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BOREL CROSS-SECTIONS AND MAXIMAL INVARIANTS

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A measurable cross-section for orbits of a sample space under a free (exact) transformation group is shown to exist under topological regularity conditions. This is used to represent the sample space as essentially the product of a maximal invariant and an equivariant part, which implies Stein's representation for the density of the maximal invariant.

0. Introduction. An increasingly common device in the study of invariant tests and equivariant estimators (e.g., [3], [6], [9], [13], [14], [19], [20], [22], [23] and [25]) is to represent the sample space $\mathscr{X}$ as a cartesian product of a single orbit Gx and the space $\mathscr{X}/G$ of values of the maximal invariant. In the present paper we prove two general representation theorems of this type (Theorems 2 and 3) assuming only topological conditions on the orbits. When such representation is valid, many interesting integrals over $\mathscr{X}$ may be obtained by a double integral, first along orbits, and then over the set of orbits. General results of this form, known as *disintegration theorems*, have been extensively studied in the context of integration theory (Bourbaki (1962) Section 3, no. 3); the ones in this paper are more suited to statistical problems invariant under transformation groups. These results are closely related to those of Wijsman ([21], [22], [23]) and of Koehn ([14]).

As an application we prove the validity of Stein's method for finding the marginal density of the maximal invariant by integrating the density of x over the group (Corollary 2 of Theorem 3). The key to this and all other proofs known to the author ([13], [22]) is to find a cross-section for the orbits and use it to get a homeomorphism, locally at least, between $\mathscr{X}$ and $(Gx) \times (\mathscr{X}/G)$. Stein is reported as remarking that since the cross-section does not occur in the statement of the theorem, it is unaesthetic and possibly restrictive to use it in the proof. However, Theorem 2 of this paper shows that if G acts without fixed points, all we need is a *measurable* cross-section, and a result of Baker, Effros and Glimm (Theorem 1) tells us that such a cross-section automatically exists under weak regularity conditions. Thus, use of a cross-section in the proof does not require strong additional hypotheses to ensure its existence.

1. Results and examples. In the following, G is a group of transformations acting on a topological space $\mathscr{X}$. G is said to act *freely* if $g \neq e$ implies $gx \neq x$ for all x in $\mathscr{X}$; equivalently, if the *stability subgroup* of x, namely $G_x = \{g \mid gx = x\}$, is $\{e\}$ (the identity of G) for all x in $\mathscr{X}$. Subgroups H and J of G are *conjugate* if $H = gJg^{-1}$ for some g in G; for any x and g, G_x and G_{gx} are conjugate.

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"*Measurable*" means "Borel-measurable" and the class $\mathscr{B}$ of *Borel* sets is the smallest σ-field generated by the open sets of $\mathscr{X}$. A *bimeasurable* function is a measurable function with measurable inverse. $\mathscr{X}/G$ refers to the set of orbits of $\mathscr{X}$, which is given the quotient topology (S in $\mathscr{X}/G$ is open iff the set of orbits in S form an open set when considered as a subset of $\mathscr{X}$); π is the canonical map $\pi(x) = Gx$. A measure λ is *relatively invariant* with modulus δ if $\lambda(gE) = \delta(g)\lambda(E)$ for g in G and E in $\mathscr{B}$. The orbit of x is $Gx = \{gx \mid g \in G\}$. A Borel *cross-section* is a Borel subset of $\mathscr{X}$ which intersects each orbit Gx precisely once. A *Borel measure* is one defined on the Borel sets which gives finite values to compact sets.

We shall assume that G and $\mathscr{X}$ satisfy

CONDITION A. G is a separable complete metrisable locally compact topological group acting continuously on $\mathscr{X}$ (i.e., the multiplication map $(g, x) \to gx$ is continuous on $G \times \mathscr{X}$); $\mathscr{X}$ is a separable complete metrisable locally compact space.

The special product structure of $\mathscr{X}$ mentioned in the introduction is

CONDITION B. There is a 1–1 bimeasurable mapping $\sigma^{-1}: \mathscr{X} \to (G/H) \times (\mathscr{X}/G)$ where H is a closed subgroup of G; if $\sigma^{-1}(x) = (\tau(x), a(x))$, then $\sigma^{-1}(gx) = (g\tau(x), a(x))$ for all $g \in G$ and $x \in \mathscr{X}$, where τ takes values in G/H and a takes values in $\mathscr{X}/G$.

