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STOPPING A SUM DURING A SUCCESS RUN[1]

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Let $\{Z_i\}$ be i.i.d., let $\{\varepsilon_i\}$ be i.i.d. Bernoulli, independent of $\{Z_i\}$, let $T_0 = z$ and $T_n = \varepsilon_n(T_{n-1} + Z_n)$ for $n \geqq 1$. Under a moment condition, optimal stopping rules are found for stopping $T_n - nc$ where $c > 0$ (the cost model), and for stopping $\beta^n T_n$ where $0 < \beta < 1$ (the discount model). Special cases are treated in detail. The cost model generalizes results of N. Starr, and the discount model generalizes results of Dubins and Teicher.

1. Introduction and summary. Let $Z, Z_1, Z_2, \cdots$ be independent identically distributed (i.i.d.) random variables with known distribution. Let $\varepsilon, \varepsilon_1, \varepsilon_2, \cdots$ be i.i.d. Bernoulli random variables, independent of $Z_1, Z_2, \cdots$ with probability p of success, $p = P(\varepsilon = 1) = 1 - P(\varepsilon = 0)$, $0 < p < 1$. In a given economic system, Z_i represents your return for the ith time period provided there is a success during the ith period, that is provided $\varepsilon_i = 1$. As long as successes occur consecutively, your returns accumulate, but when there is a failure your accumulated return drops to zero. A failure does not remove you from the system; you are allowed to accumulate future returns until the next failure drops you to zero again, and so on. The problem is to choose a time to stop, that is to withdraw from the system and be content with what you have accumulated.

Let z denote your initial accumulated return (a constant) and let T_n denote your accumulated return at the end of the nth period, so that $T_0 = z$ and

$$T_n = \varepsilon_n(T_{n-1} + Z_n) \quad n = 1, 2, \cdots .$$

We consider two models, the cost model and the discount model. In the cost model there is a cost $c > 0$ that represents the cost of living per period. If you stop at the end of the nth period your net return is

$$X_n = T_n - nc . \tag{1}$$

In the discount model, there is a discount factor β by which capital is discounted during each period, $0 < \beta < 1$. If you stop at the end of the nth period, your net return (or rather the present value thereof) is

$$X_n = \beta^n T_n . \tag{2}$$

The problem is to choose a stopping rule, N, to maximize the expected net return, EX_N.

The general theory of stopping rule problems may be found in the excellent book of Chow, Robbins, and Siegmund [1]. The problem may be described as

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follows. For given random variables $Y_1, Y_2, \cdots$ with known joint distribution, and given measurable functions $X_n = f_n(Y_1, Y_2, \cdots, Y_n)$ $n = 0, 1, \cdots$, and $X_\infty = f_\infty(Y_1, Y_2, \cdots)$, the problem is to choose an extended-valued stopping rule N to maximize EX_N. An extended-valued stopping rule N is a random variable with values in $\{0, 1, 2, \cdots\}$ such that the event $\{N = n\}$ is in the Borel field $\mathscr{F}_n$ generated by $Y_1, \cdots, Y_n$, $\mathscr{F}_n = \mathscr{B}(Y_1, \cdots, Y_n)$. Thus the decision to stop at time n is a function of $Y_1, \cdots, Y_n$, and does not depend on future observables $Y_{n+1}, \cdots$. The following theorem is known. (See, for example, Theorem 4.5′ of Chow, Robbins and Siegmund [1]. Substitution of the weaker hypothesis $\limsup X_n \leqq X_\infty$ for their implicit assumption $\limsup X_n = X_\infty$ may be made without difficulty.)

THEOREM 1. *If* $E \sup X_n < \infty$ *and* $\limsup X_n \leqq X_\infty$ a.s., *then the stopping rule*

$$N^* = \min \{n \geqq 0 : X_n = \operatorname{ess\,sup}_{N \geqq n} E\{X_N \mid \mathscr{F}_n\}\} \tag{3}$$

is optimal.

In this formula, the essential supremum is taken over all stopping rules N such that $N \geqq n$ a.s. The rule N^* is the rule obtained by the principle of optimality in dynamic programming.

In the application of this theorem to the stopping rule problems considered here, we may take $Y_i = (Z_i, \varepsilon_i)$ for $i = 1, 2, \cdots$. To allow for randomized stopping rules, we could take $Y_i = (Z_i, \varepsilon_i, U_i)$ where $\{U_i\}$ are i.i.d. uniform (0, 1) random variables. However, it is known from the general theory that there is no gain in generality. We treat the initial fortune z as a parameter of the problem and occasionally write our expectation symbol as E_z, even though, strictly speaking, it is the functions X_n that depend on z rather than the distributions of the random quantities X_n.

In Section 2, we treat the cost model where X_n is related to the $\{Y_i\}$ by (1), and $X_\infty \equiv -\infty$. We assume that $E(Z^+)^2 < \infty$, and show that the hypotheses of Theorem 1 are satisfied. The rule N^* is evaluated and seen to have a very simple form, namely, $N^* = \min \{n \geqq 0 : T_n \geqq s\}$ for some number $s \geqq -c/(1 - p)$. This rule is stationary: one does not have to keep track of time or of all the past history but only of the present accumulated return, T_n. Formulae to aid the computation of the optimal value of s are given, and in some special cases the details are worked out fairly explicitly—in the exponential case, where the positive part of the distribution of Z is exponential, and in the geometric case, where Z is integer-valued and the positive part of the distribution of Z is geometric. This generalizes results of N. Starr [5] who treated the case Z degenerated at 1. If Z is nonnegative and $EZ \leqq c/p$, then the problem is monotone (see Chow, Robbins and Siegmund [1]) and the one-stage look-ahead rule is optimal.

