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ASYMPTOTIC DISTRIBUTIONS OF MULTIVARIATE RANK ORDER STATISTICS

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By means of a general weak convergence theorem some invariance principles are proven for the multivariate sequential empirical process and for the multivariate rank order process w.r.t. stronger metrics than the generalized Skorohod metric. The underlying random variables are neither assumed to be independent nor to be stationary. These results are then applied to derive convergence of the weighted empirical cumulatives and for the weighted rank order process. Finally by a new representation asymptotic normality is proven for a general class of linear multivariate rank order statistics.

1. Introduction. In the literature, asymptotic results for multivariate rank order statistics are derived in several papers for independent, stationary random variables and under contiguous alternatives. For the case of testing the independence hypothesis cf. e.g., Ruymgaart, Shorack, van Zwet [10], Ruymgaart [11] and Behnen [1].

The aim of this paper is to develop an approach which under comparable weak boundedness assumptions concerning the scores functions and under weak boundedness conditions for the regression constants yields asymptotic normality of linear rank order statistics also for nonstationary and dependent random variables (including e.g., outlier models).

To that purpose we prove in Section 2 a theorem which allows us to obtain weak convergence for processes with values in the generalized Skorohod space w.r.t. stronger metrics than the generalized Skorohod metric. By a generalization of the Birnbaum–Marshall inequality and by a theorem of Garsia [5] it can be shown, that the assumption of this theorem is fulfilled in many interesting situations.

In Section 3 we apply this method to the multivariate sequential empirical process and obtain in the special case of i.i.d. random variables weak convergence w.r.t. functions $r(s, t_1, \cdots, t_k)$ of the type $(s \prod_{i=1}^k t_i(1 - \prod_{i=1}^k t_i))^{\frac{1}{2}-\varepsilon}$ which is for $k = 1$ essentially the result of Wellner [13]. By means of a Pyke–Shorack type representation of the multivariate sequential rank order process as function of the random time transformed empirical process we obtain further weak convergence results for the rank order process under stronger metrics.

In Section 4 we apply these results to the weighted empirical cumulatives and

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to a weighted rank order process which has been treated in the one-dimensional case by Hájek, Šidák [6] and Koul, Stoudte [7].

Finally in Section 5 we derive asymptotic normality of general linear rank order statistics; we do not assume only a finite number of discontinuities or a product form of the scores functions as, for example, in [10], but we do need a differentiability condition for the df's. For the proof we use a new representation of multivariate rank order statistics, which allows application of results for the sequential rank order process.

2. Weak convergence in stronger metrics. Let (D_k, d_k) denote the generalized Skorohod space of 'left side' continuous functions with discontinuities only of the first kind (for definition cf. [2]). The following lemma is the main tool for proving weak convergence for a D_k-valued process $(X_n)_{n\in \mathbb{N}_0}$ w.r.t. stronger metrics than d_k. It generalizes Theorem 2 of Chentsov [3] and weakens his assumptions a bit. For $r\colon [0, 1]^k \to \mathbb{R}_+$, r continuous, define the measurable set

$$U(r) = \{g \in D_k;\ |g| < r\}\,. \tag{2.1}$$

LEMMA 2.1. *Let $r\colon [0, 1]^k \to \mathbb{R}_+$ be continuous, let $X_n \Rightarrow X_0$ on (D_k, d_k) (weak convergence) and let X_0 have continuous finite dimensional distributions, then*

$$\lim_{n\to\infty} P(X_n \in U(r)) = P(X_0 \in U(r))\,. \tag{2.2}$$

PROOF. The proof of Theorem 2 of Chentsov [3] can be generalized to the present situation to yield (2.2), observing that Chentsov's condition (17) can be weakened to the tightness of $(X_n)_{n\in\mathbb{N}_0}$ (cf. relation (36) of Chentsov [3]) and further that Chentsov's condition (18) can be replaced by the assumption of continuous finite dimensional distributions since in this case, relation (29) of Chentsov is fulfilled, while relation (32) of Chentsov is fulfilled automatically by the new definition of $U(r)$ and the monotonicity of probability measures. □

THEOREM 2.2. *Let $r\colon [0, 1]^k \to \mathbb{R}_+$ be continuous, $r > 0$ on $(0, 1]^k$ and let further $X_n \Rightarrow X_0$ on (D_k, d_k), where X_0 is* a.s. *continuous, has continuous finite dimensional distributions and is such that for $\delta > 0$*

$$\lim_{\vartheta\to 0,\,\vartheta>0} P\left(\sup_{0\le t\le\vartheta} \frac{|X_0(t)|}{r(t)} \geqq \delta\right) = 0\,, \tag{2.3}$$

then

$$X_n \Rightarrow X_0 \quad on \quad (D_k, d_r)\,, \qquad where\ \ d_r(f, g) = d_k(f/r, g/r)\,. \tag{2.4}$$