The startingpoint for the present work is:

THEOREM 1 (Glimm–Baker–Effros). *Let the group G acting on the space $\mathscr{X}$ satisfy assumption A. The following seven statements are then equivalent to each other:*

(i) *$\mathscr{X}/G$ is T_0.*

(ii) *Each orbit is relatively open in its closure.*

(iii) *Each orbit is locally closed in $\mathscr{X}$*; i.e., *every x in $\mathscr{X}$ has an open neighbourhood E such that $Gx \cap E$ is closed in the relative topology of E.*

(iv) *The smallest σ-algebra containing the quotient topology on $\mathscr{X}/G$ is precisely the collection of sets E for which $\pi^{-1}(E)$ is measurable.*

(v) *For each $x \in \mathscr{X}$, the map $gG_x \to gx$ from G/G_x onto the orbit Gx (quotient topology on G/G_x; relative topology on Gx) is a homeomorphism.*

(vi) *Each orbit is locally compact in the relative topology.*

(vii) *There exists a Borel cross-section Z for the orbits of G in $\mathscr{X}$.*

This is the union of Theorem 2.1 (1 and 4), Theorem 2.6 (5, 6) and Theorem 2.9 (12) of Effros ([8]) with the facts that any first countable locally compact group is metrisable and complete ([7], Proposition 12.9.5), that a subset of $\mathscr{X}$ is locally closed iff it is locally compact (Willard [24], Section 18.4) and Effros' comment ([8], page 47) that a first countable locally compact Hausdorff space satisfies his "Condition D." Baker ([2], Theorem 2) shows that completeness of $\mathscr{X}$ is not necessary for equivalence of (i), (ii), (v) and (vii); we shall not pursue such generalisations here.

Using Theorem 1 we will prove the following in Section 2:

THEOREM 2. *If the group G acts freely on the space $\mathscr{X}$, if Condition* A *and any of the conditions* (i) *to* (vii) *of Theorem* 1 *hold, then G and $\mathscr{X}$ satisfy Condition* B *and there exists a Borel cross-section. Further, any Borel cross-section Z is the bi-measurable image of $\mathscr{X}/G$, and if f is a real-valued function which is integrable with respect to a relatively invariant Borel measure m, then*

$$\int_{\mathscr{X}} f(x)\, dm\,(x) = \int_Z \alpha(dz) \int_G f(gz)\delta(g)\mu(dg) \tag{1}$$

for some Borel (hence σ-finite) measure α on Z; μ is a left invariant measure on G and δ is the modulus of m.

We can weaken the assumption that G is free at some cost:

THEOREM 3. *If G is a Lie group of nonzero dimension acting so that all stability subgroups are compact and conjugate to each other; if Condition* A *and any of* (i) *to* (vii) *of Theorem* 1 *hold and m is a relatively invariant Borel measure, then there exists an m-null invariant set $\mathscr{N}$ such that $\mathscr{X} - \mathscr{N}$ and G satisfy Condition* B *and there exists a Borel cross-section Z for the orbits in $\mathscr{X} - \mathscr{N}$ such that Z is a bi-measurable image of $(\mathscr{X} - \mathscr{N})/G$. Further, if f is a real function integrable with respect to m, then equation* (1) *holds for some σ-finite measure α on Z; μ and δ are as in Theorem* 2.

COROLLARY 1. *If $\mathscr{X}$, G and m satisfy the conditions of Theorem* 2 *or of* 3, *if f is the density of x* w.r.t. *m, then the density of the maximal invariant $x \to Gx$* w.r.t. *α is*

$$f^{Gx}(Gx) = \delta([x])\Delta([x]) \int_G \delta(g)f(gx)\mu(dg) \tag{2}$$

where Δ is the modulus of G; $[x]$ is any element of G such that for some $z \in Z$, $[x]z = x$; δ is the modulus of m.

COROLLARY 2. *If G, $\mathscr{X}$ and m satisfy the assumptions of Theorem* 2 *or of* 3, *if P_1 and P_2 are probabilities on $\mathscr{X}$ with densities f_1 and f_2* w.r.t. *m, then the ratio of the densities for the maximal invariant is*

$$\frac{dP_1^{Gx}(Gx)}{dP_2^{Gx}(Gx)} = \frac{\int_G \delta(g)f_1(gx)\mu(dg)}{\int_G \delta(g)f_2(gx)\mu(dg)} \tag{3}$$

independent of choice of m.

COROLLARY 3. *For an invariant statistical model $(\mathscr{X}, \mathscr{B}, P_\theta, G, \Omega)$, whose $\mathscr{X}$ and G obey the conditions of Theorems* 2 *or* 3, *with density $f(x\,;\,\theta)$* w.r.t. *a relatively invariant Borel measure, the marginal likelihood (see Fraser* (1968), *page* 188 *or Dawid* et al. (1973), *Appendix* 3 *for definition) for $G\theta$ at x is*

$$\propto \int_G \delta(g)f(gx\,;\,\theta)\mu(dg)\,. \tag{4}$$

EXAMPLES. Most of the groups, sample spaces and parameter spaces of parametric statistics are locally compact and Hausdorff; any separable differentiable manifold, and any manifold that can be embedded in E^n is metrisable, complete and separable as well. Any closed subgroup of $GL(n)$ (the nonsingular transformations) or of $GA(n)$ (the translations plus $GL(n)$) is all those and a Lie group, too. The main difficulty in using the theorems lies in verifying one of the conditions (i) to (vii) of Theorem 1. The following examples illustrate some ways of overcoming this problem. In all examples save 5, Lebesgue measure on the sample space will be the natural relatively invariant measure; the modulus $\delta(g)$ is the absolute value of the determinant of the linear part of g.

