbins and Siegmund's proof of convergence of the R–M process (1971).

THEOREM 1. *Let $\sigma^2(x)$ and $M(x)$ be measurable functions such that*

$$\sigma(x) \leqq c + d|x| \qquad \text{for some constants} \quad c \quad \text{and} \quad d > 0\,, \tag{3}$$

$$|M(x)| \leqq K|x - \theta| \qquad \text{for some} \quad K > 0\,, \tag{4}$$

$$\inf_{\varepsilon<|x-\theta|<\varepsilon^{-1}} |M(x)| > 0 \qquad \text{for every} \quad 0 < \varepsilon < 1\,, \tag{5}$$

$$(x - \theta)M(x) > 0 \qquad \text{for all} \quad x \neq \theta\,. \tag{6}$$

Let $\{a_n\}$ and $\{b_n\}$ be $\mathscr{F}_n$-measurable functions with $a_n \geqq 0$ and

$$\sum a_n^2 < \infty \qquad \text{and} \qquad \sum |a_n b_n| < \infty \quad \text{a.s.} \tag{7}$$

$$\sum a_n = \infty \qquad \text{on} \quad \sup|X_n| < \infty\,. \tag{8}$$

Then, the modified R–M *process defined in* (2) *converges to θ with probability one.*

PROOF. The theorem is proved by applying Theorem 1 of Robbins and Siegmund (1971). Let $U_n = (X_n - \theta)^2$. Then

$$\begin{aligned} E(U_{n+1} | \mathscr{F}_n) &= U_n + a_n^2[\sigma^2(X_n) + (M(X_n) + b_n)^2] \\ &\quad - 2a_n(X_n - \theta)(M(X_n) + b_n) \\ &= U_n + a_n^2[\sigma^2(X_n) + M^2(X_n)] + 2a_n^2 b_n M(X_n) \\ &\quad + a_n^2 b_n^2 - 2a_n|X_n - \theta||M(X_n)| - 2a_n b_n(X_n - \theta)\,. \end{aligned}$$

Conditions (3) and (4) imply that there exist constants a and b such that $\sigma(X) + |M(X)| \leqq a + b|X|$. The inequalities $(u + v)^2 \leqq 2(u^2 + v^2)$, $2|uv| \leqq u^2 + v^2$

and $u^2 + v^2 \leqq (|u| + |v|)^2$ imply

$$\sigma^2(X_n) + M^2(X_n) \leqq (\sigma(X_n) + |M(X_n)|)^2 \leqq (a + b|X_n|)^2$$
$$\leqq (a + b|\theta| + b|X_n - \theta|)^2 \leqq 4(a^2 + b^2\theta^2) + 2b^2U_n .$$
$$2a_n{}^2b_n M(X_n) \leqq 2a_n{}^2b_n K|X_n - \theta| \leqq a_n{}^2b_n{}^2 + a_n{}^2K^2U_n .$$
$$2|a_n b_n(X_n - \theta)| \leqq |a_n b_n| + |a_n b_n|U_n .$$

Thus

$$E(U_{n+1} | \mathscr{F}_n) \leqq [1 + (2b^2 + K^2)a_n{}^2 + |a_n b_n|]U_n + a_n{}^2(4a^2 + 4\theta^2b^2 + 2b_n{}^2) + |a_n b_n| - 2a_n|X_n - \theta||M(X_n)| .$$

Setting

$$\beta_n = (2b^2 + K^2)a_n{}^2 + |a_n b_n|$$
$$\xi_n = a_n{}^2(4a^2 + 4\theta^2b^2 + 2b_n{}^2) + |a_n b_n|$$
$$\zeta_n = 2a_n|X_n - \theta||M(X_n)| ,$$

it follows that

$$\sum \beta_n < \infty \quad \text{a.s.} \qquad \text{and}$$
$$\sum \xi_n < \infty \quad \text{a.s.}$$

Hence by Robbins and Siegmund's theorem U_n converges a.s. to a random variable and $\sum \zeta_n < \infty$ a.s. This contradicts (5) and (8) unless $X_n \to \theta$ a.s.

3. Auxiliary lemmas. In this section, some of the tools which are needed in later sections are presented. It is assumed without loss of generality that $\theta = 0$.

