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SHARP UPPER BOUNDS FOR PROBABILITY ON AN INTERVAL WHEN THE FIRST THREE MOMENTS ARE KNOWN

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The subject of this research is the maximum probability assignable to closed subintervals of a closed, bounded, nondegenerate interval by distributions on that interval whose first three moments are specified. This maximum probability is explicitely displayed as a function of both the moments and the subintervals. The ready application of these results is illustrated by numerical examples.

0. Summary. This paper studies the maximum probability that distributions on a closed, bounded, nondegenerate interval can assign to closed subintervals when their first 3 moments are specified. We treat this maximum probability as a function both of the moments and the subintervals. In Section 1, moment normalization and background structure is defined. Some basic theorems relating to the sharp upper bound which hold for moment spaces of arbitrary dimension are stated. The several forms which the sharp upper bound function assumes are displayed in Section 2. Section 3 exhibits the moment-space partitions, on the sets of which these forms obtain, in terms of their dependency on the subintervals. In Section 4 procedures are provided to facilitate identification of the sets in these partitions which contain a given moment point. Several illustrative numerical examples are given in Section 5. A technical restriction on the subintervals allowed for consideration is removed in Section 6. Results are specialized in Section 7 to degenerate intervals and intervals with left endpoint zero or one, with specific numerical examples of direct and indirect application of these results given in Section 8. Lemmas which underly the proofs and a detailed proof for one form of the sharp upper bound function are given in Section 9.

The format of this paper has been designed with utility of these results as a primary objective. Some proofs not essential to continuity have been deleted and most of the proofs that are given have been reserved to the final section.

1. Some theorems for moment spaces of dimension n. Let

$$M_n = \{(\int x\, dP(x), \int x^2\, dP(x), \cdots, \int x^n\, dP(x)) : P \in \mathscr{P}\},$$

where n is an arbitrary positive integer, the integrals are taken over $[0, 1]$, and $\mathscr{P}$ is the class of all probability measures on (the Borel subsets of) $[0, 1]$. These

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moment spaces are studied in [1] and [2]. We note here the well-known fact that M_n is the convex hull of the space curve

$$\{(t, t^2, \cdots, t^n) : 0 \leqq t \leqq 1\} . \tag{1.1}$$

This paper is concerned with a collection of sharp upper bound functions on M_n; specifically, with the collection

$$\{\mathscr{U}_{a,b}^{(n)} : (a, b) \in T\} ,$$

where the index set

$$T = \{(a, b) : 0 \leqq a \leqq b \leqq 1\}$$

is identified one-one with the closed subintervals of $[0, 1]$;

$$\mathscr{U}_{a,b}^{(n)}(\mathbf{c}) = \max \{P([a, b]) : P \in V_n(\mathbf{c})\} ,$$

and

$$V_n(\mathbf{c}) = \{P \in \mathscr{P} : \textstyle\int x^i \, dP(x) = c_i, i = 1, 2, \cdots, n\} ,$$

for each $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ in M_n.

Let $M_n{}^\circ \subset M_n$, for $n > 1$, consist of those M_n points $(c_1, c_2, \cdots, c_{n-1}, c_n)$ for which $(c_1, c_2, \cdots, c_{n-1})$ belongs to the interior of M_{n-1}. Take $M_1{}^\circ = M_1$. $M_n{}^\circ$ contains all interior points of M_n and some of its boundary points as well. Restricting the domain of the sharp upper bound functions $\mathscr{U}_{a,b}^{(n)}$ to $M_n{}^\circ$ entails no loss in generality and as a technical convenience we shall henceforth consider this restriction to hold.

For each positive integer j define

$$\nu_j(c_1, c_2, \cdots) = c_j$$

$$\nu_j{}^{\pm}(c_1, c_2, \cdots) = \begin{matrix}\max\\ \min\end{matrix} \{d : (c_1, c_2, \cdots, c_{j-1}, d) \in M_j\} ,$$

for each moment sequence $(c_1, c_2, \cdots)$ corresponding to a $P \in \mathscr{P}$. Note that the ν_j are simply coordinate functions and that the $\nu_j{}^{\pm}$ depend only on the first $j - 1$ moments. Of course, the range function