PROPOSITION 1. *If* $(G, \mathscr{X})$ *is a Cartan G-space* (*definitions in Palais* (1961) *page* 297, *or Wijsman* (1967) *page* 392) *satisfying Condition* A, *then a Borel cross-section exists*; *if G is a Lie group and the* f_i *are probability densities with respect to a relatively invariant Borel measure*, *then expression* (3) *is valid for the ratio of the densities of the maximal invariant.*

PROOF. By Palais Proposition 1.1.4, orbits are closed, and so (ii) of Theorem 1 holds, proving the first statement. By Palais Theorem 2.3.3 Corollary 2, every point in $\mathscr{X}$ has a neighbourhood N in which all the stability subgroups are conjugate. We apply Theorem 3 to GN and the conditional probabilities given GN. □

EXAMPLE 1. If any compact group acts continuously on a completely regular $\mathscr{X}$, then a cross-section exists, for we then have a Cartan G-space, almost by definition.

PROPOSITION 2. *If G is a group of transformations on the topological spaces* $\mathscr{X}$ *and* $\mathscr{Y}$, *if G acts freely on* $\mathscr{X}$, *and if there exists a Borel cross-section for the orbits in* $\mathscr{X}$, *then there is a Borel cross-section for the orbits of G in* $\mathscr{X} \times \mathscr{Y}$ *under the action* $g(x, y) = (gx, gy)$.

PROOF. Let Z be a Borel cross-section for the orbits of G in $\mathscr{X}$; then $\{(z, y) \mid z \in Z, y \in \mathscr{Y}\}$, which is clearly Borel, intersects the orbit of (x_1, y_1) at the single point $(z(x_1), [x_1]^{-1}y_1)$ where $[x_1]$ is the element of G (unique since G acts freely) such that $[x_1]z(x_1) = x_1$. Thus we have a Borel cross-section. □

EXAMPLE 2. In a certain multivariate model before a sufficiency reduction is made, the sample space is the $n$fold product of E^p with itself; G is a closed subgroup of $GL(p)$ with action $g(\mathbf{x}_1, \cdots, \mathbf{x}_n) = (g\mathbf{x}_1, \cdots, g\mathbf{x}_n)$. If $p \leqq n$ we shall consider $\mathscr{X}$ to be the set of nonsingular $p \times p$ matrices and $\mathscr{Y}$ to be the $(n-p)$-fold product of E^p with itself. Then $\mathscr{X} \times \mathscr{Y}$ equals the sample space E^{np} minus a set of measure zero. The orbits of $(G, \mathscr{X})$ are the cosets of $\mathscr{X}/G$ which are closed (since they are the right translates of G and thus closed, right translation being a homeomorphism), hence (iii) of Theorem 1 applies to $\mathscr{X}$, and Proposition 2 yields a cross-section for $(G, \mathscr{X} \times \mathscr{Y})$. In this example, Theorem 7.1 of Wijsman (1970) also applies, and yields a somewhat related result.

PROPOSITION 3. *If F is a group acting on a space* $\mathscr{X}$ (*not necessarily transitively*) *such that* $(F, \mathscr{X})$ *satisfies Condition* A *and any of* (i) *to* (vii) *of Theorem* 1, *and if*

G is a closed subgroup of F, then *there is a Borel cross-section for the orbits under G in $\mathscr{X}$ if GF_x is closed for all x in $\mathscr{X}$.*

PROOF. First we prove the proposition when F acts transitively on $\mathscr{X}$. For some x in $\mathscr{X}$ and any f in F let $\tau(f) = fF_x$ in F/F_x. Then the map taking $\tau(f)$ into fx is a homeomorphism of F/F_x (with quotient topology) onto $\mathscr{X}$ (Helgason (1962), Chapter II, Theorem 3.2), hence the orbit Gx is homeomorphic to GF_x/F_x. Now, the complement of $\tau(GF_x)$ is $\tau(GF_x)^c = \tau((GF_x)^c)$ which is open since τ is open by the definition of the quotient topology on F/F_x. Therefore $\tau(GF_x)$ is closed, hence Gx is closed. The argument holds for all x, so by Theorem 1 (iii) there is a Borel cross-section for the G-orbits.

Now we drop the restriction that F be transitive on $\mathscr{X}$. By Theorem 1 (vi) applied to $(F, \mathscr{X})$, any orbit Fx is locally compact (when endowed with the relative topology induced from $\mathscr{X}$); it inherits separability from $\mathscr{X}$; with this topology Fx is homeomorphic to F/F_x (by (v) of Theorem 1) which is metric and complete (Dieudonné (1970) **2** 12.11.3). Thus the transitive case of the proposition, already proven, applies to F and Fx to give Borel cross-sections for the G-orbits in Fx and hence Gx is locally closed in Fx (Theorem 1 (iii)). Since Gx is a locally closed subset of the locally closed subset Fx, then Gx is locally closed in $\mathscr{X}$ (a proof may be constructed directly from the definition or we may use the fact that a subset of a locally compact space is locally closed iff it is locally compact). Thus every Gx is locally closed in $\mathscr{X}$ and by Theorem 1 (iii) there is a Borel cross-section for the G-orbits. □

