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A CONDITION UNDER WHICH THE PITMAN AND BAHADUR APPROACHES TO EFFICIENCY COINCIDE

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The approximate Bahadur efficiency and the Pitman efficiency for hypothesis testing problems are considered. A theorem is stated and proved which gives a condition under which the existence of the limiting (as the alternative approaches the hypothesis) approximate Bahadur efficiency implies the existence of the limiting (as the significance level approaches 0) Pitman efficiency and the equality of the two limits. Several examples are then given to show how the theorem may be used in computing previously unknown limiting Pitman efficiencies using the Bahadur approach.

1. Introduction. In [3], Bahadur defined an exact and approximate measure of efficiency between two sequences of statistics used to test the same hypothesis. He showed that when two statistics had normal limiting distributions and satisfied a few minor restrictions, the limit of the approximate Bahadur efficiency (as the alternative approached the hypothesis) was equal to the Pitman efficiency.

It generally is not possible to achieve this strong a result for statistics which don't have normal limiting distributions since the Pitman efficiencies may fail to exist or may depend on the significance level (α) and the power (β). However, in Section 3, a theorem is stated and proved which assures us that if two sequences of statistics satisfy Bahadur's conditions for the existence of the approximate Bahadur efficiency and one new condition (III*), the existence of the limiting (as the alternative approaches the hypothesis) approximate Bahadur efficiency implies the existence of the limiting (as $\alpha \to 0$) Pitman efficiency and the equality of the two limits.

A useful application of the theorem is that it allows the computation of a limiting exact Pitman efficiency using the approximate Bahadur efficiency (which is generally easier to compute). In Sections 4 and 5, this procedure is carried out for some specific cases. The t, Kolmogorov–Smirnov, and Cramér–von Mises statistics are shown to satisfy Condition III* in Section 4. In Section 5, two tables are given which permit the immediate computation of the limiting Pitman efficiencies of several well-known statistics, including those mentioned above, for location and scale alternatives.

2. Definitions. Consider a set of probability measures $\{P_\theta, \theta \in \Omega\}$ defined on a space (X, β). Let H be the hypothesis that $\theta \in \Omega_0$ where Ω_0 is some subset of Ω and let $\{T_N\}$ be a sequence of real-valued statistics (based on a random sample

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of size N) defined on (X, β). Bahadur (1960) defined $\{T_N\}$ to be a standard sequence if the following three conditions are satisfied:

I. There exists a continuous probability distribution function F such that, for each $\theta \in \Omega_0$ and $x \in \mathscr{R}_1$, $\lim_{N\to\infty} P_\theta(T_N < x) = F(x)$.

II. There exists a constant a, $0 < a < \infty$, such that $\log(1 - F(x)) = -(ax^2/2)(1 + o(1))$ where $o(1) \to 0$ as $x \to \infty$.

III. There exists a real-valued function $b(\theta)$ on $\Omega - \Omega_0$, with $0 < b(\theta) < \infty$, such that, for each $\theta \in \Omega - \Omega_0$,

$$\lim_{N\to\infty} P_\theta(|(T_N/N^{\frac{1}{2}}) - b(\theta)| > x) = 0 \qquad \text{for every} \quad x > 0\,.$$

(If $\theta \in \Omega_0$, $b(\theta) = 0$ by Condition I.)

Conditions I and III indicate that if T_N is a standard sequence for a given hypothesis, tests with a rejection region of $(T_N > K)$ are intuitively reasonable. In order to compare two tests based on standard sequences, Bahadur normalizes the tests so that they both have the same limiting distributions under H_0. For any standard sequence he lets $K_N = -2\log(1 - F(T_N))$, then shows that K_N is asymptotically distributed as a $\chi^2(2)$ random variable and notes that for $\theta \in \Omega - \Omega_0$, $(K_N/N) \to ab^2(\theta)$ in probability. He defines the approximate slope of the sequence $\{T_N\}$ to be $c(\theta) = ab^2(\theta)$ and the approximate efficiency of two standard sequences $\{T_N^{(1)}\}$ to $\{T_N^{(2)}\}$ to be $E_{12}(\theta) = c_1(\theta)/c_2(\theta)$.

REMARK. When two standard sequences are to be compared, they will be denoted by $\{T_N^{(1)}\}$ and $\{T_N^{(2)}\}$. For symbols such as a_i, $b_i(\theta)$, $c_i(\theta)$, N_i, F_i, and $o_i(1)$, the subscript i refers to $\{T_N^{(i)}\}$. $E_{12}(\theta)$ will denote the approximate Bahadur efficiency of $\{T_N^{(1)}\}$ to $\{T_N^{(2)}\}$ when θ is the true parameter. $e_{12}(\alpha, \beta)$ will represent the Pitman efficiency of $\{T_{N)}^{1)}\}$ to $\{T_N^{(2)}\}$ when both tests are of size α and power β in the limit.