PROOF. According to Skorohod's theorem (cf. [4], Theorem 3), there are a.s. converging versions of $(X_n)_{n\in\mathbb{N}_0}$ w.r.t. the uniform metric ρ on $[0, 1]^k$. Working with these versions and using that $r(t) \geqq \delta' > 0$ for $t \notin [0, \vartheta)$ we get

$$\begin{aligned} \rho_r(X_n, X_0) &= \rho\left(\frac{X_n}{r}, \frac{X_0}{r}\right) \\ &\leqq \sup_{0\le t\le\vartheta} \frac{|X_n(t)| + |X_0(t)|}{r(t)} + \frac{1}{\delta'}\,\rho(X_n, X_0)\,. \end{aligned} \tag{2.5}$$

By assumption the second term converges to 0 a.s. Further (2.2) applied to $U(\varepsilon r/2)$ implies for $n \geqq n_0$.

$$(2.6) \qquad P\left(\sup_{0\leqq t\leqq \vartheta} \frac{|X_n(t)| + |X_0(t)|}{r(t)} \geqq \varepsilon\right) \leqq 2P\left(\sup_{0\leqq t\leqq \vartheta} \frac{|X_0(t)|}{r(t)} \geqq \frac{\varepsilon}{2}\right) + \delta$$

which can be made small by (2.3) for suitable ϑ. (2.5) and (2.6) imply weak convergence of X_n to X_0 on (D_k, d_r). □

The following two lemmas supply conditions under which (2.3) is satisfied. The first lemma generalizes the Birnbaum–Marshall inequality and is concerned with martingales (MG) or positive submartingales (SMG) with multidimensional time. The proof of this lemma is based essentially on a recent inequality derived by Shorack and Smythe [12]. In this connection $(Y(t))_{t\in[0,1]^k}$ is called a SMG (MG) if

$$(2.7) \qquad E(Y(t)\,|\,\mathfrak{U}_s) \geqq Y(t \wedge s) \qquad (\text{'=' in the MG case})$$

for all $t, s \in [0, 1]^k$, where the infimum $t \wedge s$ is to be understood coordinatewise and where

$$\mathfrak{U}_s = \mathfrak{U}(Y(u);\, u \in [0, 1]^k, u \leqq s)\,.$$

We assume that $\mathfrak{U}_s$ has a product representation for all s. Let Δ denote the multidimensional difference operator and let $\mathfrak{F}_k$ denote the class of all separable MG's or SMG's Y with k-dimensional time $[0, 1]^k$, such that $EY(t) = 0$ and $\Gamma(t) = EY^2(t)$ is of bounded variation. By induction in k it can be shown, that

$$(2.8) \qquad \begin{aligned} &\Delta\Gamma \geqq E(\Delta Y)^2 \geqq 0 \quad \text{if} \quad Y \quad \text{is a} \quad \text{SMG} \quad \text{and} \\ &\Delta\Gamma = E(\Delta Y)^2 \quad \text{if} \quad Y \quad \text{is a} \quad \text{MG}. \end{aligned}$$

Therefore for $Y \in \mathfrak{F}_k$ with $\Gamma(t) = EY^2(t)$ the following definition is meaningful.

$$(2.9) \qquad \mathfrak{R}_\Gamma = \{r\colon [0, 1]^k \to \mathbb{R}_+;\ r \text{ continuous, } \Delta r \geqq 0 \text{ and } \textstyle\int_{[0,1]^k} r^{-2}(t)\, d\Gamma(t) < \infty\}\,.$$

LEMMA 2.3. *For $Y \in \mathfrak{F}_k$ with $\Gamma(t) = EY^2(t)$ and for $r \in \mathfrak{R}_\Gamma$*

$$(2.10) \qquad P\left(\sup_{t\in[0,1]^k} \frac{|Y(t)|}{r(t)} \geqq 1\right) \leqq \int_{[0,1]^k} r^{-2}(t)\, d\Gamma(t) < \infty\,.$$

PROOF. For $l = (l_1, \cdots, l_k) \in \mathbb{N}^k$, $c \in \mathbb{R}_+$ define $cl = (cl_1, \cdots, cl_k)$ and further $e_k = (1, \cdots, 1) \in \mathbb{R}_k$. Continuity of r and separability of Y imply

$$(2.11) \qquad \begin{aligned} &P\left(\sup_{t\in[0,1]^k} \frac{|Y(t)|}{r(t)} \geqq 1\right) \\ &\quad = \lim_{h\to\infty} P\left(\max_{l\in\mathbb{N}^k, l\leqq 2^h e_k} \frac{|Y(l/2^h)|}{r(l/2^h)} \geqq 1\right) \\ &\quad = \lim_{h\to\infty} P\left(\max_{l\leqq 2^h e_k} \frac{|\sum_{i\leqq l} \Delta^{i/2^h}_{(i-e_k)/2^h} Y|}{r(l/2^h)} \geqq 1\right). \end{aligned}$$

According to (8) in Shorack–Smythe [12] we get

$$(2.12)\qquad P\left(\sup \frac{|Y(t)|}{r(t)} \geqq 1\right) \leqq \lim_{h\to\infty} \sum_{l\leqq 2^h e_k} r^{-2}\left(\frac{l}{2^h}\right) E(\Delta^{l/2^h}_{(l-e_k)/2^h} Y)^2$$
$$\leqq \int_{[0,1]^k} r^{-2}(t)\, d\Gamma(t) \quad \text{by (2.8).} \qquad \square$$

