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EXTENSIONS OF MILLIKEN'S ESTIMABILITY CRITERION

BY J. K. BAKSALARY AND R. KALA

Academy of Agriculture, Poznań

Some generalizations of Milliken's necessary and sufficient condition for estimability of linear parametric functions in linear models are established. The more universal character of the present theorems consists in avoiding the assumption of the linear independence of examined functions, in using any generalized inverse instead of the Moore-Penrose inverse, and in extending the criterion on more general linear models.

1. Introduction. Let the triplet

$$(\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V}) \tag{1}$$

denote the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$, where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{X}$ is an $n \times p$ known matrix of any rank, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, and $\mathbf{e}$ is an $n \times 1$ vector of errors with expectation zero and with covariance proportional to the $n \times n$ known nonnegative definite matrix $\mathbf{V}$. In the case of $\mathbf{V}$ singular it is assumed, in addition, that $\mathbf{y}$ belongs to the space spanned by the columns of the partitioned matrix $(\mathbf{X} \vdots \mathbf{V})$, which in the view of Rao (1973) constitutes the consistency condition of a model.

If the parameters in model (1) are subject to consistent linear restrictions $\mathbf{R}\boldsymbol{\beta} = \mathbf{c}$, then the notation

$$(\mathbf{y}, \mathbf{X}\boldsymbol{\beta} \mid \mathbf{R}\boldsymbol{\beta} = \mathbf{c}, \sigma^2\mathbf{V}) \tag{2}$$

is used. It is known that (2) admits the representation

$$\left(\begin{pmatrix}\mathbf{y}\\ \mathbf{c}\end{pmatrix}, \begin{pmatrix}\mathbf{X}\\ \mathbf{R}\end{pmatrix}\boldsymbol{\beta}, \sigma^2\begin{pmatrix}\mathbf{V} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\right). \tag{3}$$

Here, as for (1) with singular $\mathbf{V}$, the consistency assumption on the vector $(\mathbf{y}' \vdots \mathbf{c}')'$ must be made.

For the case of model (1) with $\mathbf{V} = \mathbf{I}$, the identity matrix, Milliken (1971) has formulated an estimability criterion in terms of the ranks of some matrices. It is clear that his result remains true for any $\mathbf{V}$, since the structure of $\mathbf{V}$ has an effect on the form of best linear unbiased estimators, but not on the estimability itself.

Milliken's criterion is applicable only when the examined parametric functions $\mathbf{A}\boldsymbol{\beta}$ are linearly independent, i.e., when $\mathbf{A}$ is of full row rank. In this paper we extend the criterion to the case of an arbitrary $\mathbf{A}$. In addition, we replace the Moore-Penrose generalized inverse, $\mathbf{A}^+$, occurring in Milliken's result, by any generalized inverse, $\mathbf{A}^-$, (i.e., such a matrix that $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$). This

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has an advantage from the computational viewpoint. Moreover, we develop a similar criterion for model (2).

2. Results. First we establish the estimability criterion for model (1) The symbol $r(\cdot)$ denotes the rank of a matrix argument.

THEOREM 1. *The linear parametric functions* $\mathbf{A}\boldsymbol{\beta}$, *with coefficient matrix* $\mathbf{A}$ *of arbitrary rank, are unbiasedly estimable in the model* $(\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$ iff *for some* $\mathbf{A}^-$

$$r\{\mathbf{X}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\} = r(\mathbf{X}) - r(\mathbf{A}) .$$

PROOF. It is well known that $\mathbf{A}\boldsymbol{\beta}$ are estimable iff $\mathscr{R}(\mathbf{A}) \subset \mathscr{R}(\mathbf{X})$, where $\mathscr{R}(\cdot)$ denotes the row space of the matrix in brackets. We utilize the following result (see, e.g., Stoll and Wong, 1968): if $\mathbf{T}$ is a linear transformation of a vector space $\mathscr{V}$ onto a vector space $\mathscr{W}$, then

$$\dim \mathscr{V} = \dim \mathscr{W} + \dim (\text{null space of } \mathbf{T}) .$$