For the definition of Pitman efficiency we will use an extension to the concept as defined in Fraser (1957). The extension will allow us to consider cases when the Pitman efficiency as defined in Fraser fails to exist. Let $\{T_N^{(1)}\}$ and $\{T_N^{(2)}\}$ be two sequences of statistics used to form tests of size α for testing $H: \theta = \theta_0$ versus $A: \theta = \theta_j$ where $\theta_j \neq \theta_0$.

For $0 < \beta < 1$ and sequences $\theta_j \to \theta_0$, $\beta_j^{(1)} \to \beta$ and $\beta_j^{(2)} \to \beta$, we will define $N(i, j)$ to be the smallest integer with the property that for every $N \geqq N(i, j)$

$$P_{\theta_j}(T_N^{(i)} > t_N^{(i)}) \geqq \beta_j^{(i)}\,, \qquad i = 1, 2, j = 1, 2, \cdots,$$

where $t_N^{(i)}$ is determined by

$$P_{\theta_0}(T_N^{(i)} > t_N^{(i)}) = \alpha\,, \qquad i = 1, 2, N = 1, 2, \cdots.$$

If there is randomization the definition is modified in the obvious manner.

In other words, $N(i, j)$ is the first sample size to "guarantee" a power $\beta_j^{(i)}$ for all $N \geqq N(i, j)$, rather than just for $N = N(i, j)$ which is the usual definition. If the power increases with sample size, the two definitions are equivalent.

We define the Pitman efficiency of the sequence $T_N^{(1)}$ with respect to $T_N^{(2)}$ by

$$e_{12}(\alpha, \beta) = \lim_{j\to\infty} N(2, j)/N(1, j)$$

provided this limit exists and is independent of the choice of the sequences θ_j and $\beta_j^{(i)}$. If this is not the case we let

$$e_{12}^{+}(\alpha, \beta) = \sup_{\{\text{II}\}} \limsup_{j\to\infty} (N(2, j)/N(1, j))$$

and

$$e_{12}^{-}(\alpha, \beta) = \inf_{\{\text{II}\}} \liminf_{j\to\infty} (N(2, j)/N(1, j)) .$$

Here $\sup_{\{\text{II}\}}$ $(\inf)_{\{\text{II}\}}$ represents the sup (inf) over all sequences $\{\theta_j\}$, $\{\beta_j^{(1)}\}$, $\{\beta_j^{(2)}\}$, where $\theta_j \to \theta$ and $\beta_j^{(i)} \to \beta$, $i = 1, 2$.

3. Statement and proof of theorem. Throughout this section it will be assumed that Ω is an interval and Ω_0 is a single point θ_0 contained in Ω, possibly as an endpoint, i.e. we are testing $H_0\colon \theta = \theta_0$ and the alternative may be one- or two-sided. In order to simplify the notation, we will let $\mathscr{N}(\theta_0, \theta^*)$ refer to the interval $(\theta_0, \theta_0 + \theta^*)$ when dealing with one-sided alternatives and $(\theta_0 - \theta^*, \theta_0 + \theta^*)$ with θ_0 deleted in the two-sided case.

We begin by defining

CONDITION III*. Suppose for a standard sequence $\{T_N\}$ there is a $\theta^* > 0$, such that for every $\varepsilon > 0$ and $\delta \in (0, 1)$, there is a C such that for all $\theta \in \mathscr{N}(\theta_0, \theta^*)$ and $N > (C/b^2(\theta))$ we have

$$P_\theta\{|T_N/N^{\frac{1}{2}} - b(\theta)| < \varepsilon b(\theta)\} > 1 - \delta .$$

Then T_N is said to satisfy Condition III*. Note that in the above definition C may depend on θ^* but is otherwise independent of θ.

On $\mathscr{N}(\theta_0, \theta^*)$, this condition is stronger than Bahadur's Condition III since it requires the convergence of $T_N/N^{\frac{1}{2}}$ in probability to $b(\theta)$ at a specified rate. An immediate consequence is that if we verify Condition III* for a given function $b(\theta)$ we will be verifying Bahadur's Condition III simultaneously (locally).