$$R_{j-1} = \nu_j{}^+ - \nu_j{}^- ,$$

also depends only on the first $j - 1$ moments. Viewed as a function on $M_j{}^\circ$ which is independent of its last argument, it is readily seen to be everywhere positive on this domain (e.g., see [2], Corollaries 1.1 and 2.2b in Chapter 4). The normalized jth moment function $p_j = 1 - q_j$ is defined on $M_j{}^\circ$ by taking

$$p_j = (\nu_j - \nu_j{}^-)/R_{j-1} .$$

In [5], it was shown that

$$R_j = \textstyle\prod_{i=1}^{j} p_i q_i . \tag{1.2}$$

It was established in [6] that the vector valued onto mapping $\mathbf{p}_n = (p_2, p_2, \cdots, p_n)$,

$$\mathbf{p}_n : M_n{}^\circ \to (0, 1)^{n-1} \times [0, 1] \tag{1.3}$$

is one-one and explicit forms for the inverse mapping were given. Here, of course, we regard each p_i as a function on $M_n{}^\circ$ which is independent of its last $n - i$ arguments.

Let $M^\circ_{n,J}$ be the strict analogue of $M_n{}^\circ$ relative to an arbitrary, nondegenerate, closed interval

$$J = [\alpha, \beta] .$$

Let $\mathbf{p}_{n,J}$ be the vector valued map defined on $M^\circ_{n,J}$ in strict analogy to the definition of $\mathbf{p}_n$ on $M_n{}^\circ$ and $\mathscr{U}^{(n)}_{a,b,J}$, the similarly defined sharp upper bound function. Note that $\mathbf{p}_{n,J}$ is always (for every J) one-one onto $(0, 1)^{n-1} \times [0, 1]$.

The following theorem shows that we lose nothing by restriction to the case $J = [0, 1]$, and motivates our interest in normalized moment functions as they apply to the calculation of sharp upper bounds. $\circ$ denotes composition.

THEOREM 1.1. *Let $J = [\alpha, \beta]$ be an arbitrary, closed, nondegenerate interval, then for each a, b such that $\alpha \leqq a \leqq b \leqq \beta$,*

$$\mathscr{U}^{(n)}_{a,b,J} \circ p^{-1}_{n,J} = \mathscr{U}^{(n)}_{(a-\alpha)/(\beta-\alpha),(b-\alpha)/(\beta-\alpha)} \circ p_n{}^{-1} .$$

A proof of the above theorem is given in [7]. It follows in a straightforward way from Theorem 5 in [6].

A basic symmetry property possessed by the sharp upper bound function appears in an easy and natural way when expressed in terms of normalized moment functions. Let

$$\mathbf{k} : M_n{}^\circ \to M_n{}^\circ$$

be defined for each $\mathbf{c}$ in $M_n{}^\circ$ by the requirement that

$$p_i(\mathbf{K}(\mathbf{c})) = p_i(\mathbf{c}) \quad \text{or} \quad q_i(\mathbf{c}) ,$$

for $i = 1, 2, \cdots, n$, according as i is even or odd. $\mathbf{K}$ so defined is a one-one onto map because $\mathbf{p}_n = (p_1, p_2, \cdots, p_n)$ is one-one onto its range. It is clear that composing $\mathbf{K}$ with moment functions can lead to no confusion concerning the order of the space on which the moment function is defined. Moreover, $\mathbf{K}$ is obviously idempotent. Composition of $\mathbf{K}$ with the sharp upper bound function yields

THEOREM 1.2. *For each positive integer n and for each (a, b) in T*

$$\mathscr{U}^{(n)}_{a,b} = \mathscr{U}^{(n)}_{1-b,1-a} \circ \mathbf{K} .$$

Proof of this theorem is straightforward and has been given in [7].

For each (a, b) in T, let

$$M_n{}^{[a,b]} = \{(\textstyle\int x \, dP(x), \int x^2 \, dP(x), \cdots, \int x^n \, dP(x)) : P \in \mathscr{P}, P([a, b]) = 1\} .$$

Observe that $M_n{}^{[a,b]}$ is equal to $M_{n,[a,b]}$ which we take to be the strict analogue of M_n with $[0, 1]$ replaced by $[a, b]$; that it is the convex hull of $\{(t, t^2, \cdots, t^n) : a \leqq t \leqq b\}$; that of course it is a subset of M_n. It is evident that

$$(1.4) \qquad \mathscr{U}^{(n)}_{a,b}(\mathbf{c}) = 1 , \qquad \mathbf{c} \in M_n{}^{[a,b]} , \qquad (a, b) \in T .$$

Below we state without proof (specializing to the case at hand) a theorem due to J. H. B. Kemperman [3] which underlies the specific results of the following sections.