COROLLARY 1. *If F, G and $\mathscr{X}$ satisfy the hypotheses of Proposition* 3 *and if F_x is compact for all x in $\mathscr{X}$, then there is a Borel cross-section for the orbits of $\mathscr{X}$ under G.*

PROOF. In a metric group, any closed set times a compact set is closed (Dieudonné, 12.10.5). □

EXAMPLE 3. A certain simple MANOVA model after sufficiency reduction has as its sample space the set of (Y, U, W) where Y, U and W are (resp.) $p \times r$, $p \times s$ and $p \times p$ matrices and W is symmetric and nonsingular; $G = L(p) \times 0(r)$ where $L(p)$ is a closed subgroup of $GL(p)$ and $0(r)$ is the group of $r \times r$ orthogonal matrices; the group action is given by $(A, H)(Y, U, W) = (AYH^t, AU, AWA^t)$ where $A \in L(p)$ and $H \in 0(r)$. We now assume that $r \geqq p$ or $s \geqq p$. If $r \geqq p$, let Z be the $p \times p$ matrix formed by the first p columns of Y; if $r < p$ but $s \geqq p$, let Z be the matrix formed by the first p columns of U. In either case let $\mathscr{X}_0$ be the (probability one) subset of $\mathscr{X}$ where Z is nonsingular.

Now we shall apply Corollary 1 of Proposition 3 to $(G, \mathscr{X}_0)$, using $GL(p) \times 0(r)$ as our F. To see that Corollary 1 applies, first we note that $F_{(Y,U,W)}$ is a closed subset of $\{(A, H) \mid AWA^t = W\} = (W^{\frac{1}{2}}0(p)W^{-\frac{1}{2}}) \times 0(r)$ and is compact since closed subsets of compacta are compact. Now for any $x \in \mathscr{X}_0$, the orbit $GL(p) \cdot x$ is closed (since if $g_1x, g_2x, \cdots$ has a limit x_L in $\mathscr{X}_0$, then $g_1Z, g_2Z, \cdots$ has a limit

z_L, say; now $x_L \in \mathscr{X}_0$, so Z_L must be nonsingular, hence $g_1, g_2, \cdots$ has the limit $h = Z_L Z^{-1} \in GL(p)$, so that $x_L = hx \in GL(p) \cdot x)$. Since a compact set times a closed set is closed (Dieudonné, 12.10.5), the F-orbit of x, which is $(GL(p) \times 0(r)) \cdot x = 0(r) \cdot (GL(p) \cdot x)$ is closed. Thus Corollary 1 of Proposition 3 applies and there is a Borel cross-section for the G-orbits. Further, if the G_x (for all x in $\mathscr{X}_0$) are conjugate, then Theorem 3 applies.

CONJECTURES. Proposition 1 shows how useful it is to prove that a model is a Cartan G-space. The papers of Wijsman ([21], [22], [23]) show many of the models of normal multivariate analysis to be Cartan G-spaces, and raise the conjecture that if G_x is everywhere compact, then $(G, E^n$, matrix multiplication) is a Cartan G-space. If this conjecture were proved true, it would simplify the application of Proposition 1.

Theorem 3 suggests a related conjecture which would be useful if true: if G is a closed subgroup of $GL(n)$ acting on E^n by matrix multiplication, are the orbits locally closed?

Next, we show an example in which none of conditions (i) to (vii) are satisfied.

COUNTEREXAMPLE. (The irrational flow on the torus). Consider $T^2 = \{(x, y) \mid x, y \in [0, 1)\}$; x and y are real numbers modulo 1. The group action $t: (x, y) \rightarrow (x + t, y + t(2)^{\frac{1}{2}})$ (mod 1) defines a group action of the reals R on T^2. Any orbit of this group is dense in any open subset of T^2 and so the orbits are not locally closed or locally compact. The quotient topology on T^2/R is $\{\phi, T^2\}$, which means the Borel field of T^2/R is the same, and does not correspond to the projections of the invariant Borel sets (for example, an individual orbit is Borel in T^2), so (iv) is false and the invariant statistical decision procedures (those based on measurable invariant statistics) are trivial. The conclusion of Theorem 2 fails in this example, for if m is invariant measure on T^2 and $f(x) \equiv 1$, then the left side of equation (1) is finite, but the inner integral on the right side is infinite. Thus it is impossible to define an α for which (1) holds true.