THEOREM. *If $\{T_N^{(1)}\}$ and $\{T_N^{(2)}\}$ are sequences of statistics such that*

(1) *$\{T_N^{(1)}\}$ and $\{T_N^{(2)}\}$ are standard sequences with F_1 and F_2 strictly increasing on right tails,*

(2) *$\{T_N^{(1)}\}$ and $\{T_N^{(2)}\}$ satisfy Condition* III*,

(3) $\lim_{\theta\to\theta_0} b_i(\theta) = 0$ *for $i = 1, 2$, and*

(4) $\lim_{\theta\to\theta_0} ([a_1^{\frac{1}{2}} b_1(\theta)]/[a_2^{\frac{1}{2}} b_2(\theta)])$ *exists and is finite;*

then for β bounded away from 0 *or* 1, *we have* (i) $\lim_{\theta\to\theta_0} E_{12}(\theta) = \lim_{\alpha\to 0} e_{12}^{+}(\alpha, \beta) = \lim_{\alpha\to 0} e_{12}^{-}(\alpha, \beta)$ *where the existence of the first limit implies that of the latter two, and* (ii) $\lim_{\theta\to\theta_0} E_{12}(\theta) = \lim_{\alpha\to 0} e_{12}(\alpha, \beta)$ *if $e_{12}(\alpha, \beta)$ exists for all α in an interval of the form* $(0, \alpha')$.

REMARK. In the proof, it will only be necessary to consider one sequence $\{T_N\}$ initially, hence the subscript (superscript) i shall be dropped. We will define $F_N(x) = P_{\theta_0}(T_N \leqq x)$ and let F_N^{-1} be any version of its inverse.

PROOF. For given numbers $\beta \in (0, 1)$ and $\varepsilon \in (0, 1)$ and sequences $\theta_j \to \theta_0$ and $\beta_j \to \beta$, we will find bounds which will contain N_j for large j, where N_j is equivalent to the $N(i, j)$ defined previously. The ratio of these bounds will converge as $\alpha \to 0$ and the result will follow.

Since $\beta \in (0, 1)$, we can choose $\delta > 0$ and an integer j_0 such that for every $j \geqq j_0$, $\delta \leqq \beta_j \leqq 1 - \delta$, $b(\theta_j) \leqq 1$ and $\theta_j \in \mathscr{N}(\theta_0, \theta^*)$ with θ^* as in Condition III*. Take any fixed $\varepsilon \in (0, 1)$ and let C be as in Condition III*. Take $x_0 > 2(C + 1)^{\frac{1}{2}}$ such that F is strictly increasing on (x_0, ∞) and that

$$\text{(1)} \qquad \tfrac{1}{2}ax^2(1 - \varepsilon) \leqq -\log(1 - F(x)) \leqq \tfrac{1}{2}ax^2(1 + \varepsilon) \qquad \text{for} \quad x \geqq x_0 .$$

Define $\alpha^* > 0$ by $2\alpha^* = 1 - F(x_0)$ and take any fixed $\alpha \in (0, \alpha^*)$. Because of Condition I and the strict monotonicity of F around $F^{-1}(1 - \alpha)$, there exists an integer $N' > 1/\varepsilon$ such that $1 - F_N(2(C + 1)^{\frac{1}{2}}) > \alpha^* > \alpha$ and

$$\text{(2)} \qquad \frac{F^{-1}(1 - \alpha)}{1 + \varepsilon} \leqq F_N^{-1}(1 - \alpha) \leqq \frac{F^{-1}(1 - \alpha)}{1 - \varepsilon}$$

for all $N \geqq N'$. Choose an integer $j_1 \geqq j_0$ such that $C/b^2(\theta_j) \geqq N'$ for all $j \geqq j_1$. It follows that for $j \geqq j_1$ and $N \in (C/b^2(\theta_j), C/b^2(\theta_j) + 1]$, the rejection region (randomization included) of the level α test based on T_N is contained in the interval $(2(C + 1)^{\frac{1}{2}}, \infty)$ which is a subset of $(2b(\theta_j)N^{\frac{1}{2}}, \infty)$ because $2b(\theta_j)N^{\frac{1}{2}} \leqq 2(C + b^2(\theta_j))^{\frac{1}{2}} \leqq 2(C + 1)^{\frac{1}{2}}$. As $0 < \varepsilon < 1$ and $\beta_j \geqq \delta$, Condition III* ensures that the power of this test is strictly smaller than β_j and hence that $N_j > N > C/b^2(\theta_j) \geqq N'$ for all $j \geqq j_1$. This implies that for $N = N_j$ and $\theta = \theta_j$ both the estimate of Condition III* and (2) are applicable, provided $j \geqq j_1$. Hence the requirements of level α and power β_j imply that, for $j \geqq j_1$,