THEOREM 1.3 (Kemperman). *Let* $(a, b) \in T$. *Let* B_0 *denote the intersection between* M_n *and a hyperplane of support for* M_n; B_1, *the intersection between* $M_n^{[a,b]}$ *and a hyperplane of support for* $M_n^{[a,b]}$ *which is parallel to the first hyperplane and which separates it from* $M_n^{[a,b]}$. *Then for each* $\mathbf{c}$ *interior to* M_n *such that*

$$\mathbf{c} \in \operatorname{conv}(B_0 + B_1)\,, \tag{1.5}$$

$$\mathscr{U}_{a,b}^{(n)}(\mathbf{c}) = D(B_0, \mathbf{c})/D(B_0, B_1)\,,$$

where conv *denotes convex hull,* $+$ *denotes union for disjoint sets, and D denotes ordinary Euclidean distance.*

Moreover, for each $\mathbf{c}$ *interior to* M_n (i.e., *for which* $0 < p_n(\mathbf{c}) < 1$), *but not in* $M_n^{[a,b]}$, *there exists at least one, and for almost all (n-dimensional Lebesgue measure) such* $\mathbf{c}$, *at most one pair of hyperplanes as above described such that* (1.5) *holds.*

In the terminology of [3], pairs of support hyperplanes with the properties cited in the above theorem are referred to as "admissible." See pages 101, 102 and Section 6 of [3] for the proof of this theorem and discussion in more general context.

We conclude this section by introducing three basic function sequences. Let

$$\zeta_j = q_{j-1}p_j\,, \qquad \gamma_j = p_{j-1}q_j\,, \qquad j = 2, 3, \cdots,$$

and let

$$\zeta_1 = p_1 = \nu_1\,, \qquad \gamma_1 = q_1\,.$$

Define the partial sums

$$S_j = \zeta_1 + \zeta_2 + \cdots + \zeta_j\,, \qquad j = 1, 2, \cdots.$$

It is an immediate consequence of (1.2) that

$$\textstyle\prod_{j=1}^n \zeta_j = \nu_n - \nu_n^-\,, \qquad \prod_{j=1}^n \gamma_j = \nu_n^+ - \nu_n\,.$$

We shall have frequent use only for the first three terms of the above sequences. Only some of these in fact suffice to exhibit the inverse map to (1.3) for $n = 3$. Thus

$$\nu_1 = \zeta_1\,, \qquad \nu_2 = \zeta_1\zeta_2 + \zeta_1^2 = \zeta_1 S_2\,, \qquad \nu_3 = \zeta_1\zeta_2\zeta_3 + \zeta_1 S_2^2\,. \tag{1.6}$$

2. The sharp upper bound function. The upper bound function $\mathscr{U}_{a,b}^{(3)}$, which we view as a composite function through the normalizing map

$$\mathbf{p}_3 = (p_1, p_2, p_3)\,,$$

assumes a number of distinct analytic forms on the several sets of an M_3° partition determined by the index (a, b). (By a partition of a set A we mean here a nonempty disjoint class of A-subsets whose union is A. We allow that some of these subsets might be empty.) These forms and the partitions on whose sets they hold vary in the degree of their complexity as (a, b) varies over T.

For example, when $a = 0$, $b = 1$, the partition above referred to contains only one nonempty set, namely $M_3{}^\circ$ itself, and the upper bound function is constant on this set. That is,

$$\mathscr{U}_{0,1}^{(3)} = 1 .$$

At its most complex, the partition contains 5 nonempty sets.

In this and subsequent sections we shall make use of the following notation. For each (a, b) in T, let

$$\begin{aligned} s(a, b) &= a + b + (ab)^{\frac{1}{2}} , \\ s^*(a, b) &= a + b - 1 - ((1 - a)(1 - b))^{\frac{1}{2}} \end{aligned} \tag{2.1}$$

and for each triple a, b, t such that $0 \leqq a, b, t \leqq 1$ let $w_{a,b}$, $\xi_{a,b}$, $\hat{\xi}_{a,b}$, $\xi^*_{a,b}$, $\xi_{a,b,t}$ denote the functions on $M_3{}^\circ$ here defined.

$$w_{a,b} = \zeta_1\zeta_2 - (\zeta_1 - a)(b - \zeta_1) = \nu_2 - (a + b)\nu_1 + ab . \tag{2.2}$$

$$\xi_{a,b} = \frac{(S_2 - a)(b - S_2)}{\zeta_2 q_2} , \qquad \hat{\xi}_{a,b} = 1 - \xi^*_{a,b} = \frac{(\gamma_2 - a)(b - \gamma_2)}{\gamma_2 p_2} . \tag{2.3}$$

$$\xi_{a,b,t} = (1 - t)\xi_{a,b} + t\xi^*_{a,b} . \tag{2.4}$$

It is easy to show that each of the above functions is invariant under permutation of its indices.