The second lemma concerning condition (2.3) is useful when X_0 has a Gaussian distribution. Let $(Y(t))_{t\in[0,1]^k}$ be a Gaussian process with mean 0 and continuous covariance function K and let $p = p_K$ be defined by

$$(2.13)\qquad p(u) = \max_{|t-s|\leqq uk^{\frac12}} E^{\frac12}(Y(t) - Y(s))^2 ,$$
$$= \max_{|t-s|\leqq uk^{\frac12}} (K(t, t) + K(s, s) - 2K(s, t))^{\frac12} , \qquad 0 \leqq u \leqq 1 ,$$

where $|t - s| = \max |t_i - s_i|$.

The proof of the following lemma is immediate from Theorem 1 of Garsia [4].

LEMMA 2.4. *If* $\int_0^1 (\log 1/u)^{\frac12}\, dp(u) < \infty$, *then for a version* $(X(t))_{t\in[0,1]^k}$ *of the process* $(Y(t))_{t\in[0,1]^k}$

$$(2.14)\qquad \sup_{|t-s|\leqq\nu} |X(t) - X(s)| \leqq 16(\log B)^{\frac12} p(\nu) + 16(2k)^{\frac12} \int_0^\nu \left(\log \frac{1}{u}\right)^{\frac12} dp(u)$$

where $EB \leqq 4(2)^{\frac12}$, $B \geqq 0$.

3. The sequential empirical process and the multivariate rank order process. For every $n \geqq 1$ let $X_{nj} = (X^j_{n1}, \cdots, X^j_{nk})$, $1 \leqq j \leqq n$ be k-dimensional random variables with continuous df's $F_n{}^j$, F^j_{ni} of X_{nj}, X^j_{ni} respectively. Let F_{ni} denote the empirical df of $(X^1_{ni}, \cdots, X^n_{ni})$, $\bar F_{ni} = n^{-1} \sum_{j=1}^n F^j_{ni}$ and let further $R^j_{ni} = nF_{ni}(X^j_{ni})$ be the rank of X^j_{ni} in the corresponding n-tuple. Then we define the multivariate rank order process $L_n(s, t)$ for $s \in [0, 1]$ and $t = (t_1, \cdots, t_k) \in [0, 1]^k$ by

$$(3.1)\qquad L_n(s, t) = n^{-\frac12} \sum_{j=1}^{[ns]} \{I(R^j_{n1} \leqq nt_1, \cdots, R^j_{nk} \leqq nt_k) - H_{nj}(t)\}$$

where $I(A_1, \cdots, A_k)$ is 1 if $A_1, \cdots, A_k$ are fulfilled and 0 else, and where

$$H_{nj}(t) = F_n{}^j(\bar F^{-1}_{n1}(t_1), \cdots, \bar F^{-1}_{nk}(t_k)) .$$

In order to prove weak convergence results of the D_{k+1}-valued process L_n we apply in the first part of this section the results of Section 2 to the multivariate sequential empirical process V_n defined by

$$(3.2)\qquad V_n(s, t) = n^{-\frac12} \sum_{j=1}^{[ns]} \{I(\bar F_{n1}(X^j_{n1}) \leqq t_1, \cdots, \bar F_{nk}(X^j_{nk}) \leqq t_k) - H_{nj}(t)\} .$$

The following theorem is immediate from an obvious modification of Theorem 2.2 and from Lemma 2.4.

THEOREM 3.1. *If* V_n *converges weakly to an* a.s. *continuous Gaussian process* V_0 *with continuous covariance function* K_0, $K_0(t, t) \neq 0$ *for* t *in the interior of* $[0, 1]^{k+1}$, *if further* $r\colon [0, 1]^{k+1} \to \mathbb{R}_+$ *is continuous*, $r > 0$ *in the interior of* $[0, 1]^k$ (*cf.* *Remark* 3.1 c) *and if*

$$(3.3)\qquad \int_0^1 \left(\log \frac{1}{u}\right)^{\frac12} dp_M(u) < \infty ,$$

where $M(s, t) = K_0(s, t)/r(s)r(t)$, *then*

$$V_n \Rightarrow V_0 \quad \text{on} \quad (D_{k+1}, d_r) . \tag{3.4}$$

REMARK 3.1.

a) Weak convergence of V_n w.r.t. (D_k, d_k) has been proved under general conditions in the literature. If for example, K_n denotes the covariance function of V_n, and if K_n is pointwise convergent to K_0 and if further there exists a measure defining function Γ of bounded variation such that $\Delta H_{nj} \leqq \Delta\Gamma$, then $V_n \Rightarrow V_0$ on (D_{k+1}, d_{k+1}) by Theorem 1 of Rüschendorf [9] in the φ-mixing case. In the special case of independent random variables, where

$$K_n((s_1, t_1), (s_2, t_2)) = \frac{1}{n} \sum_{j=1}^{[n(s_1 \wedge s_2)]} (H_{nj}(t_1 \wedge t_2) - H_{nj}(t_1)H_{nj}(t_2)) ,$$

the assumption $K_n \to K_0$ is also necessary by a result of Neuhaus [8]; this is true since

$$\lim K_n((s_1, t_1), (s_2, t_2)) = K_0((s_1, t_1), (s_2, t_2))$$

if and only if $\lim_{n\to\infty} K_n((1, t_1), (1, t_2)) = K_0((1, t_1), (1, t_2))$, and in this case $K_0((s_1, t_1), (s_2, t_2)) = (s_1 \wedge s_2)K_0((1, t_1), (1, t_2))$.