$$\begin{aligned} F_{N_j-1}^{-1}(1 - \alpha) &\geqq (1 - \varepsilon)(N_j - 1)^{\frac{1}{2}}b(\theta_j) \geqq (1 - \varepsilon)(N_j - N_j\varepsilon)^{\frac{1}{2}}b(\theta_j) \\ &\geqq (1 - \varepsilon)^{\frac{3}{2}}N_j^{\frac{1}{2}}b(\theta_j) \end{aligned}$$

and

$$F_{N_j}^{-1}(1 - \alpha) \leqq (1 + \varepsilon)N_j^{\frac{1}{2}}b(\theta_j) \leqq (1 + \varepsilon)^{\frac{3}{2}}N_j^{\frac{1}{2}}b(\theta_j) ,$$

so

$$(1 - \varepsilon)^{\frac{5}{2}}N_j^{\frac{1}{2}}b(\theta_j) \leqq F^{-1}(1 - \alpha) \leqq (1 + \varepsilon)^{\frac{5}{2}}N_j^{\frac{1}{2}}b(\theta_j) ,$$

and, in view of (1),

$$(1 - \varepsilon)^3N_j^{\frac{1}{2}}b(\theta_j) \leqq \left(\frac{-2\log\alpha}{a}\right)^{\frac{1}{2}} \leqq (1 + \varepsilon)^3N_j^{\frac{1}{2}}b(\theta_j) ,$$

or

$$\frac{-2\log\alpha}{(1 + \varepsilon)^6ab^2(\theta_j)} \leqq N_j \leqq \frac{-2\log\alpha}{(1 - \varepsilon)^6ab^2(\theta_j)} .$$

Thus for every $\varepsilon \in (0, 1)$ there exists $\alpha^* > 0$ such that for every $\beta \in (0, 1)$ and every $\alpha \in (0, \alpha^*)$

$$\left(\frac{1 - \varepsilon}{1 + \varepsilon}\right)^6 \lim_{\theta\to 0} \frac{c_1(\theta)}{c_2(\theta)} \leqq e_{12}^-(\alpha, \beta) \leqq e_{12}^+(\alpha, \beta) \leqq \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^6 \lim_{\theta\to 0} \frac{c_1(\theta)}{c_2(\theta)}$$

and the result follows.

4. Verification of Condition III*. In this section, three examples are given to indicate how to verify that a particular statistic satisfies Condition III*. Each of the examples will deal with the $H_0: X_1, \cdots, X_N$ i.i.d. with continuous distribution functions $H(x)$ versus $A: X_1, \cdots, X_N$ i.i.d. with continuous distribution functions $G_\theta(x)$, $\theta > \theta_0$, where $\theta > \theta_0$ implies $\sup_x (H(x) - G_\theta(x)) > 0$ and $\lim_{\theta\to\theta_0^+} \sup_x (H(x) - G_\theta(x)) = 0$.

The following lemma will be useful when considering the examples.

LEMMA. *Suppose there is a family of sequences of statistics $U_{(N,\theta)}$ which satisfy $\lim_{N\to\infty} P_\theta\{U_{(N,\theta)} < z\} = Q(z)$ for every real number z where Q is a continuous distribution function and where the rate of convergence is independent of θ in some neighborhood, $\mathscr{N}(\theta_0, \theta')$, of θ_0. Then given any $\varepsilon_1 > 0$ and $\delta_1 \in (0, 1)$, there is a C' such that if $\theta \in \mathscr{N}(\theta_0, \theta')$, $b(\theta) < 1$ and $N > C'/b^2(\theta)$, then $P_\theta\{|U_{(N,\theta)}/N^{\frac{1}{2}}| < \varepsilon_1 b(\theta)\} > 1 - \delta_1$,*