Our development will now proceed as follows. We introduce a T-indexed partition of $M_3{}^\circ$ consisting of five sets named respectively

$$\Lambda_1(a, b), \Lambda_2(a, b), \cdots, \Lambda_5(a, b) . \tag{2.5}$$

It is respectively upon these five sets that the five distinct forms for $\mathscr{U}_{a,b}^{(3)}$ are assumed. We will defer explicit definition of these sets to Section 3. Suffice it here to say that for each (a, b) in T, each of them (provided we add it to a set of 3-dimensional Lebesgue measure zero), is a simplex which has vertices on the curve (1.1) or else it is a union of such simplices as one or more vertices vary over part of (1.1).

For a technical reason which derives from the method of proof and need not be developed here we shall *for the remainder of this section and throughout Sections* 3 *and* 4 *as well assume that* (a, b) *in* T *satisfies the inequality*

$$s^*(a, b) \leqq 0 . \tag{2.6}$$

That is, we shall assume that (a, b) belongs to the region below (and including) the upper curve in Figure 1. For all such (a, b), we proceed here to exhibit the five distinct forms for the sharp upper bound function to which reference has been made. The results which obtain when (2.6) does not hold are deferred to Section 6 following explicit definition of the sets (2.5) for such (a, b). The division of intervals into two types (those which satisfy (2.6) and those which do not) simplifies definiton of the sharp upper bound function and the sets (2.5) on which its distinct forms are assumed.

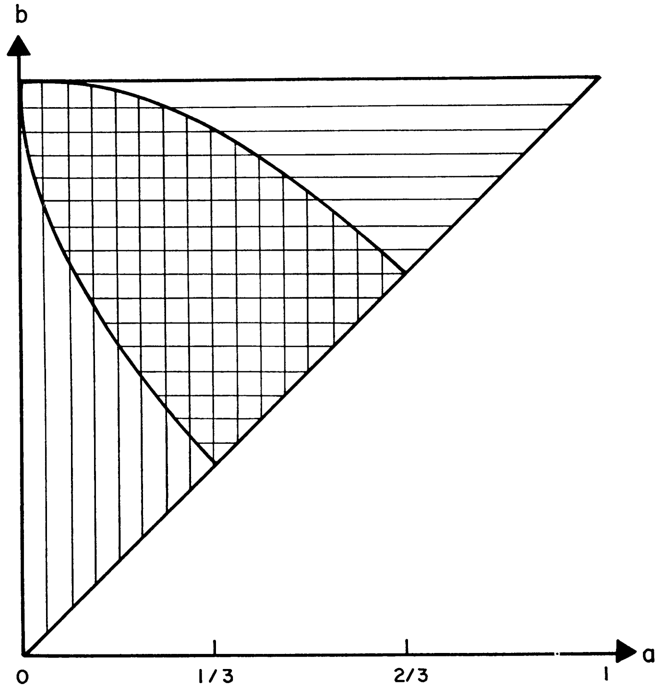

FIG. 1. Subsets of $T = \{(a, b) : 0 \leqq a \leqq b \leqq 1\}$.
$s^*(a, b) \leqq 0$ in region marked by vertical lines.
$s(a, b) \geqq 1$ in region marked by horizontal lines.

The main results of this section now follow.