EXAMPLE 4. (Dynamical systems). Consider some system (economic, physical, biological or what-have-you) whose state at time t can be described by a "state-vector" $\mathbf{x}(t) = (x^{(1)}(t), \cdots, x^{(p)}(t))$ in E^p (the "phase space"). If changes in the system obey the differential equation $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ (the dot denotes differentiation with respect to time) where f does not depend on time, then it is well known in the theory of dynamical systems that if solutions to the differential equation are unique for every initial value problem $\mathbf{x}(t_0) = \mathbf{x}_0$, then time acts as a group of transformations on E^p in the following sense: define $\phi_s(\mathbf{x}) = \mathbf{x}(t + s)$ where $\mathbf{x} = \mathbf{x}(t)$. Then $\phi_s: E^p \rightarrow E^p$ is well-defined and $(s, \mathbf{x}) \rightarrow \phi_s(\mathbf{x})$ $(s \in R, \mathbf{x} \in E^p)$ defines a group action by R on E^p; it is easily seen that $\phi_{s+t} = \phi_s(\phi_t)$, $\phi_0(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$, $\phi_{-s}(\phi_s) = \phi_0$, and $\phi_s(\phi_t \phi_u) = (\phi_s \phi_t)\phi_u$.

In applications it is sometimes of interest to take an observation (subject to random error) at a known time t_1, for example

$$\mathbf{y} = \mathbf{x}(t_1) + \boldsymbol{\epsilon}$$

where $\boldsymbol{\varepsilon}$ is a $N(\mathbf{0}, I_{p\times p})$ random variable. This is not an invariant statistical model; nevertheless Theorem 2 tells us that if the system is nonperiodic (free) and the orbits (known in the trade as the "trajectories") are locally closed, then the likelihood function for a trajectory may be obtained by integrating the $N(\mathbf{y}, I_{p\times p})$ density function over that trajectory.

If Z is a Borel cross-section for the orbits, G is free and Lebesgue measure λ is quasi-invariant ($\lambda(gE) = 0$ implies $\lambda(E) = 0$ for any $g \in G$ and $E \subset E^p$), then we may get an invariant measure as follows: define the g-translate of λ by $\lambda_g(E) = \lambda(g^{-1}E)$ and let $J(x)$ be the Radon–Nikodym derivative $d\lambda/d\lambda_{[x]}$ evaluated at $\mathbf{x}$ where $[x]$ is such that $[x]z = x$ for some $z \in Z$. Then $d\lambda(x)/J(x)$ is an invariant measure element. By Theorem 2, the likelihood function of the trajectory through a point $\mathbf{w}$ in the phase space is

$$\int_{-\infty}^{+\infty} \exp[-\tfrac{1}{2}\textstyle\sum_{i=1}^{p} (\mathbf{y}^{(i)} - \phi_t^{(i)}(\mathbf{w}))^2] J(\phi_t(\mathbf{w}))^{-1}\, dt\,.$$

EXAMPLE 5. If $p = 2$ and the differential equation discussed in the previous example is

$$\begin{aligned} \dot{x}^{(2)} &= 1 && \text{if } \ x^{(1)} > 0\,, \\ &= -1 && \text{if } \ x^{(1)} \leqq 0\,, \\ \dot{x}^{(1)} &= 1\,, \end{aligned}$$

the trajectories are then "L-shaped" with cusps on the line $x^{(1)} = 0$ and thus cannot be diffeomorphic to R either in the differential structure induced by $t \to \mathbf{x}(t)$ or that induced by inclusion in E^p (compare this with the situation in Koehn (1970), Theorem 2). We have Borel cross-sections and Theorem 2 applies. $J(\mathbf{x}) = 1$ in this example.

2. Proofs. The key to Theorem 2 is representing $\mathscr{X}$ as a bimeasurable image of a cartesian product. This is done as follows: we take the Borel cross-section Z which is given to us by (vii) of Theorem 1. If $z(x)$ is the element of Z which lies in the orbit of x, we define $[x]$ to be the (unique by the assumption that $G_x = \{e\}$) element of G which takes $z(x)$ into x. In other "words," $[x]z(x) = x$ where $z(x) = Z \cap Gx$. Consider the map $\sigma\colon ([x], \pi(x)) \to x$. We shall prove Theorem 2 by using Fubini's theorem on $G \times (\mathscr{X}/G)$ and carrying it over to $\mathscr{X}$ via σ.

PROOF OF THEOREM 2. First we show that σ is a 1–1 bimeasurable map from $G \times (\mathscr{X}/G)$ onto $\mathscr{X}$. Consider Π, the restriction to Z of $\pi\colon x \to Gx$. Π is continuous, since for any open $V \subset \mathscr{X}/G$, $\Pi^{-1}(V) = \pi^{-1}(V) \cap Z$ which is open by definition of the relative topology of Z. $\mathscr{X}/G$ has a countable base for its topology (Effros (1965), Lemma 2.3) and is almost Hausdorff (Effros, Theorem 2.9 (10)), hence Π^{-1} is measurable (by Baker (1965), Lemma 2, which gives a condition for inverses of 1–1 Borel functions to be Borel). The map $(g, a) \to (g, \Pi^{-1}(a))$ is measurable on $G \times (\mathscr{X}/G)$ since each coordinate is measurable. $(g, \Pi^{-1}(a)) \to g(\Pi^{-1}(a))$ is continuous since G is a topological transformation group.