PROOF. We select an M such that $Q(\varepsilon_1 M^{\frac{1}{2}}) > 1 - \delta_1/4$. We then choose a $C' \geqq M$ such that $N > C'$ implies $|P_\theta\{U_{(N,\theta)} < \varepsilon_1 M^{\frac{1}{2}}\} - Q(\varepsilon_1 M^{\frac{1}{2}})| < \delta_1/4$ for all $\theta \in \mathscr{N}(\theta_0, \theta')$. Then for θ satisfying $b(\theta) < 1$, $N > C'/b^2(\theta) \Rightarrow P_\theta\{U_{(N,\theta)} < \varepsilon_1 M^{\frac{1}{2}}\} > 1 - \delta_1/2 \Rightarrow P_\theta\{U_{(N,\theta)} < \varepsilon_1 C'^{\frac{1}{2}}\} > 1 - \delta_1/2 \Rightarrow P_\theta\{U_{(N,\theta)}/N^{\frac{1}{2}}\} < \varepsilon_1 b(\theta)\} > 1 - \delta_1/2$. Similar reasoning shows that M and C' can be chosen such that $N > C'$ implies $P_\theta\{U_{(N,\theta)} > -\varepsilon M^{\frac{1}{2}}\} > 1 - \delta_1/2$ for $\theta \in \mathscr{N}(\theta_0, \theta')$ which leads to $P_\theta\{U_{(N,\theta)}/N^{\frac{1}{2}} > -\varepsilon_1 b(\theta)\} > 1 - \delta_1/2$ and the result follows.

EXAMPLE 1. The first statistic to be considered will be the t-statistic, $T_N = N^{\frac{1}{2}}(\bar{X} - \mu)/S_N$ where $S_N = [\sum (X_i - \bar{X})^2/(N-1)]^{\frac{1}{2}}$. For this statistic it will be assumed that $G_\theta(x) = H(x - \theta)$, $\int x\, dH(x) = \mu$ (known), $\int x^2\, dH(x) - \mu^2 = \sigma^2$ (possibly unknown), and $\theta_0 = \theta$. It is fairly clear that $b(\theta) = \theta/\sigma$, but we want to show that T_N satisfies Condition III*.

To begin we choose a θ' such that $\theta'/\sigma < 1$ and pick an $\varepsilon > 0$ and a $\delta \in (0, 1)$. Letting $U_{(N,\theta)} = N^{\frac{1}{2}}[\bar{X} - (\mu + \theta)]/S_N$, $Q(z) = \Phi(z)$ where $\Phi(z) = \int_{-\infty}^{z} (2\pi)^{-\frac{1}{2}} \exp\{-x^2/2\}\, dx$, $\varepsilon_1 = \varepsilon/2$, and $\delta_1 = \delta/4$, we can apply the lemma to see that there is a C' such that $\theta < \theta'$ and $N > C'/b^2(\theta)$ implies $P_\theta\{|\bar{X} - \mu - \theta)/S_N| < \varepsilon b(\theta)/2\} > 1 - \delta/4$. Furthermore, since S_N converges in probability to σ and is independent of θ, we know there is a C'' such that $N > C''$ implies $P_\theta\{|S_N - \sigma|/(S_N\sigma) < \varepsilon/(2\sigma)\} > 1 - \delta/4$ for all θ. Letting $C^* = \max(C', C'')$ and $\theta^* = \theta'$, we have $\theta < \theta^*$ and $N > C^*/b^2(\theta)$

$$
\begin{aligned}
&\Rightarrow P_\theta\{(\bar{X} - \mu - \theta)/S_N + \theta/\sigma < \varepsilon b(\theta)/2 + \theta/\sigma\} > 1 - \delta/4 \\
&\Rightarrow P_\theta\{\bar{X} - \mu)/S_N - \theta(\sigma - S_N)/(\sigma S_N) < \varepsilon b(\theta)/2 + b(\theta)\} > 1 - \delta/4 \\
&\Rightarrow P_\theta\{T_N/N^{\frac{1}{2}} - (\theta/\sigma)(\varepsilon/2) < \varepsilon b(\theta)/2 + b(\theta)\} > 1 - \delta/2 \\
&\Rightarrow P_\theta\{T_N/N^{\frac{1}{2}} - b(\theta) < \varepsilon b(\theta)\} > 1 - \delta/2\,.
\end{aligned}
$$

By similar reasoning, we have $P_\theta\{|T_N/N^{\frac{1}{2}} - b(\theta)| < \varepsilon b(\theta)\} > 1 - \delta$, so Condition III* is satisfied.

It should be noted that "non-Studentized" t-statistics also satisfy Condition III* when σ is known. In this case S_N is replaced by σ which simplifies the proof.