LEMMA 1. *Let $X_1, X_2 \cdots$ be a modified* R–M *process. Assume that conditions* (3), (4) *and* (6) *of Theorem* 1 *are satisfied, and*

(9) *There exists* $K_1 > 0$ *such that* $|M(x)| > K_1|x|$ *for all* x.

(10) $a_n = An^{-1}$ *with* $2AK_1 > 1$.

Let b_n be $\mathscr{F}_n$-measurable functions with

(11) $Eb_n{}^2 \leqq C \log_2 n/n$ *for some* $C > 0$ *and all* $n \geqq 3$ *where* $\log_2 n \equiv \log(\log n)$.

Then there exists $C_1 > 0$ such that for all $n \geqq 3$

$$EX_n{}^2 < C_1 \log_2 n/n .$$

PROOF. By (2)

$$X_{n+1}^2 = X_n{}^2 + A^2n^{-2}Y_n{}^2 + A^2n^{-2}b_n{}^2 - 2An^{-1}X_n Y_n - 2An^{-1}b_n X_n + 2A^2n^{-2}b_n Y_n .$$

Since

$$E(Y_n{}^2 | \mathscr{F}_n) = \sigma^2(X_n) + M^2(X_n) \leqq (\sigma(X_n) + |M(X_n)|)^2$$
$$\leqq (a + b|X_n|)^2 \leqq 2(a^2 + b^2X_n{}^2) ,$$

it follows that

$$EY_n{}^2 \leqq 2a^2 + 2b^2EX_n{}^2 . \tag{12}$$

The inequality $2|uv| \leqq u^2 + v^2$ implies

(13) $$2An^{-1}|b_n X_n| \leqq (A/2K_1\varepsilon)n^{-1}b_n^{\,2} + 2AK_1\varepsilon n^{-1}X_n^{\,2}$$
$$2A^2n^{-2}|b_n Y_n| \leqq A^2n^{-2}b_n^{\,2} + A^2n^{-2}b_n^{\,2}, \qquad \text{where} \quad \varepsilon > 0 .$$

Now by (6) and (9)

(14) $$E(X_n Y_n) = E(E(X_n Y_n \mid \mathscr{F}_n)) = E(X_n M(X_n))$$
$$= E|X_n M(X_n)| = E|X_n|\,|M(X_n)| \geqq K_1 EX_n^{\,2} .$$

Denote $c_n^{\,2} = Eb_n^{\,2}$. Let ε in (13) be such that $2AK_1(1 - \varepsilon) > 1$. Then combining (12), (13) and (14) one obtains

$$EX_{n+1}^2 \leqq (1 - 2AK_1(1 - \varepsilon)n^{-1} + 4b^2A^2n^{-2})EX_n^{\,2} + 4a^2A^2n^{-2}$$
$$+ 2A^2n^{-2}c_n^{\,2} + (A/2K_1\varepsilon)n^{-1}c_n^{\,2} .$$

Let $\varepsilon_1 > 0$ be such that $2AK_1(1 - \varepsilon - \varepsilon_1) > 1$.

Let $N_0 \geqq \max(2b^2A/\varepsilon_1 K_1, 4AK_1\varepsilon)$. Then if $n \geqq N_0$

(15) $$EX_{n+1}^2 \leqq (1 - dn^{-1})EX_n^{\,2} + d_1 n^{-2} + d_2 n^{-1}c_n^{\,2}$$

with $d = 2AK_1(1 - \varepsilon - \varepsilon_1) > 1$, $d_1 = 4a^2A^2$ and $d_2 = A/K_1\varepsilon$.

Let

$$\beta_{mn} = 1 \qquad \text{if} \quad m = n$$
$$= \prod_{j=m+1}^{n} (1 - dj^{-1}) \qquad \text{if} \quad m < n .$$

Iterating (15) yields

(16) $$EX_{n+1}^2 \leqq \beta_{N_0-1,n}EX_{N_0}^2 + d_1 \sum_{k=N_0}^{n} \beta_{kn} k^{-2} + d_2 \sum_{k=N_0}^{n} \beta_{kn} k^{-1}c_k^{\,2} .$$

Now, $\beta_{mn} \leqq d_3 m^d n^{-d}$. Hence

$$EX_{n+1}^2 \leqq d_3 N_0^{\,d} n^{-d} EX_{N_0}^2 + d_1 d_3 n^{-d} \sum_{k=N_0}^{n} k^{d-2} + d_2 d_3 C n^{-d} \sum_{k=N_0}^{n} k^{d-2} \log_2 k$$
$$\leqq d_4 n^{-d} + d_5 n^{-1} + d_6 n^{-1} \log_2 n .$$

This completes the proof since $d > 1$.