b) In the independent stationary case we have

$$K_0((s_1, t), (s_2, v)) = (s_1 \wedge s_2)\{\textstyle\prod_{i=1}^k (t_i \wedge v_i) - \prod_{i=1}^k t_i v_i\} . \tag{3.5}$$

With $r(s_1, t_1, \cdots, t_k) = [s_1 \prod_{i=1}^k t_i(1 - \prod_{j=1}^k t_j)]^{(1-\varepsilon)/2}$ and $s = (s_1, t_1, \cdots, t_k)$, $t = (s_1 + h_1, \cdots, t_k + h_k)$, $M(s, t) = K_0(s, t)/r(s)r(t)$

$$\begin{aligned} &M(t, t) + M(s, s) - 2M(s, t) \\ &\quad \leqq (s_1 + h_1)^\varepsilon \textstyle\prod_{i=1}^k (t_i + h_i)^\varepsilon - s_1^\varepsilon \prod_{i=1}^k t_i^\varepsilon \\ &\quad \leqq ((s_1 + h_1) \textstyle\prod_{i=1}^k (t_i + h_i) - s_1 \prod_{i=1}^k t_i)^\varepsilon \leqq K \max_{1\leqq i\leqq k} h_i^\varepsilon . \end{aligned} \tag{3.6}$$

Therefore,

$$p_M(u) \leqq K' u^{\varepsilon/2} \qquad \text{which implies} \quad \int_0^1 \log\left(\frac{1}{u}\right)^{\frac{1}{2}} dp_M(u) < \infty .$$

This example generalizes a recent result of Wellner [13] for the case $k = 1$. It shows, that in this case both methods lead to results of about equal strength.

c) Define $B = \{(t_1, \cdots, t_{k+1}); \exists\, i, t_i = 0\}$ and define the interior of $[0, 1]^{k+1}$ by $[0, 1]^{k+1} - (B \cup e_{k+1})$. □

In the one-dimensional independent case we can also use Lemma 2.3 to obtain weak convergence w.r.t. stronger metrics. Let v denote the df of the one dimensional Lebesgue measure on $[0, 1]$. For monotone increasing functions Γ on $[0, 1]$ of bounded variation define

$$\begin{aligned} Q_\Gamma = \{r\colon [0, 1]^2 \to \mathbb{R}^+,\ &r \text{ continuous},\ \Delta r_i \geqq 0,\ i = 1, 2 \\ &\text{where } r_1(t_1, t_2) = r(t_1, t_2),\ r_2(t_1, t_2) = r(t_1, 1 - t_2) \\ &\text{for } t_1 \leqq 1,\ t_2 \leqq \tfrac{1}{2},\ \textstyle\int_{[0,1]^2} r^{-2}(t)\, dv\Gamma(t) < \infty\} . \end{aligned} \tag{3.7}$$

THEOREM 3.2. *Let $(X_{nj})_{1\leq j\leq n}$ be independent, one-dimensional and let K_n be pointwise convergent to K_0. If there exists a $\delta > 0$ and a monotone increasing continuous function Γ on $[0, 1]$ of bounded variation such that*

$$H_{nj}(\delta) \leq \tfrac{1}{2}, \qquad H_{nj}(1-\delta) \geq \tfrac{1}{2}, \qquad \forall j, n \tag{3.8}$$

$$H_{nj}(t_1) - H_{nj}(t_2) \leq \Gamma(t_1) - \Gamma(t_2), \qquad \forall t_2 \leq t_1, \tag{3.9}$$

then there exists an a.s. *continuous Gaussian process V_0, such that*

$$V_n \Rightarrow V_0 \quad \text{on} \quad (D_2, d_r) \quad \text{for} \quad r \in Q_\Gamma. \tag{3.10}$$

PROOF. By Theorem 1 in [9], there exists an a.s. continuous Gaussian process V_0, such that $V_n \Rightarrow V_0$ on (D_2, d_2). Now

$$W_n(s, t) = n^{-\frac{1}{2}} \textstyle\sum_{j=1}^{[ns]} \dfrac{I\{\bar{F}_{n1}(X_{n1}^j) \leq t) - H_{nj}(t)\}}{1 - H_{nj}(t)}$$

is a martingale. Therefore, by (3.7) and Lemma 2.3 for $\vartheta_2 \leq \delta$

$$P\left(\sup_{0\leq t\leq \vartheta_2, 0\leq s\leq \vartheta_1} \frac{|V_n(s,t)|}{r(s,t)} \geq \lambda\right) \leq P\left(\sup_{0\leq t\leq \vartheta_2, 0\leq s\leq \vartheta_1} \frac{|W_n(s,t)|}{r(s,t)} \geq \frac{\lambda}{2}\right) \tag{3.11}$$
$$\leq \frac{16}{\lambda^2} \int_{[0,1]^2} r^{-2}(s,t)\, dv\Gamma(s,t).$$