Let $\mathscr{V} = \mathscr{R}(\mathbf{X})$ and let it be projected by $\mathbf{T} = \mathbf{I} - \mathbf{A}^-\mathbf{A}$ onto $\mathscr{W} = \mathscr{R}\{\mathbf{X}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\}$. Then

$$\begin{aligned} r(\mathbf{X}) &= r\{\mathbf{X}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\} + \dim \{\boldsymbol{\lambda} \in \mathscr{R}(\mathbf{X}) : \boldsymbol{\lambda}(\mathbf{I} - \mathbf{A}^-\mathbf{A}) = \mathbf{0}\} \\ &= r\{\mathbf{X}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\} + \dim \{\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\mathbf{A})\} . \end{aligned}$$

Since $\dim \{\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\mathbf{A})\} = r(\mathbf{A})$ iff $\mathscr{R}(\mathbf{A}) \subset \mathscr{R}(\mathbf{X})$, the theorem results.

Now we develop an analogous estimability criterion for the restricted linear model. Applying Theorem 1 to the model of the form (3) we can state that the parametric functions $\mathbf{A}\boldsymbol{\beta}$ are estimable in model (2) iff

$$r\{\binom{\mathbf{X}}{\mathbf{R}}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\} = r\{\binom{\mathbf{X}}{\mathbf{R}}\} - r(\mathbf{A}) .$$

To reformulate this equality, we note that the relation given in Theorem 1 is, in fact, a necessary and sufficient condition for the inclusion $\mathscr{R}(\mathbf{A}) \subset \mathscr{R}(\mathbf{X})$. Therefore, since obviously $\mathscr{R}(\mathbf{R}) \subset \mathscr{R}\{(\mathbf{X}' \vdots \mathbf{R}')'\}$, we get

$$r\{\binom{\mathbf{X}}{\mathbf{R}}(\mathbf{I} - \mathbf{R}^-\mathbf{R})\} = r\{\binom{\mathbf{X}}{\mathbf{R}}\} - r(\mathbf{R}) ,$$

and hence

$$r\{\binom{\mathbf{X}}{\mathbf{R}}\} = r\{\mathbf{X}(\mathbf{I} - \mathbf{R}^-\mathbf{R})\} + r(\mathbf{R}) .$$

So we have proved the following.

THEOREM 2. *The linear parametric functions* $\mathbf{A}\boldsymbol{\beta}$ *are unbiasedly estimable in the model* $(\mathbf{y}, \mathbf{X}\boldsymbol{\beta} \mid \mathbf{R}\boldsymbol{\beta} = \mathbf{c}, \sigma^2\mathbf{V})$ iff *for some* $\mathbf{A}^-$, $\mathbf{R}^-$,

$$r\{\binom{\mathbf{X}}{\mathbf{R}}(\mathbf{I} - \mathbf{A}^-\mathbf{A})\} = r\{\mathbf{X}(\mathbf{I} - \mathbf{R}^-\mathbf{R})\} + r(\mathbf{R}) - r(\mathbf{A}) .$$

REMARK. It is obvious that the estimability conditions stated in Theorems 1 and 2 can be reformulated in terms of the traces of some matrices, in the manner of Milliken (1971), but using arbitrary generalized inverses instead of the Moore–Penrose inverses.

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REFERENCES

[1] MILLIKEN, G. A. (1971). New criteria for estimability for linear models. *Ann. Math. Statist.* **42** 1588–1594.

[2] RAO, C. R. (1973). Representations of best linear unbiased estimators in the Gauss-Markoff model with a singular dispersion matrix. *J. Multivariate Anal.* 3 276–292.

[3] STOLL, R. R. and WONG, E. T. (1968). *Linear Algebra.* Academic Press, New York.

ACADEMY OF AGRICULTURE
DEPARTMENT OF MATHEMATICAL AND STATISTICAL METHODS
60-637 POZNAŃ
WOJSKA POLSKIEGO 28
POLAND