In Section 3, the discounted model is treated wherein X_n is related to the $\{Y_i\}$ by (2) and $X_\infty \equiv 0$. We assume $EZ^+ < \infty$, and show that the hypotheses of

Theorem 1 are satisfied. The rule N^* is seen to have the simple form $N^* = \min\{n \geqq 0 : T_n \geqq s\}$ for some number $s \geqq 0$. Again, formulae to aid the computation of the optimal s are given, and the exponential and geometric cases are worked out in detail. In this section, the value $p = 1$ is allowed. When $p = 1$, the problem reduces to that solved by Dubins and Teicher [2].

When $p = 1$, this problem is equivalent to the burglar problem mentioned by G. Haggstrom [3]. In this problem, a burglar makes a sequence of burglaries with i.i.d. yields $Z_1, Z_2, \cdots$ until he gets caught or decides to stop. Let $\{\delta_i\}$ be i.i.d. independent of $\{Z_i\}$, with $P\{\delta_i = 1\} = \beta = 1 - P\{\delta_i = 0\}$, and let $\{\delta_i = 0\}$ represent the event that the burglar gets caught during the ith job. It is assumed that if the burglar is ever caught, his return is fixed at zero. The burglar's problem is to choose a stopping rule N to maximize $E(\prod_1^N \delta_i) \sum_1^N Z_j$. But since for any stopping rule N, the rule N', which acts as N would if all $\delta_i = 1$, has the same expected return and is independent of $\{\delta_i\}$, and

$$E(\textstyle\prod_1^N \delta_i) \sum_i^N Z_i = E(\prod_1^{N'} \delta_i) \sum_1^{N'} Z_j = E\{E\{(\prod_1^{N'} \delta_i) \sum^{N'} Z_j \mid \{Z_j\}, N'\}\}$$
$$= E\{\beta^{N'} \textstyle\sum_1^{N'} Z_j\} ,$$

this problem is equivalent to the discounted stopping of a sum problem of Dubins and Teicher.

In the burglar problem, it is usually assumed that the Z_i are nonnegative. In addition to making the burglar interpretation of the model more "realistic," this assumption has the advantage of making the problem monotone, as Haggstrom observes, so that the simple one-stage look-ahead rule is optimal.

The general discount model treated here has a similar interpretation: the $\{Z_i\}$ represent the returns to the burglar, $1 - \beta$ represents the probability he is caught during any burglary, reducing his return to zero permanently, and $1 - p$ represents the probability that the burglar himself is burglarized, reducing his return to zero temporarily but allowing him to continue his occupation.

2. The cost model. Let $\{Z_i\}$, $\{\varepsilon_i\}$, $0 < p < 1$, z, $c > 0$, and $\{T_n\}$ be as defined in Section 1, and consider the problem of finding a stopping rule N to maximize EX_N where X_n is given by (1), and $X_\infty \equiv -\infty$.

2.1. *Form of the optimal rule.* We assume that $E(Z^+)^2 < \infty$ and show that under this assumption the hypotheses of Theorem 1 are satisfied.

We first show that $E(Z^+)^2 < \infty$ is necessary and sufficient for $E \sup X_n < \infty$. The proof of necessity, for which we are indebted to Thomas Liggett, uses the following known results.

LEMMA 1. *Let* $Z, Z_1, Z_2, \cdots$ *be* i.i.d. *with* $EZ < 0$, *let* $c > 0$, *and let* $S_n = \sum_1^n Z_i$. *Then* $E \sup (S_n - nc) < \infty$ *if and only if* $E(Z^+)^2 < \infty$.

LEMMA 2. *Let* $Z, Z_1, Z_2, \cdots$ *be* i.i.d. *and let* $c > 0$. *Then* $E \sup(Z_n - nc) < \infty$ *if and only if* $E(Z^+)^2 < \infty$.

Proofs of these lemmas may be found in Chow, Robbins and Siegmund [1].

Lemma 1 is contained in their Theorem 4.13. The proof of the "only if" part of Lemma 2 may be found in their proof of Theorem 4.13 since in this part their hypothesis that $EX = 0$ is not used. The direct half of Lemma 2 is contained in their Lemma 4.7.

THEOREM 2. *In order that* $E\sup_n (T_n - nc) < \infty$, *it is necessary and sufficient that* $E(Z^+)^2 < \infty$.

PROOF. To simplify the computation, we put $z = 0$ and $c = 1$. *Sufficiency.* Note that T_n has the distribution of $\sum_1^{\min(K,n)} Z_i$, where K is independent of the $\{Z_i\}$ and $P(K = k) = (1 - p)p^k$, $k = 0, 1, 2, \cdots$. Therefore $T_n \leqq Q_n$, where Q_n has the distribution of $Q = \sum_1^K Z_i^+$, and

$$
\begin{aligned}
E\sup_n (T_n - n) &\leqq \textstyle\sum_{x=0}^\infty P\{\sup_n (T_n - n) \geqq x\} \\
&\leqq \textstyle\sum_{x=0}^\infty \sum_{n=0}^\infty P\{T_n - n \geqq x\} \\
&\leqq \textstyle\sum_{x=0}^\infty \sum_{n=0}^\infty P\{Q \geqq x + n\} \\
&= \textstyle\sum_{x=0}^\infty (x+1)P\{Q \geqq x\} .
\end{aligned} \tag{4}
$$