On $\Lambda_1(a, b)$,

$$\mathscr{U}_{a,b}^{(3)} = \frac{\zeta_1}{\delta}\left(1 + \frac{\xi_{\delta,\delta}}{p_3 - \xi_{\lambda,\delta}}\right),$$

where $\lambda = \lambda_{a,b}$ and $\delta = \delta_{a,b}$ are respectively 3 and 2 valued functions on $\Lambda_1(a, b)$ as defined below. Specifically

$$\Lambda_1(a, b) = \textstyle\sum_{j=1}^{3} \Lambda_{1,j}(a, b)\,,$$

while

$$\begin{aligned} \lambda(\mathbf{c}) &= a, 0, \text{ or } b\,, \quad \text{according as } \mathbf{c} \in \Lambda_{1,1}(a, b), \Lambda_{1,2}(a, b), \text{ or } \Lambda_{1,3}(a, b) \\ \delta(\mathbf{c}) &= a \text{ or } b\,, \quad \text{according as } \mathbf{c} \in (\Lambda_{1,1}(a, b) + \Lambda_{1,2}(a, b)) \text{ or } \Lambda_{1,3}(a, b)\,. \end{aligned}$$

The sets $\Lambda_{1,j}(a, b)$ are explicitly defined in Section 3.

On $\Lambda_2(a, b)$,

$$\mathscr{U}_{a,b}^{(3)} = 1\,.$$

On $\Lambda_3(a, b)$,

$$\begin{aligned} \mathscr{U}_{a,b}^{(3)} &= R_2(p_3 - \xi_{r,\hat{s}})/d\,, \quad a + b \neq 1 \\ &= \gamma_1\gamma_2/a(1 - a)\,, \quad a + b = 1\,, \end{aligned}$$

where

$$d = d(a, b) = ab(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 \quad \text{or} \quad ab(1-a)(1-b)/(1-a-b)$$
$$r = r(a, b) = s(a, b) \quad \text{or} \quad (a^2 + ab + b^2 - a - b)/(1-a-b)$$
$$\hat{s} = \hat{s}(a, b) = s(a, b) \quad \text{or} \quad 1 ,$$

all respectively, according as

$$s(a, b) \leqq \quad \text{or} \quad \geqq 1 .$$

(Note that the set of (a, b) for which $s(a, b) \leqq 1$ is the region in Figure 1 below and including the lower curve, and that $\hat{s}(a, b)$ is simply the minimum of $s(a, b)$ and 1.)

On $\Lambda_4(a, b)$,

$$\mathscr{U}_{a,b}^{(3)} = 1 - \frac{(w_{a,b})^3}{R_2^2(p_3 - \xi_{a,a,b})(p_3 - \xi_{a,b,b})} .$$

On $\Lambda_5(a, b)$,

$$\mathscr{U}_{a,b}^{(3)} = \frac{\gamma_1}{1 - \delta^*}\left(1 + \frac{\hat{\xi}_{\delta^*,\delta^*}}{q_3 - \hat{\xi}_{\delta^*,\lambda^*}}\right),$$

where $\lambda^* = \lambda_{a,b}^*$ and $\delta^* = \delta_{a,b}^*$ are respectively 3 and 2 valued functions on $\Lambda_5(a, b)$ as defined below. Specifically,

$$\Lambda_5(a, b) = \textstyle\sum_{j=1}^3 \Lambda_{5,j}(a, b) ,$$

while

$$\lambda^*(\mathbf{c}) = a, 1, \quad \text{or} \quad b , \quad \text{according as} \quad \mathbf{c} \in \Lambda_{5,1}(a, b), \Lambda_{5,2}(a, b), \quad \text{or} \quad \Lambda_{5,3}(a, b) ,$$
$$\delta^*(\mathbf{c}) = a \quad \text{or} \quad b , \quad \text{according as} \quad \mathbf{c} \in \Lambda_{5,1}(a, b) \quad \text{or} \quad (\Lambda_{5,2}(a, b) + \Lambda_{5,3}(a, b)) .$$

The sets $\Lambda_{5,j}(a, b)$ are explicitly defined in Section 3.

3. Partitions of $M_3°$. In this section we define the five members of the $M_3°$ partition, listed in (2.5), on which the distinct forms for $\mathscr{U}_{a,b}^{(3)}$ given in Section 2 obtain.