LEMMA 2. *Let $p > \frac{1}{2}$ be a fixed number. Then under the conditions of Lemma* 1

(17) $$n^{-p+\frac{1}{2}} \sum_{k=1}^{n} k^{p-1}X_k^{\,2} \to 0 \qquad \text{a.s. } as \quad n \to \infty .$$

PROOF. Let $\varepsilon > 0$ be an arbitrary real number. It is sufficient to prove that

$$P\{n^{-p+\frac{1}{2}} \sum_{k=1}^{n} k^{p-1}X_k^{\,2} > \varepsilon, \ \text{i.o.}\} = 0 .$$

$$P\{n^{-p+\frac{1}{2}} \sum_{k=1}^{n} k^{p-1}X_k^{\,2} > \varepsilon, \ \text{i.o.}\} = P\{\max_{2^m < j \leqq 2^{m+1}} j^{-p+\frac{1}{2}} \sum_{k=1}^{j} k^{p-1}X_k^{\,2} > \varepsilon, \ \text{i.o.}\} .$$

By Chebychev's inequality and Lemma 1,

$$P\{\max_{2^m < j \leqq 2^{m+1}} j^{-p+\frac{1}{2}} \sum_{k=1}^{j} k^{p-1}X_k^{\,2} > \varepsilon\}$$
$$\leqq P\{2^{-m(p-\frac{1}{2})} \sum_{k=1}^{2^{m+1}} k^{p-1}X_k^{\,2} > \varepsilon\}$$
$$\leqq (C/\varepsilon)2^{-m(p-\frac{1}{2})} \sum_{k=3}^{2^{m+1}} k^{p-2}(\log_2 k)$$
$$< C'2^{-m/2} \log(m + 1) \qquad \text{if} \quad p > 1$$
$$< C''2^{-m/2}(m + 1) \log(m + 1) \qquad \text{if} \quad p = 1$$
$$< C'''2^{-m(p-\frac{1}{2})} \log(m + 1) \qquad \text{if} \quad p < 1 .$$

Hence by the Borel–Cantelli lemma

$$P\{\max_{2^m < j \leq 2^{m+1}} j^{-p+\frac{1}{2}} \sum_{k=1}^{j} k^{p-1} X_k^2 > \varepsilon, \text{ i.o.}\} = 0$$

LEMMA 3. *Let* $\delta_1(x)$ *be a measurable function such that* $\lim_{x\to 0} \delta_1(x)/x^2 = 0$. *Then under the conditions of Lemma* 2

$$n^{-p+\frac{1}{2}} \sum_{k=1}^{n} k^{p-1} \delta_1(X_k) \to 0 \qquad \text{a.s. } as \quad n \to \infty .$$

PROOF. This is an obvious consequence of Lemma 2 since the conclusion of Lemma 3 holds for every fixed ω for which the conclusion of Lemma 2 holds and for which $X_n(\omega) \to 0$ as $n \to \infty$.

4. An approximation procedure approaching from below. In this section, the results of the previous sections are used to construct a modified R–M process which converges to 0 a.s. from below. The b_n's are taken to be constants. Assume the following:

(18) $$M(x) = \alpha x + \delta(x) \qquad where \quad \delta(x) = \alpha_1 x^2 + \delta_1(x)\,, \qquad \delta_1(x) = o(x^2) \quad as \quad x \to 0\,, \quad and \quad \alpha > 0\,.$$

(19) (a) $\sup_x E|Y(x) - M(x)|^{2+\eta} < \infty$ *for some* $\eta > 0$.

(b) $\lim_{x\to 0} \sigma^2(x) = \sigma^2$.