$\sigma: (g, a) \to g(\Pi^{-1}a)$ is the composition of Borel functions and is thus Borel. By Theorem 1 of Baker, $x \to [x]$ is measurable (note that our $[x]$ is Baker's $\varphi(x)^{-1}$). π is continuous, hence $\sigma^{-1}: x \to ([x], \pi(x))$ is measurable.

A technical complication arises which requires us to restrict attention to cross-sections Z with the following "local boundedness" property: for any x in $\mathscr{X}$ there is a nonempty open invariant neighbourhood U of x and a compact set K such that $Z \cap U \subset K$. This restriction causes no loss of generality, for if any Borel cross-section Z exists, then $\mathscr{X}$, being separable metric, can be covered with precompact open spheres $S_1, S_2, \cdots$. For all $1 \leqq n$, let $N_n = GS_n - \bigcup_{j=1}^{n-1} GS_j$. The N's are thus a partition of $\mathscr{X}$ whose members are invariant. Let $g_1, g_2, \cdots$ be a sequence dense in G. Define $Z_n = Z \cap N_n (1 \leqq n)$ so that $Z = \bigcup Z_n$. For $n, m \geqq 1$ define $Y_{nm} = Z_n \cap g_m S_n$ and $Z_{nm} = Y_{nm} - \bigcup_{j=1}^{m-1} Y_{nj}$. Now $GS_n = \bigcup_{m=1}^{\infty} g_m S_n$ (since $x \in GS_n$ implies $g^{-1}x \in S_n$ for some $g \in G$, therefore there is a subsequence $\{g_{m_i}: i = 1, 2, \cdots\}$ with $g_{m_i} \to g$; $g_{m_i}{}^{-1}x$ is eventually in S_n since S_n is open), hence $Z_n = Z_n \cap GS_n = Z_n \cap (\bigcup_{m=1}^{\infty} g_m S_n) = \bigcup_{m=1}^{\infty} Y_{nm} = \bigcup_{m=1}^{\infty} Z_{nm}$ (disjoint union). Define $\eta(z) = g_m{}^{-1}z$ for $z \in Z_{nm}$. $\eta(Z) = \bigcup g_m{}^{-1}Z_{nm}$ is thus a countable sum of Borel sets and intersects each Gz precisely once. Any $x \in \mathscr{X}$ lies in some N_k, hence lies in the invariant set $U = \bigcup_{n=1}^{k} GS_n$ which is open since the S_n and their translates are open, and $\eta(Z) \cap U$ lies in $\bigcup_{n=1}^{k} \bar{S}_n$ which is compact since it is a finite union of closed balls. Thus $\eta(Z)$ is a locally bounded Borel cross-section.

We next show that if $\sigma^{-1}(E)$ is of the form $F \times A$ (F and A open), then $m(E) = \lambda(F)\alpha(A)$ where λ is relatively invariant on G (with the same modulus δ as m) and α is some measure on $\mathscr{X}/G$. If we show that $\lambda \times \alpha$ is σ-finite, it then follows by the uniqueness-of-extension theorem (Halmos 13.A) that m and $(\lambda \times \alpha)\sigma^{-1}$ agree on the smallest σ-field generated by the $\sigma(F \times A)$'s (this latter field is precisely the Borel field of $\mathscr{X}$ since σ is bimeasurable). As our λ we shall use the measure $\lambda(E) = \int_E \delta(g)\mu(dg)$, which is relatively invariant with modulus δ (Nachbin (1965), Chapter 2, Proposition 26).

$m\sigma$ is relatively invariant since

$$(m\sigma)(g\sigma^{-1}E) = (m\sigma)(\sigma^{-1}(gE)) = m(gE) = \delta(g)m(E) = \delta(g)(m\sigma)(\sigma^{-1}E) . \tag{1}$$

Take a measurable A in $\mathscr{X}/G$, then for any measurable F in G, define the measure

$$\lambda_A(F) = (m\sigma)(F \times A) .$$

This is relatively invariant by (1), hence by the uniqueness of relatively invariant measures of given modulus (Nachbin, page 138, Theorem 1), λ_A is a constant times λ:

$$\lambda_A(F) = K(A)\lambda(F) .$$

$K(A)$ is clearly nonnegative and easily shown to be σ-additive, hence is the measure α that we want. Because the canonical map $\Pi: z \to Gz$ is bimeasurable, it induces a measure α^* on Z; we shall not distinguish betweeen α and α^*.

To prove $\lambda \times \alpha$ is σ-finite, we show that every point in $G \times (\mathscr{X}/G)$ lies in a set $F \times A$ (F and A open) for which $m\sigma(F \times A) < \infty$, and it will then follow by second countability of G and $\mathscr{X}/G$ that $G \times (\mathscr{X}/G)$ may be covered by a countable family of such rectangles $F_i \times A_n$, which proves σ-finiteness. By earlier remarks we can restrict Z to be locally bounded, whence for each x in $\mathscr{X}$ there is an invariant open neighbourhood U of x such that $Z \cap U \subset K$, a compact set. Let F be the interior of a compact neighbourhood of $[x]$ in G; then FK is contained in a compact set (since the product of compact sets is compact, Dieudonné (1970), 12.10.5) and therefore is of finite measure. So $m(F(Z \cap U)) \leqq m(FK) < \infty$, and $\sigma^{-1}(F(Z \cap U)) = \{(g, z) \mid g \in F, z \in \pi(U)\}$ which is a product of open sets as desired. To prove Borelness of α, observe that $\mathscr{X}/G$ is covered by the open sets A_n, therefore if $K \subset \mathscr{X}/G$ is compact, it can be covered by a finite number of A_n, each of which has finite α-measure.