The last sum is finite since Q has a finite second moment:

$$EQ^2 = E\{E(Q^2 \mid K)\} = E\{KE(Z^+)^2 + K(K-1)(EZ^+)^2\} < \infty .$$

Necessity. Let $I(A)$ represent the indicator random variable of the event A, and note that $\sup_n (T_n - n) \geqq \sup_n I(\varepsilon_n = 1, \varepsilon_{n-1} = 0)(Z_n - n)$ which is equivalent in distribution to $\sup_n (Z_n - K_n)$, where $K_1 = \min\{n \geqq 1 : \varepsilon_n = 1, \varepsilon_{n-1} = 0\}$ (let $\varepsilon_0 \equiv 0$), and for $j > 1$ $K_j = \min\{n > K_{j-1} : \varepsilon_n = 1, \varepsilon_{n-1} = 0\}$. Then $K_2 - K_1$, $K_3 - K_2, \cdots$ are i.i.d. with finite variance. Let $\mu = E(K_2 - K_1)$. It is sufficient to show that $E\sup(Z_n - K_n) < \infty$ implies $E\sup(Z_n - c'n) < \infty$ for some $c' > 0$ since then by the "only if" part of Lemma 2, $E(Z_1^+)^2 < \infty$. But $\sup(Z_n - c'n) \leqq \sup(Z_n - K_n) + \sup(K_n - c'n)$ and $E\sup(K_n - c'n) < \infty$ when $c' = 2\mu$, say, by Lemma 1. □

It is easy to see that $\lim X_n = X_\infty$ a.s. or, equivalently, that $P(T_n \geqq nc + a$ i.o.$) = 0$ for any real number a. For example, by the Borel-Cantelli lemma, it is sufficient to show that $\sum_n P(T_n \geqq nc + a) < \infty$, and, with Q defined as in the proof of Theorem 1, $\sum_n P(T_n \geqq nc + a) \leqq \sum_n P(Q \geqq nc + a)$ which is finite since Q has a finite first moment. Therefore, the stopping rule N^* of Theorem 1 is optimal.

We now show that the rule N^* has a very simple form. For this purpose, let $V^*(z)$ denote the optimal return starting with initial fortune z,

$$V^*(z) = \sup_N E_z X_N = E_z X_{N^*} .$$

LEMMA 3. *V^* is convex and nondecreasing. There is a unique number s such that*

$$z < V^*(z) \leqq s \quad \text{for} \quad z < s \qquad \text{and}$$
$$V^*(z) = z \quad \text{for} \quad z \geqq s .$$

Furthermore, $V^*(z) \geqq -c/(1-p)$ *for all* z.

(Note that s may be negative, but $s \geqq -c/(1-p)$.)

PROOF. For a fixed rule N, $E_z X_N = zP(N < K) + E_0 X_N$, where $K = \min\{k \geqq 1 : \varepsilon_k = 0\}$. Therefore, $V^*(z)$ is the supremum of linear affine functions with nonnegative slopes and so is convex and nondecreasing. Except for the stopping rule $N \equiv 0$, whose return is z, the slope of $E_z(X_N)$ is less than $P(K > 1) = p$. (That we may restrict attention to nonrandomized stopping rules was mentioned in the introduction.) Therefore for some s, $V^*(z) = z$ for $z \geqq s$, and $V^*(z) > z$ for $z < s$. Finally, since the rule $N' = \min\{n \geqq 1 : \varepsilon_n = 0\}$ has expected return $-cEN' = -c/(1-p)$, we have $V^*(z) \geqq -c/(1-p)$ for all z. □

If $P(Z_1 \leqq 0) = 1$, then the rule N' in the proof of Lemma 3 is optimal when $z \leqq -c/(1-p)$, and $V^*(z) = \max(z, -c/(1-p))$.

LEMMA 4. $\operatorname{ess\,sup}_{N \geqq n} E(X_N \mid \mathscr{F}_n) = V^*(T_n) - nc$ a.s.

PROOF. For fixed z, let $N(z)$ be an optimal rule starting at z, and define $\psi_k(z, Y_1, \cdots, Y_k) = P(N(z) = k \mid \mathscr{F}_k)$ where $Y_i = (Z_i, \varepsilon_i)$. The ψ_k are not necessarily measurable in z but for any $\delta > 0$ the functions $\psi_k'(z, Y_1, \cdots, Y_k) = \sum_{j=-\infty}^{\infty} I_{[j\delta,(j+1)\delta)}(z)\psi_k(j\delta, Y_1, \cdots, Y_k)$ are measurable. Let N^δ be the rule for which $P(N^\delta = k \mid \mathscr{F}_k) = 0$ for $k < n$, and $P(N^\delta = k \mid \mathscr{F}_k) = \psi'_{k-n}(T_n, Y_{n+1}, \cdots, Y_k)$ for $k \geqq n$. Let $T_0' = j\delta$ and $T_k' = \varepsilon_{n+k}(T'_{k-1} + Z_{n+k})$ for $k \geqq 1$. Then $N^\delta \geqq n$ and for $k \geqq n$ and $j\delta \leqq T_n < (j+1)\delta$, $T_k \geqq T'_{k-n}$, so that

$$
\begin{aligned}
E(Y_{N^\delta} \mid \mathscr{F}_n) &= \textstyle\sum_{j=-\infty}^{\infty} I_{[j\delta,(j+1)\delta)}(T_n) \sum_{k=n}^{\infty} E\{X_n \psi_{k-n}(j\delta, Y_{n+1}, \cdots, Y_k) \mid \mathscr{F}_n\} \\
&\geqq \textstyle\sum_{j=-\infty}^{\infty} I_{[j\delta,(j+1)\delta)}(T_n) \\
&\qquad \times \textstyle\sum_{k=n}^{\infty} E\{T_k' - kc)\psi_{k-n}(j\delta, Y_{n+1}, \cdots, Y_k) \mid \mathscr{F}\} - nc \\
&= \textstyle\sum_{j=-\infty}^{\infty} I_{[j\delta,(j+1)\delta)}(T_n) V^*(j\delta) - nc \\
&\geqq \textstyle\sum_{j=-\infty}^{\infty} I_{[j\delta,(j+1)\delta)}(T_n)(V^*(T_n) + \delta) - nc \qquad \text{(Lemma 3)} \\
&= V^*(T_n) + \delta - nc \quad \text{a.s.}
\end{aligned}
$$

Therefore, $\operatorname{ess\,sup}_{N \geqq n} E(X_N \mid \mathscr{F}_n) \geqq V^*(T_n) - nc$ a.s.