Consider the modified R–M procedure defined by

(20) $$X_{n+1} = X_n - An^{-1}(Y_n + b_n)\,, \qquad n = 1, 2, \cdots$$

where X_1 is an arbitrary random variable. Let $Z_n = M(X_n) - Y_n$. Clearly $E(Z_n \mid \mathscr{F}_n) = 0$ so that the Z_n's are martingale differences. Substituting (18) in (20) yields

$$\begin{aligned} X_{n+1} &= X_n - An^{-1}(\alpha X_n + \delta(X_n) - Z_n + b_n) \\ &= (1 - \alpha A n^{-1}) X_n - An^{-1}\delta(X_n) + An^{-1} Z_n - An^{-1} b_n\,. \end{aligned}$$

By iteration

(21) $$X_{n+1} = \beta_{0n} X_1 - A \sum_{m=1}^{n} m^{-1} \beta_{mn} \delta(X_m) + A \sum_{m=1}^{n} m^{-1} \beta_{mn} Z_m - A \sum_{m=1}^{n} m^{-1} \beta_{mn} b_m\,,$$

where

$$\begin{aligned} \beta_{mn} &= \prod_{k=m+1}^{n} (1 - \alpha A k^{-1}) \qquad \text{if} \quad m < n \\ &= 1 \qquad \text{if} \quad m = n\,. \end{aligned}$$

Let $D_n = A \sum_{m=1}^{n} m^{-1} \beta_{mn} b_m$.
Thus

$$\begin{aligned} &P\{X_{n+1} > 0, \text{ i.o.}\} \\ &\quad = P\{A \sum_{m=1}^{n} m^{-1}\beta_{mn} Z_m > A \sum_{m=1}^{n} m^{-1} \beta_{mn} \delta(X_m) + D_n - \beta_{0n} X_1, \text{ i.o.}\}\,. \end{aligned}$$

As was shown by Heyde (1974), under Conditions (4), (6), (9) and (19) with $2AK_1 > 1$

$$\limsup_{n\to\infty} \{n^{\frac{1}{2}} (\log_2 n)^{-\frac{1}{2}} \sum_{m=1}^{n} m^{-1} \beta_{mn} Z_m\} = \sigma(2\alpha A - 1)^{-\frac{1}{2}} \quad \text{a.s.}$$

Now there exists a constant C such that $|\beta_{mn}| \leqq Cn^{-\alpha A}m^{\alpha A}$ for all m and n. Thus from Lemmas 2 and 3 it follows that

$$\sum_{m=1}^{n} m^{-1}\beta_{mn}\delta(X_n) = o(n^{-\frac{1}{2}}) .$$

Clearly

$$\beta_{0n}X_1 = o(n^{-\frac{1}{2}}) .$$

Hence

$$P\{X_{n+1} > 0, \ \text{i.o.}\} = P\{A \sum_{m=1}^{n} m^{-1}\beta_{mn}Z_m > D_n + o(n^{-\frac{1}{2}}), \ \text{i.o.}\} .$$

Thus if the b_n's are chosen so that

$$D_n \geqq Dn^{-\frac{1}{2}}(\log_2 n)^{\frac{1}{2}} + o(n^{-\frac{1}{2}}(\log_2 n)^{\frac{1}{2}}) \tag{22}$$

with $D > A\sigma(2\alpha A - 1)^{-\frac{1}{2}}$ then

$$P\{X_n > 0, \ \text{i.o.}\} = 0 .$$

This is summarized in the following

THEOREM 2. *Let* $X_1, X_2, \cdots$ *be a modified* R–M *process given by* (20). *If the conditions of Theorem* 1 *together with* (9), (10), (19) *and* (22) *hold with* $2AK_1 > 1$ *then* $X_n \to \theta$ a.s. *and with probability one* $X_n > \theta$ *only finitely many times.*

For example if one chooses

$$b_1 = b_2 = 0 \quad \text{and} \quad b_k = h(k) \quad \text{for} \quad k \geqq 3 \tag{23}$$

where

$$h(x) = D'x^{-\frac{1}{2}}(\log_2 x)^{\frac{1}{2}}[1 + 1/(\log x)(\log_2 x)] \quad [x \geqq 3]$$

with

$$D' > \tfrac{1}{2}\sigma(2\alpha A - 1)^{\frac{1}{2}} ,$$

a simple calculation shows that D_n satisfies (22).

REMARK. It is easy to see that the asymptotic normality result of Sacks (1958) applies to the process (2) with a very minor modification, i.e., $n^{\frac{1}{2}}(X - \theta + D_n) \to_{\mathscr{L}} N(0, \sigma_A^2)$ where $\sigma_A^2 = A^2\sigma^2(2\alpha A - 1)^{-1}$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OHIO 44106