We employ two notational conventions: 1. Functions defined on the interior of M_2 or M_1 will hereafter be viewed as functions on $M_3°$ which are independent of their third or their second and third arguments. 2. $M_3°$ subsets determined by functional relations on $M_3°$ will be denoted by parentheses about the statement of relation with function arguments omitted. *All sets so represented are to be interpreted as $M_3°$ subsets.* Thus for example consider the following "cylinder" set whose base is a subset of the interior of M_2.

$$\begin{aligned}(w_{a,b} < 0) &= \{\mathbf{c} \in M_3° : w_{a,b}(\mathbf{c}) < 0\} \\ &= \{\mathbf{c} : c_2/c_1^2 \leqq c_3 \leqq c_2 - ((c_1 - c_2)^2/(1 - c_1)), \\ &\qquad c_1^2 < c_2 < (a + b)c_1 - ab, a < c_1 < b\} .\end{aligned}$$

We need the following additional notation. For each pair a, b such that $0 \leqq a, b \leqq 1$, let

$$h_{a,b} = 1 - h'_{a,b} = I_{(w_{a,b}<0)} , \tag{3.1}$$

(where I denotes an indicator function), and for each triple a, b, t such that $0 \leqq a \leqq b \leqq t \leqq 1$, let

$$h_{a,b,t} = 1 - h'_{a,b,t} = h_{a,b} + h_{b,t} \,. \tag{3.2}$$

Finally, let

$$\eta_a = \frac{a(S_2 - a)}{q_2(\zeta_1 - a)} \quad \text{on} \quad (\zeta_1 \neq a)\,, \qquad 0 \leqq a \leqq 1\,. \tag{3.3}$$

With the exception of $\Lambda_{5,1}$ and $\Lambda_{5,3}$, each of the nine sets

$$\Lambda_{1,1}, \Lambda_{1,2}, \Lambda_{1,3}, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_{5,1}, \Lambda_{5,2}, \Lambda_{5,3} \tag{3.4}$$

(the arguments, which must satisfy (2.6), are suppressed), is a set of the form

$$(B^- \leqq p_3 < B^+)\,, \tag{3.5}$$

where $B^\pm = B^\pm_{a,b}$ are functions on $M_3{}^\circ$ which depend at most on the first two moments. The exceptional sets $\Lambda_{5,1}$ and $\Lambda_{5,3}$ are sets of this form intersected respectively with the sets $(\gamma_2 < a)$ and $(\gamma_2 > b)$. Recall that Λ_1 is the union of the first three sets in (3.4) and that Λ_5 is the union of the last three. In Table 3, we give explicit definitions for each of the nine sets in (3.4) by display of the bounding functions $B^\pm$ which determine them. In applying this table to (3.5), the two conventions listed in the second paragraph of this section should be kept in mind. Thus

$$(\eta_b \leqq p_3 < 2) = (\eta_b \leqq p_3 \leqq 1) = (p_3 \geqq \eta_b)$$

so that applying the last line of Table 1 to (3.5) and taking into account the additional condition which holds for $\Lambda_{5,3}$, we have for example that

$$\Lambda_{5,3} = (p_3 \geqq \eta_b, \gamma_2 > b)\,.$$

Establishment of the fact that for each (a, b) satisfying (2.4), these nine sets do indeed partition $M_3{}^\circ$ is a relatively straightforward though somewhat tedious exercise.

Let $a_0, a_1, \cdots, a_k$ be $k + 1$ numbers such that

$$0 \leqq a_0, a_1, \cdots, a_k \leqq 1\,.$$

We shall denote by

$$L(a_0, a_1, \cdots, a_k) \tag{3.6}$$

the intersection with $M_3{}^\circ$ of the simplex with vertices (a_i, a_i^2, a_i^3), $i = 0, 1, \cdots, k$. For $k = 1, 2, 3$ and $a_0, a_1, \cdots, a_k$ distinct, k is of course the dimensionality of this simplex. Thus, for example, $L(u, v)$ (with $0 \leqq u < v \leqq 1$) denotes the open chord joining (u, u^2, u^3), (v, v^2, v^3). It is clear that the sets (3.6) are invariant under permutation of their arguments. For $k = 2$, and $0 \leqq a, b \leqq 1$, let

$$\mathscr{L}(a, b, A) = \bigcup\nolimits_{t \in A} L(a, b, t)\,, \qquad A \subset [0, 1]\,. \tag{3.7}$$

Eight of the sets (3.4) (all but Λ_3) are essentially of this form. Λ_3 is essentially