(3.11) implies (3.10) similarly to the proof of Wellner [13] in the stationary case. □

REMARK 3.2.

a) The technique of the proof of Theorem 2.6—which is essentially the same as those of Wellner—does not generalize to the k-dimensional case, since the MG (or SMG) property is not fulfilled for $k > 1$.

b) If $1/n \sum_{j=1}^n H_{nj}(t)$ converges to $H(t)$ for all t, the result of Theorem 2.6 can be sharpened using Theorem 2.2 to yield convergence for $r \in Q_H \supset Q_\Gamma$. For the proof observe, that under this additional assumption W_n converges weakly to an a.s. continuous martingale. □

To apply the results to the multivariate rank order process L_n, we need the following two assumptions.

ASSUMPTION A. The reduced empirical process V_n converges weakly on (D_{k+1}, d_{k+1}) to an a.s. continuous Gaussian process V_0 with continuous covariance function K_0, such that $K_0(t, t) \neq 0$ for t in the interior of $[0, 1]^{k+1}$.

ASSUMPTION B. There exist partial derivatives l_{ij}^n, $1 \leq i \leq k$ for H_{nj}, $1 \leq j \leq n$ such that with $l_i^n = 1/n \sum_{j=1}^n l_{ij}^n$

$$\lim_{n\to\infty} l_i^n(u) = l_i(u), \qquad 1 \leq i \leq k \tag{3.12}$$

uniformly in $u \in [0, 1]^k$ and l_i is continuous on $[0, 1]^k$.

THEOREM 3.3. *If conditions A, B are satisfied, then the multivariate rank order* *process L_n converges in distribution to the* a.s. *continuous Gaussian process*

$$L_0(s,t) = V_0(s,t) - s\sum_{i=1}^{k} l_i(t)V_0(1,1,\cdots,t_i,\cdots,1) \tag{3.13}$$

on (D_{k+1}, d_r) for $r\colon [0,1]^{k+1} \to \mathbb{R}_+$, r continuous such that $r > 0$ in the interior of $[0,1]^{k+1}$ and

$$p_M \quad \textit{fulfills (3.3), where} \quad M(u,\nu) = \frac{EL_0(u)L_0(\nu)}{r(u)r(\nu)}. \tag{3.14}$$

PROOF. L_n has with probability 1 the following representation

$$\begin{aligned} L_n(s,t) &= V_n(s, \bar F_{n1}\circ F_{n1}^{-1}(t_1), \cdots, \bar F_{nk}\circ F_{nk}^{-1}(t_k)) \\ &\quad + n^{-\frac12}\sum_{j=1}^{[ns]}\{F_n{}^j(F_{n1}^{-1}(t_1),\cdots,F_{nk}^{-1}(t_k)) - H_{nj}(t)\} \end{aligned} \tag{3.15}$$

where F_{ni}^{-1} is defined by

$$\begin{aligned} F_{ni}^{-1}(t_i) &= -\infty &&\text{for } t_i \in \left[0, \frac1n\right) \\ &= X_{k:n}^i &&\text{for } t_i \in \left[\frac kn, \frac{k+1}{n}\right) \\ &= X_{n:n}^i &&\text{for } t_i = 1 \end{aligned} \tag{3.16}$$

$X_{k:n}^i$ denotes the kth order statistic of $\{X_{ni}^1, \cdots, X_{ni}^k\}$.
According to the Skorohod theorem there exist a.s. converging versions of $(V_n)_{n\in\mathbb{N}_0}$ w.r.t. ρ which lead to versions of F_{ni} and (in virtue of (3.15)) of L_n with this property. Working with these constructions, we get by Taylor approximation of first order

$$\begin{aligned} &\sum_{j=1}^{[ns]}\{F_n{}^j(F_{n1}^{-1}(t_1),\cdots,F_{nk}^{-1}(t_k)) - H_{nj}(t)\} \\ &\quad = \sum_{j=1}^{[ns]}\{H_n{}^j(\bar F_{n1}\circ F_{n1}^{-1}(t_1),\cdots,\bar F_{nk}\circ F_{nk}^{-1}(t_k)) - H_{nj}(t)\} \\ &\quad = \sum_{i=1}^{k}(\bar F_{ni}\circ F_{ni}^{-1}(t_i) - t_i)\sum_{j=1}^{[ns]} l_{ij}^n(\mu_n), \end{aligned} \tag{3.17}$$

where μ_n lies in the closed interval

$$\begin{aligned} &[(\min\{\bar F_{n1}\circ F_{n1}^{-1}(t_1), t_1\}, \cdots, \min\{\bar F_{nk}\circ F_{nk}^{-1}(t_k), t_k\}), \\ &\qquad (\max\{\bar F_{n1}\circ F_{n1}^{-1}(t_1), t_1\}, \cdots, \max\{\bar F_{nk}\circ F_{nk}^{-1}(t_k), t_k\})]. \end{aligned}$$