To prove the reverse inequality, we are to show for all $N \geqq n$ $E(X_N \mid \mathscr{F}_n) \leqq V^*(T_n) - nc$ a.s. For a given rule $N \geqq n$, let $\psi_k(Y_1, \cdots, Y_k) = P(N = k \mid \mathscr{F}_n)$, and define $\psi_k'(Y_{n+1}, \cdots, Y_{n+k}) = \psi_{n+k}(Y_1, \cdots, Y_n, Y_{n+1}, \cdots, Y_{n+k})$ for fixed $Y_1, \cdots, Y_n$. Then letting $T_0' = T_n$ and, for $k \geqq 1$, $T_k' = \varepsilon_{n+k}(T'_{k-1} + Z_{n+k})$, we have

$$
\begin{aligned}
E(X_N \mid \mathscr{F}_n) &= \textstyle\sum_{k=n}^{\infty} E\{(T_k - kc)\psi_k(Y_1, \cdots, Y_k) \mid \mathscr{F}_n\} \\
&= \textstyle\sum_{k=0}^{\infty} E\{(T_k' - kc)\psi_k'(Y_{n+1}, \cdots, Y_{n+k}) \mid \mathscr{F}_n\} - nc \\
&\leqq V^*(T_n) - nc \quad \text{a.s.}
\end{aligned}
$$

□

As an immediate consequence of these two lemmas, we have

THEOREM 3. *The rule* $N^* = \min\{n \geqq 0 : T_n \geqq s\}$ *is optimal.*

2.2. *Evaluation of the optimal rule.* To evaluate the optimal value, s, let N_t denote the stopping rule $N_t = \min\{n \geqq 0 : T_n \geqq t\}$, and let $V_t(z)$ denote the expected return of this rule starting at z.

(5) $$V_t(z) = E_z(T_{N_t} - cN_t) .$$

The value of t that maximizes $V_t(z)$ is equal to s provided $z \leqq s$. Since s is known to be at least $-c/(1-p)$, we may evaluate s as the value of t that maximizes $V_t(-c/(1-p))$. The following theorem aids such an evaluation since it reduces computations involving N_t and T_{N_t} to computations involving the joint distribution of N_t' and $S_{N_t'}$, where $S_n = \sum_0^n Z_i$ (with $Z_0 = z$), and

$$N_t' = \min\{n \geqq 0 : S_n \geqq t\} .$$

(See also Lemma 7.) The distribution of N_t' and $S_{N_t'}$ has been studied at length in the literature on statistical sequential analysis, and various techniques, including Wald's fundamental identity, are available for use in computations.

Since the trivial case $P(Z \leqq 0) = 1$ has $s = -c/(1-p)$, we assume that $P(Z > 0) > 0$, so that $P(N_t' = \infty) < 1$ and $Ep^{N_t'} > 0$. If $N_t' = \infty$, then both $p^{N_t'}$ and $p^{N_t'}S_{N_t'}$ are interpreted to be zero.

THEOREM 4. *For $t \leqq z$, $V_t(z) = z$. For $t > z$,*

(6) $$V_t(z) = E_z p^{N_t'} S_{N_t'} + (1 - E_z p^{N_t'})(V_t(0) - c/(1-p)) ,$$

where

(7) $$V_t(0) = \frac{E_0 p^{N_t'} S_{N_t'}}{E_0 p^{N_t'}} - c\,\frac{1 - E_0 p^{N_t'}}{(1-p)E_0 p^{N_t'}} .$$

PROOF. Let $K = \min\{k \geqq 1 : \varepsilon_k = 0\}$. Then

(8) $$\begin{aligned} V_t(z) &= E_z\{T_{N_t} - cN_t\} \\ &= E_z\{I(N_t' < K)(S_{N_t'} - cN_t')\} + E_z\{I(N_t' \geqq k)(T_{N_t} - cN_t)\} \end{aligned}$$

where $I(A)$ represents the indicator random variable of the event A. Since $P(K > n) = \sum_{k=n+1}^{\infty}(1-p)p^k = p^n$, the first term of (8) may be evaluated at

(9) $$\begin{aligned} E_z\{I(N_t' < K)(S_{N_t'} - cN_t')\} &= E_z\{(S_{N_t'} - cN_t')E\{I(N_t' < K) \mid N_t'\}\} \\ &= E_z\{(S_{N_t'} - cN_t')p^{N_t'}\} . \end{aligned}$$

To evaluate the second term of (8), note that on $\{N_t' \geqq K\}$, $T_{N_t} - cN_t = \hat{T}_{\hat{N}_t} - c\hat{N}_t - cK$, where $\hat{T}_0 = 0$ and for $n \geqq 1$, $\hat{T}_n = \varepsilon_{K+n}(\hat{T}_{n-1} + Z_{K+n})$, and $\hat{N}_t = \min\{n \geqq 0 : \hat{T}_n \geqq t\}$. Then since $I(N_t' \geqq K)$ depends only on $X_1, \cdots, X_{k-1}$, and $\hat{T}_{\hat{N}_t} - cN_t$ depends only on $X_{K+1}, X_{K+2}, \cdots$, they are conditionally independent given K, and