TABLE 1*

Set	B^-	B^+	Simplex union
$\Lambda_{1,1}$	$\xi_{a,\hat{s}}$	$\eta_a h_{0,a,1} + \xi_{a,1} h'_{0,a,1}$	$\mathscr{L}(0, a, [0, a] + [\hat{s}, 1])$
$\Lambda_{1,2}$	0	$\eta_a h_{a,b} + \xi_{a,b} h'_{a,b}$	$\mathscr{L}(0, a, [a, b])$
$\Lambda_{1,3}$	0	$\eta_b h_{b,\hat{s}} + \xi_{b,\hat{s}} h'_{b,\hat{s}}$	$\mathscr{L}(0, b, [b, \hat{s}])$
Λ_2	η_a	$\eta_b h_{a,b}$	$\mathscr{L}(a, b, [a, b])$
Λ_3	$\xi_{a,b} h_{0,b} + \xi_{b,\hat{s}} h'_{0,b}$	$\xi_{a,b,\hat{s}} h_{a,\hat{s}} + \xi_{a,\hat{s}} h'_{a,\hat{s}}$	$L(0, a, b, \hat{s})$
Λ_4	$\xi_{a,b,\hat{s}} h_{a,\hat{s}} + \eta_a h'_{a,\hat{s}}$	$\xi^*_{a,b} h_{a,1} h'_{b,1} + \eta_b h_{b,1}$	$\mathscr{L}(a, b, [\hat{s}, 1])$
$\Lambda_{5,1}$	$\eta_a h_{0,a} + \xi^*_{0,a} h'_{0,a}$	2	$\mathscr{L}(1, a, [0, a])$
$\Lambda_{5,2}$	$\eta_b h_{a,b} + \xi^*_{a,b} h'_{a,b}$	2	$\mathscr{L}(1, b, [a, b])$
$\Lambda_{5,3}$	η_b	2	$\mathscr{L}(1, b, [b, 1])$

* As a notational convenience, the arguments a, b of the sets Λ in column one and of the numbers $\hat{s}(a, b)$ have been suppressed. Recall that $\hat{s}(a, b) = \min(s(a, b), 1)$. To apply this table see preceding text, in particular (3.5) and remarks concerning $\Lambda_{5,1}$ and $\Lambda_{5,3}$.

of the form (3.6) with $k = 3$. By "essentially", we mean that they are of this form minus a set of 3-dimensional Lebesgue measure zero. The specific simplex unions (simplex in the case of Λ_3) which contain the sets (3.4) are exhibited in the last column of Table 1.

We have observed that for arbitrary $\mathbf{c}$ in $M_3°$, $\mathscr{U}^{(3)}_{a,b}(\mathbf{c})$ is given precisely (Section 2) once the set in (3.4) to which $\mathbf{c}$ belongs is ascertained. On the other hand the barycentric coordinates of $\mathbf{c}$ relative to the simplex that contains it together with the points in [0, 1] that determine the vertices of that simplex, yield in an obvious way a probability measure of finite support on [0, 1] which has the specified moments. The usefulness of the information carried in the last column of Table 3 derives from the fact that this measure, in addition, has a value at $[a, b]$ which attains the bound. Specific application to numerical examples is given in Sections 5 and 8.

4. To which set of a partition does a moment point belong? The definition of $\mathscr{U}^{(3)}_{a,b}$ for a, b which satisfy (2.6) as given in Sections 2 and 3 is complete. The properties of this sharp upper bound as a function on the moment space $M_3°$ are now open to study in detail. The definition is moreover easily specialized to intervals $[a, b]$ of particular form (e.g., symmetric about $\frac{1}{2}$, degenerate, left endpoint zero, etc.). Some specialization is carried out in Section 7.

The multiform nature of the sharp upper bound function presents a practical problem. To find the bound that corresponds to some specific moment point $\mathbf{c}$ in $M_3°$, i.e., to find $\mathscr{U}^{(3)}_{a,b}(\mathbf{c})$, we must first answer the question which heads this section. The answer of course depends upon the value at $\mathbf{c}$ of p_3 together with the values at $\mathbf{c}$ of the bounding functions in Table 1. The purpose of this section is to simplify and to facilitate the latters' evaluation.