We further have with probability 1 the following relation

$$\begin{aligned} &n^{\frac12}(\bar F_{ni}\circ F_{ni}^{-1}(t_i) - t_i) \\ &\quad = n^{\frac12}(\bar F_{ni}\circ F_{ni}^{-1}(t_i) - F_{ni}\circ F_{ni}^{-1}(t_i)) + n^{\frac12}(F_{ni}\circ F_{ni}^{-1}(t_i) - t_i) \\ &\quad = -V_n(1,1,\cdots,\bar F_{ni}\circ F_{ni}^{-1}(t_i),\cdots,1) + o(n^{-\frac12}). \end{aligned} \tag{3.18}$$

Now $\lim_{n\to\infty}\rho(\bar F_{ni}\circ F_{ni}^{-1}, 1) = 0$ a.s. and continuity of V_0 w.r.t. ρ imply

$$|V_n(s, \bar F_{n1}\circ F_{n1}^{-1}, \cdots, \bar F_{nk}\circ F_{nk}^{-1}(t_k)) - V_0(s,t)| \leqq \rho(V_n, V_0) + o(1). \tag{3.19}$$

Further $\lim_{n\to\infty}|V_n(1,\cdots,\bar F_{ni}\circ F_{ni}^{-1}(t_i),\cdots,1) - V_0(1,\cdots,t_i,\cdots,1)| = 0$ uniformly in t_i and (3.12) imply

$$\begin{aligned} \lim_{n\to\infty}|V_n(1,\cdots,\bar F_{ni}\circ F_{ni}^{-1}(t_i),\cdots,1)l_i{}^n(\mu_n) \\ - sV_0(1,\cdots,t_i,\cdots,1)l_i(t)| = 0 \end{aligned} \tag{3.20}$$

uniformly in s, t_i.

(3.15)—(3.20) imply $L_n \Rightarrow L_0$ on (D_{k+1}, d_{k+1}).

Now Theorem 2.2 and Lemma 2.4 can be applied to yield the result. □

REMARK 3.3.

a) Assumption B is satisfied in the stationary case for $k = 1$ with $l_1(t) = 1$ and in the independent stationary case for $k > 1$ with $l_i(t) = \prod_{j \neq i} t_j$. For $k = 2$ it is e.g. satisfied in the stationary case, when F is a bivariate normal distribution.

b) In the independent stationary case, $k \geqq 1$, (3.14) is fulfilled for

$$r(s, t_1, \cdots, t_k) = (s \prod_{i=1}^k t_i(1 - \prod_{j=1}^k t_j))^{\frac{1}{2}-\varepsilon} \qquad \text{(cf. Remark 3.1b).}$$

4. The weighted empirical cumulatives and the weighted rank order process. The weighted empirical cumulative function U_n^c is defined by

$$(4.1) \qquad U_n^c(t) = n^{-\frac{1}{2}} \sum_{j=1}^n c_{nj}\{I(X_{n1}^j \leqq F_{n1}^{-1}(t), \cdots, X_{nk}^j \leqq F_{nk}^{-1}(t_k)) - H_{nj}(t)\}.$$

Defining discrete signed measures μ_n on $[0, 1]\mathfrak{B}^1$ by

$$(4.2) \qquad \mu_n\left(\frac{i}{n}\right) = c_{ni}^*, \qquad 1 \leqq i \leqq n,$$

where c_{ni}^* are obtained inductively from $c_{ni} = \sum_{k=i}^n c_{nk}^*$, we get the following representation of U_n^c

$$(4.3) \qquad U_n^c(t) = \int_{[0,1]} V_n(s, t)\, d\mu_n(s).$$

Defining for $r: [0, 1] \rightarrow \mathbb{R}$ and for a signed measure μ_0

$$(4.4) \qquad \mu_n \rightarrow_r \mu_0 \quad \text{if} \quad \lim_{n\rightarrow\infty} \int fr\, d\mu_n = \int fr\, d\mu_0$$

for all $f \in C[0, 1]$ and if further $\int r d|\mu_n| \leqq M < \infty$ for all $n \in \mathbb{N}_0$, where $|\mu_n|$ denotes the measure of total variation, we get the following theorem.

THEOREM 4.1. *If Assumption* A *is fulfilled and if there exists a signed measure* μ_0 *such that* $\mu_n \rightarrow_{r_1} \mu_0$, *then the weighted empirical cumulative process* U_n^c *converges weakly to the* a.s. *continuous Gaussian process*

$$(4.5) \qquad U_0^c(t) = \int_{[0,1]} V_0(s, t)\, d\mu_0(s) \qquad on \quad (D_k, d_{r_2})$$

for $r_2: [0, 1]^k \rightarrow \mathbb{R}_+$, $r_2 > 0$ *in the interior of* $[0, 1]^k$ *and such that* $r = r_1 r_2$ *fulfills* (3.4).