(10) $$\begin{aligned} &E_z\{I(N_t' \geqq K)(T_{N_t} - cN_t)\} \\ &\quad = E_z\{E_z\{I(N_t' \geqq K) \mid K\}E\{\hat{T}_{\hat{N}_t} - c\hat{N}_t - cK \mid K\}\} \\ &\quad = E_z\{E_z\{I(N_t' \geqq K) \mid K\}(V_t(0) - cK)\} \\ &\quad = E_z\{I(N_t' \geqq K)(V_t(0) - cK)\} \\ &\quad = (1 - E_z p^{N_t'})V_t(0) - cE_z\{I(N_t' \geqq K)K\} . \end{aligned}$$

Now, since $E\{I(K \leqq n)K\} = \sum_{k=1}^{n} (1-p)p^{k-1}k = (1-p)(d/dp)\sum_0^n p^k = (1 - np^n(1-p) - p^n)/(1-p)$, we have

(11) $$E_z\{I(N_t' \geqq K)K\} = (1-(1-p)E_z\{N_t' p^{N_t'}\} - E_z p^{N_t'})/(1-p) .$$

Combining (8), (9), (10) and (11) gives formula (6) for $V_t(z)$. Formula (7) may be obtained by setting $z = 0$ in (6) and solving for $V_t(0)$. □

In addition to evaluating s as the value of t that maximizes $V_t(-c/(1-p))$ (or $V_t(0)$ if it is known that $s > 0$), one may derive the following equation and solve for s as the root. First note that (6) may be rewritten as

(12) $$V_t(z) = E_0 p^{N'_{t-z}}(S_{N'_{t-z}} + z) + (1 - E_0 p^{N'_{t-z}})(V_t(0) - c/(1-p)) .$$

We put $t = s$, and let $\delta = s - z > 0$.

(13) $$V^*(s-\delta) = E_0 p^{N_\delta'}(S_{N_\delta'} + s - \delta) + (1 - E_0 p^{N_\delta'})(V_s(0) - c/(1-p)) .$$

Let

(14) $$N'' = \min\{n \geqq 1 : \textstyle\sum_1^n Z_i > 0\} .$$

As $\delta \searrow 0$, $N_\delta' \searrow N''$, so that $E_0 p^{N_\delta'} \nearrow E_0 p^{N''}$. In addition, $S_{N_\delta'} \to S_{N''}$ and $p^{N_\delta'} S_{N_\delta'}$ is bounded by an integrable function: $p^{N_\delta'} S_{N_\delta'} = p^{N_\delta'} S^+_{N_\delta'} \leqq p^{N_\delta'} \sum_1^{N_\delta'} Z_i^+ \leqq \sum_1^{N_\delta'} p^i Z_i^+ \leqq \sum_1^\infty p^i Z_i^+$. Therefore taking the limit as $\delta \to 0$ in (13) yields

$$s = E_0 p^{N''}(S_{N''} + s) + (1 - E_0 p^{N''})(V_s(0) - c/(1-p)) .$$

Rearranging terms yields

(15) $$s = \frac{E_0 p^{N''} S_{N''}}{1 - E_0 p^{N''}} + V_s(0) - c/(1-p) .$$

2.3. *Special cases.* In certain cases, equation (15) for s simplifies considerably. We consider first the exponential case, in which the conditional distribution of Z given $Z > 0$ is exponential. Specifically, we assume that $F_Z(z) = (1-\alpha)G(z) + \alpha H(z)$, where G is a distribution function such that $G(0) = 1$, $H(z) = (1 - e^{-z/\mu})I(0 < z)$, and $0 < \alpha \leqq 1$. Let $S_n = \sum_1^n Z_i$, and N'' satisfy (14). It is well known that the conditional distribution of $S_{N''}$ given $N'' = n < \infty$ is simply exponential with distribution function H and with expectation μ. Similarly, the conditional distribution of $S_{N_t'} - t$ given $N_t' = n < \infty$ is H. Thus, from (7) and (15)

$$s = \frac{\mu E p^{N''}}{1 - E p^{N''}} + (\mu + s) - c\frac{1 - E p^{N_s'}}{(1-p)E p^{N_s'}} - c\frac{1}{1-p} .$$

Rearrangement yields

(16) $$\frac{c}{(1-p)E p^{N_s'}} = \frac{\mu}{1 - E p^{N''}} .$$

Furthermore, the distribution of N_s' is the same as the distribution of $N_1'' + \cdots + N_M''$ where $M - 1$ has a Poisson distribution with expectation s/μ, and given M, the variables $N_1'', \cdots, N_M''$ are i.i.d. with the distribution of N''. Hence, setting $\theta = Ep^{N''}$,

$$Ep^{N_s'} = Ep^{N_1''+\cdots+N_M''} = E\theta^M = \theta e^{-s(1-\theta)/\mu} . \tag{17}$$

Thus (16) may be solved for s provided θ is known, yielding

$$s = \frac{\mu}{1-\theta} \log \frac{\mu(1-p)\theta}{c(1-\theta)} .$$

Furthermore, θ can be computed from knowledge of the Laplace transform of G, $\varphi(t) = \int e^{tz}\, dG(z)$, which exists at least for all $t \geqq 0$. This may be done using a method similar to that of Lemma 8 of Dubins and Teicher [2], as follows,

$$\begin{aligned} \theta &= Ep^{N''} = E\{E\{p^{N''} \mid Z_1\}\} \\ &= E\{I(Z_1 \leqq 0)E\{p^{N''} \mid Z_1\} + I(Z_1 > 0)p\} \\ &= (1-\alpha)[\textstyle\int_{-\infty}^{0-} Ep^{1+N'_{-z}}\, dG(z) + pP(Z=0)\theta] + \alpha p \\ &= (1-\alpha)p\theta\varphi((1-\theta)/\mu) + \alpha p , \end{aligned} \tag{18}$$

where we have used (17) in the last step. If for example, G is concentrated at zero, so that $\varphi \equiv 1$, then $\theta = \alpha p/(1-(1-\alpha)p)$.