Now, we have proved

$$\int_{\mathscr{X}} f(x)m(dx) = \int_{G\times(\mathscr{X}/G)} f(\sigma y)(\lambda\times\alpha)(dy) .$$

Apply Fubini's theorem to get

$$\int_{\mathscr{X}} \alpha(da) \int_G f(g(\Pi^{-1}a))\lambda(dg) ;$$

now use the definition of λ:

$$f(g\Pi^{-1}a)\lambda(dg) = f(gz)\delta(g)\mu(dg) . \qquad \square$$

Proof of Theorem 3. Choose some x_0 in $\mathscr{X}$; let $H = G_{x_0}$, and define $\mathscr{X}_H = \{x \mid G_x = H\}$. We now prove that $\mathscr{X}_H$ is closed. Consider the sequence $x_1, x_2, \cdots$ of elements in $\mathscr{X}_H$, with limit x. Now, h in H implies $hx_i = x_i$ and also $hx_i \to hx$ (by continuity of multiplication); but $hx_i \to x$, whence $hx = x$ by the Hausdorffness of $\mathscr{X}$, hence $H \subset G_x$. Since G_x is conjugate to H, there exists $g \in G$ such that $H \subset gHg^{-1}$. A consequence of continuity of multiplication is that stability subgroups are closed, hence H and gHg^{-1} are analytic subgroups of G (Cohn (1957), Theorem 6.5.1). Therefore H is an analytic subgroup of gHg^{-1} (Helgason (1962) Chapter 2, Corollary 2.9), and since the two groups have equal dimension it follows that they have the same component of the identity (Cohn, page 53). Since G is a topological group, $h \to ghg^{-1}$ is a homeomorphism, so H and gHg^{-1} have the same number of connected components; each component of H is identical to one of gHg^{-1} (since the components of a topological group are translates of the component of the identity), and since the number of connected components is finite by compactness, we have $H = gHg^{-1}$, completing the proof that $\mathscr{X}_H$ is closed.

Take the Borel cross-section Z_1 which is given to us by Theorem 1 (vi). Define $\sigma_1: G \times Z_1 \to \mathscr{X}$ by $\sigma_1(g, z) = gz$; σ_1 is measurable because it is a restriction of the multiplication map, which is continuous. Therefore $\sigma_1^{-1}(\mathscr{X}_H)$ is measurable in $G \times Z_1$; its projection on Z_1 is all of Z_1 (since all G_x are conjugate to H, every orbit Gx in $\mathscr{X}$ has at least one point in $\mathscr{X}_H$). We now use the usual trick to exhibit a finite measure equivalent to m: by Condition A, $\mathscr{X}$ is the union of an increasing sequence of compact sets, $\mathscr{X} = \bigcup_{i=1}^{\infty} K_i$; define m_i as the restriction to K_i of m: $m_i(S) = m(S \cap K_i)$; then $P = \sum_{i=1}^{\infty} [2^i m(K_i)]^{-1} m_i$ is a probability measure equivalent to m. The restriction Π_1 of π to Z_1 is measurable, and π^{-1} takes Borel sets into Borel sets (Theorem 1 (iv)), hence the probability $Q = P\pi^{-1}\Pi_1$ obtained by projecting P onto Z_1 along orbits, is defined on the Borel sets of Z_1.

Now we take note of Aumann's selection theorem (Aumann (1969), page 17, quoted in Parthasarathy (1972), Theorem 7.2): let (Z_1, Q) be a σ-finite measure space, let G be a standard measurable space and let S be a measurable subset of $G \times Z_1$ whose projection on Z_1 is all of Z_1; then there exists a measurable function γ from Z_1 to G such that $(z, \gamma(z)) \in S$ for almost all (Q) z in Z_1, the exceptional set $\mathscr{M}$ being Borel. A *standard measurable space* is a measurable space which is a bimeasurable image of the cartesian product of $\{0, 1\}$ denumerably many times; in particular a separable Lie group of nonzero dimension is standard. Applying Aumannn's theorem, we get a measurable $\gamma: Z_1 \to G$ such that $(z, \gamma(z)) \in \sigma_1^{-1}(\mathscr{X}_H)$ for $z \in Z_1 - \mathscr{M}$. Now, $z \to (z, \gamma(z))$ is measurable and so is $(z, \gamma(z)) \to (\gamma(z))z$, hence $\beta: z \to (\gamma(z))z$ is measurable on $Z_1 - \mathscr{M}$. By Kuratowski (1966), Section 39, V, Theorem 1, β^{-1} is measurable since β itself is one-one and measurable, hence $Z = \beta(Z_1 - \mathscr{M})$ is a Borel cross-section for the orbits of $\mathscr{X} - \mathscr{N}$ where $\mathscr{N} = \pi^{-1}\pi(\mathscr{M})$ is an invariant P-null (hence m-null) Borel subset of $\mathscr{X}$, and we have $G_z = H$ for all z in Z.