PROOF. By Assumption A, V_n converges weakly to V_0 on $(D_{k+1}, d_{r_1 r_2})$. Using a.s. converging versions w.r.t. $\rho_{r_1 r_2}$, Theorem 10 follows from the condition $\mu_n \rightarrow_{r_1} \mu_0$, the a.s. continuity of V_0 w.r.t. $\rho_{r_1 r_2}$ (cf. Lemma 2.4) and the following inequality

$$(4.6) \qquad \int V_n\, d\mu_n - \int V_0\, d\mu_0 \leqq \rho_{r_1}(V_n, V_0) \int r_1\, d|\mu_n| + \int \frac{V_0}{r_1} r_1\, d(\mu_n - \mu_0). \qquad \square$$

As further application we consider the asymptotic behaviour of the following weighted rank order statistic

$$(4.7) \qquad T_n^c(t) = n^{-\frac{1}{2}} \sum_{j=1}^n c_{nj}\{I(R_{n1}^j \leqq nt_1, \cdots, R_{nk}^j \leqq nt_k) - H_{nj}(t)\}.$$

The one dimensional case $T_n^c(t_1)$ (with a different normalization) has been treated by Hájek and Sidák [6] in the i.i.d. case and by Koul and Stoudte [7] in the nonstationary, independent case. Using the a.s. representation

$$T_n^c(t) = \int_{[0,1]} L_n(s, t)\, d\mu_n(s) \tag{4.8}$$

and Theorem 3.3, the proof of the following theorem is analogously to that of Theorem 4.1.

THEOREM 4.2. *If Assumptions* A, B *are fulfilled and if there exists a signed measure* μ_0, *such that* $\mu_n \to_{r_1} \mu_0$ *and* (3.4) *is fulfilled for* $r = r_1 r_2$ *then* T_n^c *converges weakly to the* a.s. *continuous Gaussian process*

$$T_0^c(t) = \int_{[0,1]} L_0(s, t)\, d\mu_0(s) \qquad on \quad (D_k, d_{r_2})\,. \tag{4.9}$$

REMARK 4. Let under the conditions of Theorem 4.2 the covariance function K_n of V_n converge pointwise to K_0, where K_0 is of the type $K_0((s_1, t_1), (s_2, t_2)) = (s_1 \wedge s_2)\Gamma(t_1, t_1)$. Then $T_0^c(t)$ has a $N(0, \sigma^2)$ distribution with

$$\begin{aligned} \sigma^2 = {} & \left(\int_{[0,1]} \int_{[0,s]} u\, d\mu_0(u)\, d\mu_0(s)\right)\Gamma(t, t) + \left(\int_{[0,1]} u\, d\mu_0(u)\right)^2 \\ & \times \{\textstyle\sum_{i,j} l_i(t) l_j(t) \Gamma((1, \cdots, t_i, \cdots, 1), (1, \cdots, t_j, \cdots, 1)) \\ & - 2 \textstyle\sum_i l_i(t) \Gamma(t, (1, \cdots, t_i, \cdots, 1))\}\,. \end{aligned} \tag{4.10}$$

Using (4.4), (3.3)

$$\begin{aligned} 2 \int_{[0,1]} \int_{[0,s]} u\, d\mu_0(u)\, d\mu_0(s) &= \int_{[0,1]} \int_{[0,1]} s_1 \wedge s_2\, d\mu_0(s_1)\, d\mu_0(s_2) \\ &\leqq \int_{[0,1]} \int_{[0,1]} \frac{s_1}{r_1(s_1)} \wedge \frac{s_2}{r_1(s_2)} r_1(s_1) r_1(s_2)\, d\mu_0(s_1)\, d\mu_0(s_2) \\ &\leqq C\left(\int_{[0,1]} r_1(s_1)\, d|\mu_0(s_1)|\right)^2 < \infty\,. \end{aligned}$$

In the same way

$$\left(\int_{[0,1]} u\, d\mu_0(u)\right)^2 < \infty$$

and, therefore, $\sigma^2 < \infty$.

5. A direct approach to convergence results. The aim of this section is to treat rank order statistics of the type

$$\tilde{S}_n^c = \sum_{j=1}^n c_{nj} \varphi_n\left(\frac{R_{n1}^j}{n}, \cdots, \frac{R_{nk}^j}{n}\right) \tag{5.1}$$

with regression constants c_{nj} and scores functions φ_n. For the proof of convergence we use the following representation

$$\begin{aligned} \varphi_n & \left(\frac{R_{n1}^j}{n}, \cdots, \frac{R_{nk}^j}{n}\right) \\ &= \textstyle\sum_{(l_1, \cdots, l_k) \leq n e_k} \varphi_n\left(\frac{l_1}{n}, \cdots, \frac{l_k}{n}\right) I(R_{n1}^j = l_1, \cdots, R_{nk}^j = l_k) \\ &= \textstyle\sum_{(l_1, \cdots, l_k) \leq n e_k} \lambda_n\left(\frac{l_1}{n}, \cdots, \frac{l_k}{n}\right) I(R_{n1}^j \leqq l_1, \cdots, R_{nk}^j \leqq l_k)\,, \end{aligned} \tag{5.2}$$