A second case in which (15) simplifies is *the geometric case*, where Z takes on only integer values and the conditional distribution of Z given $Z > 0$ is geometric. Specifically, the frequency function of Z is $f_Z(z) = (1-\alpha)g(z) + \alpha h(z)$, where g is an arbitrary frequency function on $\{\cdots, -1, 0\}$, $h(z) = (1-\pi)\pi^{z-1}$ for $z = 1, 2, \cdots, 0 \leqq \pi < 1$, and $0 < \alpha \leqq 1$. When $\pi = 0$, the distribution of Z is termed *elementary* in Dubins and Teicher [2]. Analogous to the exponential case, the distribution of $S_{N''}$ given $N'' = n < \infty$ is simply geometric on $\{1, 2, \cdots\}$ with parameter π and expectation $1/(1-\pi)$. Furthermore, for $t > 0$, the distribution of $S_{N_t'} - ([t]-1)$ given $N_t' = n < \infty$ is also this geometric distribution. We use $[x]$ to represent the smallest integer greater than or equal to x. Hence, (15) simplifies to

$$\frac{c}{(1-p)Ep^{N_s'}} = \frac{\pi + \theta - \pi\theta}{(1-\pi)(1-\theta)} + ([s]-s) \tag{19}$$

where $\theta = Ep^{N''}$. In addition, the distribution of N_s' is the same as the distribution of $N_1'' + \cdots + N_M''$ where $M - 1$ has a binomial distribution with sample size $[s] - 1$ and probability of success $1 - \pi$ (most easily proved by induction on $[s]$), and given M, the variables $N_1'', \cdots, N_M''$ are i.i.d. with the distribution of N''. Hence,

$$Ep^{N_s'} = Ep^{N_1''+\cdots+N_M''} = E\theta^M = \theta(\pi + \theta - \pi\theta)^{[s]-1} . \tag{20}$$

Substitution into (19) yields

$$\frac{c}{(1-p)\theta}(\pi + \theta - \pi\theta)^{-[s]} = \frac{1}{(1-\pi)(1-\theta)} + \frac{[s]-s}{\pi + \theta - \pi\theta} . \tag{21}$$

Given knowledge of θ, this equation may be solved for s by first finding $[s]$ as

$$[s] = \min\left\{n : (\pi + \theta - \pi\theta)^{-n} \geqq \frac{(1-p)\theta}{c(1-\pi)(1-\theta)}\right\}, \tag{22}$$

and then adjusting s to get equality in (21). It is easily seen that this method leads to the unique solution of (21). If the initial value of z is an integer, then the rule N_s is identical to the rule $N_{[s]}$, so that the optimal rule is to stop at the first n such that $(\pi + \theta - \pi\theta)^{-T_n} \geqq (1-p)\theta/(c(1-\pi)(1-\theta))$.

Analogous to (18), the value of θ can be obtained from knowledge on the generating function of g, $M(t) = \sum_{z=-\infty}^{0} t^{-z} g(z)$, which exists at least for $0 \leqq t \leqq 1$.

$$\begin{aligned} \theta &= Ep^{N''} = E\{E\{p^{N''} \mid Z_1\}\} \\ &= (1-\alpha) \textstyle\sum_{z=-\infty}^{0} Ep^{1+N'_{1-z}} g(z) + \alpha p \\ &= (1-\alpha)p\theta M(\pi + \theta - \pi\theta) + \alpha p\,. \end{aligned} \tag{23}$$

The value of θ is the unique root of this equation between zero and one.

The case of Z degenerate at 1, treated by Starr [5], is obtained from the geometric case when $\alpha = 1$ and $\pi = 0$. In this case, (23) gives $\theta = p$, and (22) gives $[s] = \min\{n : p^{n+1} \leqq c\}$, which agrees with Starr's formula.

A third case in which equation (15) for s simplifies is when $P(Z \geqq 0) = 1$ and $pEZ \leqq c$. In this case, $N'' \equiv 1$ and (15) is satisfied by $s = (pEZ - c)/(1 - p) \leqq 0$, since $V_s(0) = 0$. In fact, in this case the stopping rule problem is monotone and N^* is the one-stage look-ahead rule. To compute the one-stage look-ahead rule, you compare what you already have at stage n, say $t - nc$, with what you expect to have if you continue just one stage and then stop, $E\varepsilon(t + Z) - (n + 1)c$. The one-stage look-ahead rule requires stopping if the former is larger than the latter, which occurs if and only if $t > (pEZ - c)/(1 - p)$. Here, we continue only if our accumulated return is sufficiently negative; we stop when our accumulated return finally exceeds $(pEZ - c)/(1 - p)$, or when we experience our first failure (which should probably be called a success in this situation).

Among the problems discussed in this paper, the only other nontrivial situation for which the one-stage look-ahead rule is optimal, is the discount model with $P(Z \geqq 0) = 1$ and $p = 1$, as pointed out by Haggstrom.

Whenever $P(Z \geqq 0) = 1$, the optimal rule for either model may be considered as a one-stage look-ahead rule for the problem modified so that one is obliged to stop at the first failure and receive $V_s(0)$, an idea due to Ross [4]. However, the problem of computing s and $V_s(0)$ still remains.