As in Theorem 2, define $z(x) = Z \cap Gx$; then we have measurable map $x \to [x]$ taking values in G such that $[x]z(x) = x$ (use the inverse of the $\varphi(x)$ of Baker's Theorem 1; contrary to the situation in our Theorem 2, φ is no longer unique). Define the map $\sigma: ([x]H, \pi(x)) \to x$ and put the quotient topology on G/H. As in Theorem 2, Π^{-1} is measurable, therefore $(gH, a) \to (gH, \Pi^{-1}(a))$ is measurable. Now $(gH, \Pi^{-1}(a)) \to gH(\Pi^{-1}a)$ (note that the latter is a single element of $\mathscr{X}$ since $\Pi^{-1}(a)$ is invariant under H) is continuous, thus $\sigma: (gH, a) \to gH(\Pi^{-1}a)$ is Borel. σ^{-1} factors as $x \to ([x], \pi(x)) \to ([x]H, \pi(x))$ and hence is measurable.

An easy corollary of Weil's theorem on relatively invariant measures says that the integral taking functions f on G/H into $\int_G f(gH)\delta(g)\mu(dg)$ is a relatively invariant integral of modulus δ (for a proof see Bondar (1972), page 330, Lemma 1). For $E \subset G/H$, define $\lambda(E)$ as the value assigned to the indicator function of E by this integral. Then λ is a relatively invariant measure of modulus δ; it is easy to see that it gives finite measure to compact sets, and nonzero measure to nonempty open sets. Again, $m\sigma$ is relatively invariant with modulus δ as in (1). Writing $\hat{F}$ for the image of F under the canonical map from G to G/H, if F is measurable in G, and A in $\mathscr{X}/G$, the uniqueness of relatively invariant measures implies that $m\sigma$ can be written $m\sigma(\hat{F} \times A) = K(A) \cdot \lambda(\hat{F})$ where, as in the proof of Theorem 2, K can be shown to be a measure which we shall write as α.

Now cover $\mathscr{X}$ by a countable family $\{S_n\}$ of precompact open balls, and cover G by an increasing sequence $\{F_i\}$ of open sets whose closures are compact. The Borel sets $F_i(Z \cap S_n))$ cover $\mathscr{X} - \mathscr{N}$, and as in Theorem 2, are of finite m-measure. $\sigma^{-1}(F_i(Z \cap S_n)) = \{(gH, a): g \in F_i,\ a \in \pi(Z \cap S_n)\}$ which is a product of measurable sets, hence $(G/H) \times (\mathscr{X}/G)$ is covered by a countable family of rectangles with finite $m\sigma$-measure, which proves that $\lambda \times \alpha$ is σ-finite. An extension of $\lambda \times \alpha$ must be unique, so $m = (\lambda \times \alpha)\sigma^{-1}$. σ-finiteness of α follows from the previous argument: if we choose any nonempty F_i, say F, then $\lambda(\hat{F}) > 0$; the foregoing shows that $(\lambda \times \alpha)\sigma^{-1}(F(Z \cap S_n)) = \lambda(F) \cdot \alpha(\pi(Z \cap S_n))$ is finite, hence $\alpha(\pi(Z \cap S_n))$ is also finite; since the collection of sets $Z \cap S_n$ covers Z, α is σ-finite.

By now we have proved

$$\int_{\mathscr{X}} f(x)m(dx) = \int_{(G/H)\times(\mathscr{X}/G)} f(\sigma y)(\lambda \times \alpha)(dy)$$

where $y = (gH, a)$ is the generic element of $(G/H) \times (\mathscr{X}/G)$. If we again abuse our notation by identifying α with $\alpha^* = \alpha\Pi$, and $a \in \mathscr{X}/G$ with the corresponding $z = \Pi^{-1}a$, Fubini's theorem gives

$$\int_Z \alpha(dz) \int_{gH\in G/H} f(gz)\lambda(dgH) .$$

Recalling the definition of λ, this becomes

$$\int_Z \alpha(dz) \int_G f(gz)\delta(g)\mu(dg) . \qquad \square$$

Proof of Corollary 1. Make a change of variable $g = h[x]$ and

$$\begin{aligned} f(gz)\delta(g)\mu(dg) &= f(h[x]z)\delta(h[x])\mu(dh[x]) \\ &= f(hx)\delta(h)\delta([x])\Delta([x])\mu(dh) \end{aligned}$$

(readers of Nachbin (1965), Chapter 2 or Fraser (1968), Chapter 2 will be familiar with these manipulations). $\square$

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