with

$$(5.3)\qquad \lambda_n\left(\frac{l_1}{n}, \cdots, \frac{l_k}{n}\right) = \Delta^{l_1/n}_{(l_1+1)/n} \cdots \Delta^{l_k/n}_{(l_k+1)/n}\varphi_n \qquad \text{for} \quad (l_1, \cdots, l_k) \neq e_k\,,$$
$$= \varphi_n\left(\frac{e_k}{n}\right) \qquad \text{for} \quad (l_1, \cdots, l_k) = e_k\,,$$

where in the right hand side expression each one-dimensional Δ-operator with upper index n/n is omitted. The last equality in (5.2) can be shown by induction in k. Define

$$(5.4)\qquad S_n^c = n^{-\frac{1}{2}}(\tilde{S}_n^c - a_n)$$

with

$$a_n = \sum_{j=1}^n c_{nj} \sum_{(l_1,\cdots,l_k)} H_{nj}\left(\frac{l_1}{n}, \cdots, \frac{l_k}{n}\right).$$

For $r\colon [0, 1]^k \to \mathbb{R}$ and for a signed measure λ_0 define $\lambda_n \to_r \lambda_0$ similarly to (4.4); then

THEOREM 5.1. *If Assumptions* A, B *are fulfilled and if there exist signed measures* μ_0, λ_0 *such that* $\mu_n \to_{r_1} \mu_0$, $\lambda_n \to_{r_2} \lambda_0$ *and* (3.14) *is fulfilled for* $r = r_1 r_2$, *then*

$$(5.5)\qquad S_n^c \to N(0, \sigma^2)$$

with

$$\sigma^2 = E\left(\int_{[0,1]^{k+1}} L_0(s, t)\, d\lambda_0(t)\, d\mu_0(s)\right)^2 .$$

PROOF. The proof follows from Theorem 3.3, using the a.s. representation

$$(5.6)\qquad S_n^c = \int_{[0,1]^{k+1}} L_n(s, t)\, d\lambda_n(t)\, d\mu_n(s)\,. \qquad \square$$

EXAMPLE 1.

a) Consider the simple linear rank statistic for $k = 1$ with $\varphi_n(x) = x$ for $x \in [0, 1]$, then

$$(5.7)\qquad \lambda_n\left(\frac{k}{n}\right) = \varphi_n\left(\frac{k}{n}\right) - \varphi_n\left(\frac{k+1}{n}\right) \quad k \leqq n-1$$
$$= \varphi_n(1) \qquad k = n$$
$$= -\frac{1}{n} \quad 1 \leqq k \leqq n-1$$
$$= 1 \qquad k = n\,.$$

Therefore, $|\lambda_n| \leqq 2$, $n \in \mathbb{N}_0$, and $\int f\, d\lambda_n \to \int f\, d\lambda_0$, $f \in C[0, 1]$, with $\lambda_0 = -\lambda_1 + \sigma_1$, where λ_1 is the restriction of the Lebesgue measure on $[0, 1]$, and where σ_1 is the one dimensional one point measure in 1.

b) In the case $k = 2$ we define a Spearman type rank correlation statistic by letting $\varphi_n(t_1, t_2) = t_1 t_2$. In this case

$$
\begin{aligned}
\lambda_n\left(\frac{k}{n}, \frac{l}{n}\right) &= \varphi_n\left(\frac{k}{n}, \frac{l}{n}\right) - \varphi_n\left(\frac{k+1}{n}, \frac{l}{n}\right) \\
&\quad - \varphi_n\left(\frac{k}{n}, \frac{l+1}{n}\right) + \varphi_n\left(\frac{k+1}{n}, \frac{l+1}{n}\right) \qquad \text{if } k, \ l \leqq n-1 \\
&= \varphi_n\left(1, \frac{l}{n}\right) - \varphi_n\left(1, \frac{l+1}{n}\right) \qquad \text{if } k = n, \ l \leqq n-1 \\
&= \varphi_n\left(\frac{k}{n}, 1\right) - \varphi_n\left(\frac{k+1}{n}, 1\right) \qquad \text{if } k \leqq n-1, \ l = n \\
&= \varphi_n(1, 1) \qquad \text{if } k = l = n\,. \\
&= \frac{1}{n^2} I(k \leqq n-1, l \leqq n-1) - \frac{1}{n}\{I(k = n, l \leqq n-1) \\
&\quad + I(k \leqq n-1, l = n)\} + I(k = n, l = n)\,.
\end{aligned} \tag{5.8}
$$

Therefore, $|\lambda_n| \leqq 4$ for all $n \in \mathbb{N}_0$ and

$$\int_{[0,1]^2} f\, d\lambda_n \to \int_{[0,1]^2} f\, d(\lambda_2 - \lambda_2^1 + \sigma_{\{1,1\}}) \qquad \text{for} \quad f \in C[0, 1]^2\,,$$

where λ_2 is the restriction of the 2-dimensional Lebesgue measure on $[0, 1]^2$, λ_2^1 is the 1-dimensional Lebesgue measure restricted on $\{1\} \times [0, 1] \cup [0, 1] \times \{1\}$ and $\sigma_{\{1,1\}}$ is the one point P-measure concentrated in $(1, 1)$.

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