The results for the cost model may be generalized by allowing the costs to be random. Thus, we take $Y_i = (Z_i, \varepsilon_i, C_i)$ where C_i are i.i.d. with $EC_1 > 0$, and C_i represents the cost during the ith period. We assume for simplicity that $\{(Z_i, C_i)\}$ and $\{\varepsilon_i\}$ are independent. If both $E(Z^+)^2 < \infty$ and $E(C^-)^2 < \infty$, then $E\sup_n (T_n - C^{(n)}) < \infty$, where $C^{(n)} = \sum_1^n C_i$. For $E\sup_n (T_n - C^{(n)}) \leqq E\sup_n (T_n - nc') + E\sup(nc' - C^{(n)})$; the first term is finite for any $c' > 0$ from

Theorem 2, and the second term is finite if $0 < c' < EC_1$ from Lemma 1. Since under these conditions we also have $\lim (T_n - C_n) = -\infty$, N^* is optimal. The proof given that N^* has the simple form of Theorem 3 still goes through, but the evaluation of the optimal s is difficult in general. However, if one assumes in addition that $\{Z_i\}$ and $\{C_i\}$ are independent, then the formulae of Sections 2.2 and 2.3 are valid as is, if c is replaced by EC_1.

3. The discount model. Let $\{Z_i\}$, $\{\varepsilon_i\}$, $0 < p \leqq 1$, z, $0 < \beta < 1$, and $\{T_n\}$ be as defined in Section 1, and consider the problem of finding a stopping rule N to maximize EX_N, where X_n is given by (2) and $X_\infty \equiv 0$. The development we give parallels that of Section 2, so we give less details. Where the proofs are essentially identical, they are omitted.

3.1. *Form of the optimal rule.* We assume $EZ^+ < \infty$ and show that the hypotheses of Theorem 1 are satisfied.

THEOREM 5. *$E \sup \beta^n T_n \leqq \infty$ if and only if $EZ^+ < \infty$.*

PROOF. (*if*) $E \sup \beta^n T_n \leqq E \sum_0^\infty \beta^n T_n^+ \leqq \sum_0^\infty \beta^n E \sum_0^n Z_k^+ \leqq \sum_0^\infty \beta^n n EZ^+ < \infty$. (*only if*) $\sup \beta^n T_n \geqq \beta^K Z_K^+$ where $K = \min \{k > 1 : \varepsilon_k = 1, \varepsilon_{k-1} = 0\}$. Since K and $\{Z_i\}$ are independent, $\infty > E \sup \beta^n T_n \geqq E\beta^K Z_K^+ = E\beta^K EZ_1^+$, completing the proof since $E\beta^K > 0$. □

It is clear that $\limsup X_n \leqq X_\infty$ a.s., since $\beta^n T_n \leqq \beta^n \sum_1^n Z_i^+ \to 0$ a.s. Therefore the stopping rule N^*, of Theorem 1 is optimal. To evaluate the rule N^*, we first find the form of $V^*(z) = \sup_N E_z X_N$.

LEMMA 5. *$V^*(z)$ is convex, nondecreasing and nonnegative. There is a unique number $s \geqq 0$ such that*

$$z < V^*(z) \leqq s \quad \textit{for} \quad z < s \qquad \textit{and}$$
$$V^*(z) = z \quad \textit{for} \quad z \geqq s\,.$$

The proof is omitted.

LEMMA 6. $\operatorname{ess\,sup}_{N \geqq n} E(X_N \mid \mathscr{F}_n) = \beta^n V^*(T_n)$ a.s.

The proof is omitted.

THEOREM 6. *The rule $N^* = \min \{n \geqq 0 : T_n \geqq s\}$ is optimal.*

This follows immediately from Lemmas 5 and 6.

3.2. *Evaluation of the optimal rule.* To evaluate the optimal value of s, we may compute $V_t(0) = E_0 \beta^{N_t} T_{N_t}$ for the rule $N_t = \min \{n \geqq 0 : T_n \geqq t\}$. The following lemma allows us to reduce computations involving the joint distribution of N_t and T_{N_t} to computations involving the joint distribution of N_t' and $X_{N_t'}$, where $S_n = \sum_1^n Z_i$ and $N_t' = \min \{n \geqq 0 : S_n \geqq t\}$. Throughout this and the following section, E represents E_0, the expectation when $z = 0$.

LEMMA 7. *Let g be a measurable real-valued function of a real variable. If $E\beta^{N_t}g(T_{N_t})$ exists and is finite, then*

$$E\beta^{N_t}g(T_{N_t}) = \frac{(1-p\beta)E(p\beta)^{N_t'}g(S_{N_t'})}{(1-\beta)+(1-p)\beta E(p\beta)^{N_t'}} .$$

PROOF. To simplify the notation, we drop the subscript t from N_t and N_t'. Let $K = \min\{n \geqq 1 : \varepsilon_n = 0\}$.

$$\begin{aligned} E\beta^N g(T_N) &= E\{I(N' < K)\beta^{N'}g(S_{N'}) + I(K \leqq N')\beta^K E\beta^N g(T_N)\} \\ &= E\{\beta^{N'}g(S_{N'})P(K > N' \mid N')\} + E\{E\{I(K \leqq N')\beta^K \mid N'\}\}E\beta^N g(T_N) . \end{aligned}$$

Now $P(K > n) = p^n$, and $E\{I(K \leqq n)\beta^K\} = (1-p)\sum_1^n \beta^k p^{k-1} = (1-p)\beta(1-(p\beta)^n)/(1-p\beta)$. Hence

$$E\beta^N g(T_N) = E(p\beta)^{N'}g(S_{N'}) + \frac{(1-p)\beta}{1-p\beta}(1 - E(p\beta)^{N'})E\beta^N g(T_N) .$$