EXAMPLE 2. Let $G_N(x)$ be the empirical distribution function, i.e., $G_N(x)$ is the proportion of observations $X_1, \cdots, X_N$ which are less than or equal to x. It is known (Doob, 1949) that $\lim_{N\to\infty} P_\theta\{N^{\frac{1}{2}} \sup_x |G_N(x) - G_\theta(x)| < z\} = K(z)$ where

$$K(z) = 1 - 2 \sum_{m=1}^{\infty} (-1)^{m+1} \exp\{-2m^2z^2\}$$

and that the rate of convergence is independent of θ (Darling, 1957). For the Kolmogorov–Smirnov statistic ($T_N = N^{\frac{1}{2}} \sup_x (H(x) - G_N(x))$, $b(\theta) = \sup_x (H(x) - G_\theta(x))$ (Bahadur, 1960). Letting $\varepsilon_1 = \varepsilon$, $\delta_1 = \delta/2$, $U_{(N,\theta)} = N^{\frac{1}{2}} \sup_x |G_N(x) - G_\theta(x)|$ and $Q(z) = K(z)$, our lemma assures us that there is a C' such that $b(\theta) < 1$ and $N > C'/b^2(\theta)$ implies $P_\theta\{\sup_x |G_N(x) - G_\theta(x)| < \varepsilon b(\theta)\} > 1 - \delta/2$. If we choose a θ^* such that $\theta \in \mathscr{N}(\theta_0, \theta^*)$ implies $b(\theta) < 1$ and let $C = C'$, then for $\theta \in \mathscr{N}(\theta_0, \theta^*)$ and $N > C/b^2(\theta)$ we have

$$P_\theta\{\sup_x (G_\theta(x) - G_N(x)) + \sup_x (H(x) - G_\theta(x)) < \varepsilon b(\theta) + b(\theta)\} > 1 - \delta/2$$
$$\Rightarrow P_\theta\{\sup_x (H(x) - G_N(x)) - b(\theta) < \varepsilon b(\theta)\} > 1 - \delta/2 .$$

Similarly, we can show $P_\theta\{\sup_x (H(x) - G_N(x)) - b(\theta) > -\varepsilon b(\theta)\} > 1 - \delta/2$ so $P_\theta\{|T_N/N^{\frac{1}{2}} - b(\theta)| < \varepsilon b(\theta)\} > 1 - \delta$ and Condition III* is satisfied.

EXAMPLE 3. The one sample Cramér–von Mises statistic is $N \int [G_N(x) - H(x)]^2 \, dH(x)$ and to apply the Bahadur theory, we let $T_N = [N \int [G_N(x) - H(x)]^2 \, dH(x)]^{\frac{1}{2}}$. T_N satisfies Condition III* with $b(\theta) = [\int [G_\theta(x) - H(x)]^2 \, dH(x)]^{\frac{1}{2}}$. To show this we again use the fact that if we choose a θ^* such that $\theta \in \mathscr{N}(\theta_0, \theta^*)$ implies $b(\theta) < 1$ and an ε and δ such that $0 < \varepsilon < 1$ and $0 < \delta < 1$, there is a C such that $\theta \in \mathscr{N}(\theta_0, \theta^*)$ and $N > C/b^2(\theta) \Rightarrow P_\theta \{\sup_x |G_N(x) - G_\theta(x)| < \varepsilon b(\theta)/4\} > 1 - \delta$.

Since $T_N^2/N = \int [G_N(x) - G_\theta(x)]^2 dH(x) + 2 \int [G_N(x) - G_\theta(x)][G_\theta(x) - H(x)] dH(x) + \int [G_\theta(x) - H(x)]^2 \, dH(x)$, we have $T_N^2/N - b^2(\theta) = \int [G_N(x) - G_\theta(x)]^2 \, dH(x) + 2 \int [G_N(x) - G_\theta(x)][G_\theta(x) - H(x)] dH(x)$ so $|T_N^2/N - b^2(\theta)| \leqq \int [G_N(x) - G_\theta(x)]^2 dH(x) + 2 \int |G_N(x) - G_\theta(x)| \, |G_\theta(x) - H(x)| \, dH(x) \leqq \sup_x |G_N(x) - G_\theta(x)|^2 + 2 \sup_x |G_N(x) - G_\theta(x)| [\int [G_\theta(x) - H(x)]^2 \, dH(x)]^{\frac{1}{2}} \leqq \sup_x |G_N(x) - G_\theta(x)|^2 + 2 \sup_x |G_N(x) - G_\theta(x)| \cdot b(\theta)$. But $P_\theta\{\sup_x |G_N(x) - G_\theta(x)|^2 + 2 \sup |G_N(x) - G_\theta(x)| \cdot b(\theta) < \varepsilon^2 b^2(\theta)/16 + 2\varepsilon b^2(\theta)/4\} > 1 - \delta$ which implies $P_\theta\{|T_N^2/[Nb^2(\theta)] - 1| < \varepsilon\} > 1 - \delta$ which shows that T_N satisfies Condition III*.

5. Computation of efficiencies. In this section, the theorem will be applied to several statistics and the values required for the computation of the limiting Pitman efficiency will be obtained. Location and scale alternatives will be considered. All the statistics discussed satisfy Condition III* for the appropriate alternatives, although this will not be verified here since the techniques required are similar to those of the previous section. More details appear in Wieand (1975).

Table 1 lists values for "a" and $b^2(\theta)$ for each statistic which is to be discussed in this section. The entries for the t-statistic and the Kolmogorov–Smirnov statistic can be found in Bahadur (1960). The entries for the scale statistic

TABLE 1
Slopes

Statistic	a	$b^2(\theta)$
t	1	$\|\theta\|^2/\sigma^2$
scale	$1/\tau^{2*}$	$\|\theta^2 - 1\|^2\sigma^4$
Kolmogorov-Smirnov (KS)	4	$\sup_x \|G_\theta(x) - H(x)\|^2$
Kuiper	4	$[\sup_x (G_\theta(x) - H(x)) - \inf_x (G_\theta(x) - H(x))]^2$
Cramér-von Mises (CVM)	π^2	$\int (G_\theta(x) - H(x))^2\, dH(x)$
Watson	$4\pi^2$	$(\int (G_\theta(x) - H(x))^2\, dH(x) - [\int (G_\theta(x) - H(x))\, dH(x)]^2)$
Rényi	$d/(1-d)$	$\sup_{H(x)>d} (\|G_\theta(x) - H(x)\|/H(x))^2$

* $\tau^2 = \int x^4\, dH(x) - \sigma^4$

($T_N = N^{\frac{1}{2}}[\sum_{i=1}^N (X_i^2/N - \sigma^2)]$) are implicit in his work. Abrahamson (1965, 1967) found the slopes of the Kuiper ($T_N = N^{\frac{1}{2}}[\sup_x (G_N(x) - H(x)) - \inf_x (G_N(x) - H(x))]$), Watson ($T_N^2 = N \int [G_N(x) - H(x) - \int (G_N(y) - H(y))\, dH(y)]^2\, dH(x)$), and Cramér–von Mises statistics. The slope of the Rényi statistic ($T_N = N^{\frac{1}{2}} \sup_{H(x)>d} [(H(x) - G_N(x))/H(x)]$) has not been computed previously, but the procedure is straightforward. To compute "a," we note that the limiting distribution of T_N under H_0 is $F(x) = 2\Phi(kx) - 1$ where $k = (d/[1-d])^{\frac{1}{2}}$ (Johnson and Kotz, 1970). Hence, $\lim_{x\to\infty} [\log (1 - F(x))/x^2] = k^2 \lim_{x\to\infty} [\log (1 - \Phi(kx))/(k^2x^2)] = k^2$. The last equality follows from the fact that $a = 1$ for the normal distribution (Bahadur, 1960). The expression for $b(\theta)$ is an immediate consequence of the definition.

We want to find $\lim_{\theta\to\theta_0} [c_1(\theta)/c_2(\theta)]$ where $c_1(\theta)$ and $c_2(\theta)$ are the slopes of two different statistics. As it is more convenient to work with one statistic at a time, we will first consider $\lim_{\theta\to\theta_0} [c(\theta)/\theta^2]$ for location alternatives and $\lim_{\theta\to 1} [c(\theta)/(1-\theta)^2]$ for scale alternatives.

If H can be expanded in the form $H(x - \theta) = H(x) - \theta h(x) + o(\theta)$ for all x where $h(x)$ is a bounded density function, then the values of $\lim_{\theta\to 0} (c(\theta)/\theta^2)$ for location alternatives ($G_\theta(x) = H(x - \theta)$) are as shown in Table 2. The normal distribution function $\Phi(x)$ and the logistic distribution function $L(x) = [1 + \exp(-x)]^{-1}$ can be expanded in this way and are included in the table.

If we have scale alternatives, i.e. $G_\theta(x) = H(x/\theta)$, and H can be expanded in the form $H(x/\theta) = H(x) + x[(1-\theta)/\theta]h(x) + o(1-\theta)$ where $xh(x)$ is bounded, then the values of $\lim_{\theta\to 1} [c(\theta)/(1-\theta)^2]$ are as shown in Table 3. Again $\Phi(x)$ and $L(x)$ have this expansion and are included in the table.

The values listed in Tables 2 and 3 under general $H(x)$ follow immediately from the expansion of H. The calculations for values under $H(x) = \Phi(x)$ and $H(x) = L(x)$ are more lengthy, particularly for the Rényi statistic, however they are straightforward and have been omitted. More of the calculations required appear in Wieand (1975).