If $E\beta^N g(T_N)$ is finite, it may be found from this equation, and the formula of the lemma results. □

THEOREM 7.

$$V_t(0) = \frac{(1-p\beta)E(p\beta)^{N_t'}S_{N_t'}}{(1-\beta)+(1-p)\beta E(p\beta)^{N_t'}} .$$

PROOF. The lemma applies, since $E|\beta^{N_t}T_{N_t}| \leqq ET_{N_t}^+ \leqq t + EZ^+$. □

A simple method to find s is to search for the value of t that maximizes $V_t(0)$. Alternately, the following equation may be used. If we start at s, we are indifferent between stopping and continuing. The rule that uses $N'' = \min\{n \geqq 0 : \sum_1^n Z_i > 0\}$ until the first failure and N_s thereafter is also optimal at s. Hence,

$$s = E\{I(K > N'')\beta^{N''}(s + S_{N''}) + I(K \leqq N'')\beta^K V_s(0)\}$$

where $K = \min\{k \geqq 1 : \varepsilon_k = 0\}$, which simplifies to

$$s = \frac{E(p\beta)^{N''}S_{N''}}{(1-E(p\beta)^{N''})} + \frac{(1-p)\beta}{1-p\beta}V_s(0) . \tag{24}$$

When $p = 1$, this reduces to the Dubins–Teicher result. If $P(Z \leqq 0) = 1$, then $s = 0$.

3.3. *Special cases.* First, consider the exponential case where $F_Z(z) = (1-\alpha)G(z) + \alpha H(z)$ with $H(z) = (1 - e^{-z/\mu})I(0 < z)$, $G(0) = 1$, and $0 < \alpha \leqq 1$. The conditional distribution of $S_{N''}$ given N'' and of $S_{N_t'} - t$ given N_t' are both H, so that (24) becomes

$$s = \frac{E(p\beta)^{N''}\mu}{1-E(p\beta)^{N''}} + \frac{(1-p)\beta E(p\beta)^{N_s'}(s+\mu)}{(1-\beta)+(1-p)\beta E(p\beta)^{N_s'}} .$$

Furthermore, similar to (17)

$$E(p\beta)^{N_s'} = \theta e^{-s(1-\theta)/\mu} ,$$

where θ denotes $E(p\beta)^{N''}$. Hence,

$$s = \frac{\theta\mu}{1-\theta} + \frac{(1-p)\beta\mu}{(1-\theta)(1-\beta)} e^{-s(1-\theta)/\mu} .$$

There exists a unique root to this equation that can be found by numerical methods.

To compute θ, one may use equation (19) with p replaced by $p\beta$, namely

$$\theta = (1-\alpha)p\beta\theta\varphi((1-\theta)/\mu) + \alpha p\beta$$

where φ is the Laplace transform of G.

Next, consider the geometric case where $f_Z(z) = (1-\alpha)g(z) + \alpha h(z)$ with $h(z) = (1-\pi)\pi^{z-1}$ for $z = 1, 2, \cdots, 0 \leqq \pi < 1$, and $g(z)$ an arbitrary frequency function on $\{\cdots, -1, 0\}$. The conditional distribution of $S_{N''}$ given N'' and of $S_{N_t'} - ([t]-1)$ given N_t' are both h, so that (24) becomes

$$s = \frac{\theta}{(1-\pi)(1-\theta)} + \frac{(1-p)\beta E(p\beta)^{N_s'}([s] - 1 + 1/(1-\pi))}{(1-\beta) + (1-p)\beta E(p\beta)^{N_s'}}$$

where $\theta = E(p\beta)^{N''}$. Also, from (20)

$$E(p\beta)^{N_s'} = \theta(\theta + \pi - \theta\pi)^{[s]-1} ,$$

so that

$$[s] - \frac{(1-\beta)\theta + (1-p)\beta\theta(\theta+\pi-\theta\pi)^{[s]}}{(1-\pi)(1-\beta)(1-\theta)}$$
$$= ([s] - s)\frac{(1-\beta) + (1-p)\beta\theta(\theta+\pi-\theta\pi)^{[s]-1}}{(1-\beta)} .$$

To solve this equation, first find the smallest integer $[s]$ such that the left side is nonnegative, and then adjust s on the right side to get equality.

If $p = 1$, we obtain $s = \theta/((1-\theta)(1-\pi))$. If $p = 1$, and $\pi = 0$, we obtain $s = \theta/(1-\theta)$, the value obtained by Dubins and Teicher for the elementary case.

Finally, to compute θ, we may use equation (23) with p replaced by $p\beta$:

$$\theta = (1-\alpha)p\beta\theta M(\pi + \theta - \pi\theta) + \alpha p\beta$$

where M is the generating function of g.

As in the paper of Dubins and Teicher, we may allow the discount factor β to be random. That is, we may assume that $\{(Z_i, \beta_i)\}$ are i.i.d. independent of $\{\varepsilon_i\}$, where β_i represents the discount factor of the ith period, and $0 < \beta_i < 1$. The hypotheses of Theorem 1 are still valid for the returns $X_n = \beta^{(n)}T_n$ where $\beta^{(n)}$ represents $\prod_1^n \beta_i$. The optimal rule N^* has the simple form of Theorem 6 but the evaluation of s is difficult in general. However, under the additional assumption that $\{Z_i\}$ and $\{\beta_i\}$ are independent, the formulae of Sections 3.2 and 3.3 hold provided β is replaced by $E\beta_1$.

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