The ratio of any two values in a column of one of the tables represents a limiting (as $\theta \to \theta_0$) approximate Bahadur efficiency, and by the theorem this is also the limiting (as $\alpha \to 0$) Pitman efficiency. For example, to find the limiting

TABLE 2

$\lim_{\theta\to 0} \frac{c(\theta)}{\theta^2}$ *for location alternatives*

Statistic	$\lim_{\theta\to 0} \frac{c(\theta)}{\theta^2}$		
	General $H(x)$	$H(x) = \Phi(x)$	$H(x) = \mathscr{L}(x)$
t	$\frac{1}{\sigma^2}$	1	$\frac{3}{\pi^2}$
KS	$4 \sup_x (h^2(x))$	$\frac{2}{\pi}$	$\frac{1}{4}$
Kuiper	$4 \sup_x (h^2(x))$	$\frac{2}{\pi}$	$\frac{1}{4}$
CVM	$\pi^2 \int h^3(x)\,dx$	$\frac{\pi}{2\cdot 3^{\frac{1}{2}}}$	$\frac{\pi^2}{30}$
Watson	$4\pi^2[\int h^3(x)\,dx - (\int h^2(x)\,dx)^2]$	$\frac{\pi(2\cdot 3^{\frac{1}{2}} - 3)}{3}$	$\frac{\pi^2}{45}$
Rényi	$\frac{d}{1-d} \sup_{H(x)>d} \left\lvert \frac{h(x)}{H(x)} \right\rvert^2$	$\frac{\phi^2(\Phi^{-1}(d))}{d(1-d)}$ *	$d(1-d)$

* $\phi(x) = \Phi'(x)$

TABLE 3

$\lim_{\theta\to 1} \frac{c(\theta)}{(1-\theta)^2}$ *for scale alternatives*

Statistic	$\lim_{\theta\to 1} (c(\theta)/(1-\theta)^2)$		
	General $H(x)$	$H(x) = \Phi(x)$	$H(x) = \mathscr{L}(x)$
scale	$\sigma^4 4/\tau^2$	2	1.25
KS	$4 \sup_x \lvert x^2h^2(x)\rvert$	$2/(\pi e)$	.200
Kuiper	$4(\sup_x [xh(x)] - \inf_x [xh(x)])^2$	$8/\pi e$	.802
CVM	$\pi^2 \int x^2h^3(x)\,dx$	$3^{\frac{1}{2}}\pi/18$	.26*
Watson	$4\pi^2(\int x^2h^3(x) - [\int xh^2(x)\,dx]^2)$	$2\cdot 3^{\frac{1}{2}}\pi/9$	1.04
Rényi	$[d/(1-d)] \sup_{H(x)>d} [xh(x)/H(x)]^2$	$.087d/(1-d)$	$.0775d/(1-d)$
		$(.38 < d < .79)$	$(.39 < d < .78)$
		$\frac{(\phi'[\Phi^{-1}(d)])^2}{d(1-d)}$	$d(1-d)\ln^2(d/[1-d])$
		($d < .38$ or $d > .79$)	($d < .39$ or $d > .78$)

* A computer program, run at the Computer Center of the University of Pittsburgh, was used in obtaining this value.

Pitman efficiency of the Kolmogorov–Smirnov statistic relative to the Cramér–von Mises statistic for general scale alternatives, we divide $4 \sup_x \lvert x^2h^2(x)\rvert$ by $\pi^2 \int x^2h^3(x)\,dx$. To find the same efficiency for normal scale alternatives ($G_\theta(x) = \Phi(x/\theta)$) we divide $2/\pi e$ by $3^{\frac{1}{2}}\pi/18$ and obtain $36/(3^{\frac{1}{2}}\pi^2 e)$ or .77.

6. Comments. Although the tables were constructed using the one-sided, one-sample problem, the efficiencies are the same in the two-sample and/or two-sided case with the appropriate form of the statistic (Wieand, 1974).

It should be noted that Capon (1965) found bounds for the Pitman efficiency of the one-sample Kolmogorov–Smirnov statistic relative to the t-statistic which are close (as $\alpha \to 0$) to the values we obtain from the tables. Yu (1971) found bounds in the two-sample version of the same efficiency, but his bounds do not close, hence cannot be "sharp." None of the other pairs of statistics has known bounds for the Pitman efficiency, hence the theorem is required to get even the limiting Pitman efficiency.

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