The class of all the intersections of cylinder subsets of $M_3°$ whose base sets are projections on M_2 of the sets in (3.4) is a partition of $M_3°$. It is easy to see that each set in this partition may itself be partitioned into its intersections with the sets in (3.4). This demonstrates (provided one accepts that the sets (3.4) are disjoint) that M_3 is a subset of (hence equal to) the union of the sets in (3.4). Such a demonstration, as carried out in Figure 2 and Table 2 allows one (by classifying the first two moments according to specified cylinder sets) to describe the sharp upper bound function in terms of p_3, essentially as is done in Table 1, but now with the bounding functions greatly simplified. To illustrate: $(w_{a,b} < 0)$ is one set in the above described partition of $M_3{}^\circ$ into cylinder sets. It is in its turn partitioned by its intersections with $\Lambda_{1,2}(a, b)$, $\Lambda_2(a, b)$, and $\Lambda_{5,2}(a, b)$, these being the same as its intersections with

$$(p_3 < \eta_a)\,, \qquad (\eta_a \leqq p_3 < \eta_b)\,, \qquad (p_3 \geqq \eta_b)\,.$$

It is disjoint from all other sets in (3.4). Thus following Section 2 we have

$$\mathscr{U}^{(3)}_{a,b} = \frac{\zeta_1}{a}\left(1 + \frac{\xi_{a,a}}{p_3 - \xi_{0,a}}\right), \quad 1\,, \quad \text{or} \quad \frac{\gamma_1}{1-b}\left(1 + \frac{\hat{\xi}_{b,b}}{q_3 - \hat{\xi}_{b,1}}\right),$$

according as $p_3 < \eta_a$, $\eta_a \leqq p_3 < \eta_b$, or $p_3 \geqq \eta_b$, whenever $w_{a,b} < 0$. An additional illustration is given as part of the explanation for Table 2. Some numerical examples follow in Section 5.

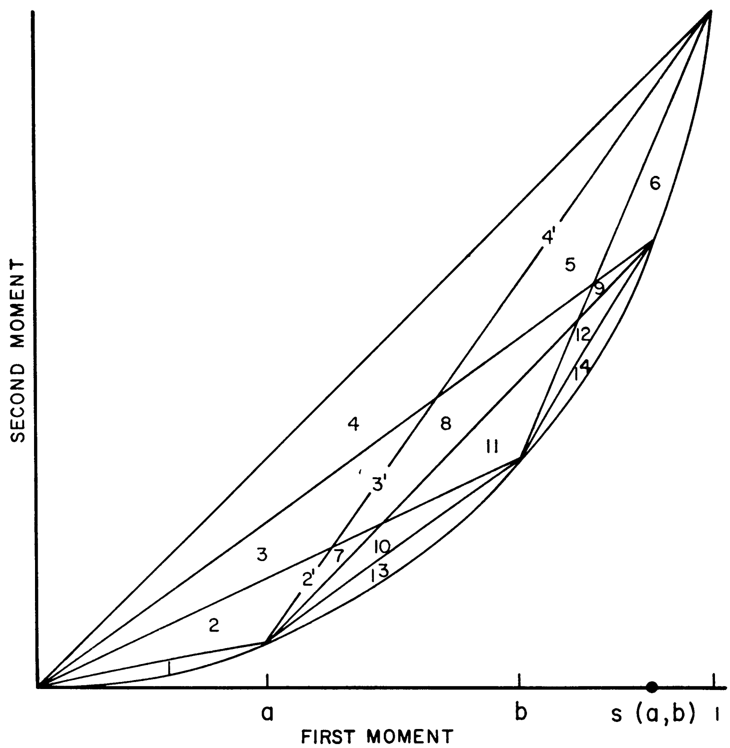

FIG. 2. Intersections of projections on M_2 of the nine sets $\Lambda_{1,1}(a, b), \cdots, \Lambda_{5,3}(a, b)$. See explanation in text.

Explanation and definitions for numbered regions in Figure 2. Below, explicit definitions are given for $M_3°$ cylinder sets determined by intersections of the projections on M_2 of the nine sets $\Lambda_{1,j}(a, b)$, $\Lambda_2(a, b)$, $\Lambda_3(a, b)$, $\Lambda_4(a, b)$, $\Lambda_{5,j}(a, b)$, $j = 1, 2, 3$, for all a, b such that $s^*(a, b) \leqq 0$. For the purpose of the illustration in the figure, a, b are taken so that $s(a, b) < 1$. Thus in the illustration $\hat{s}(a, b) = s(a, b)$. The shape of M_2 in the figure is somewhat distorted to allow for greater clarity. As a notational convenience below and in Table 2 which this figure serves, the arguments a, b of $\hat{s}(a, b)$ are suppressed.

1. $(S_2 \leqq a)$
2. $(\gamma_2 < a < S_2 \leqq b)$, 2′. $(\gamma_2 = a, S_2 \leqq b)$
3. $(\gamma_2 < a, b < S_2 \leqq \hat{s})$, 3′. $(\gamma_2 = a, b < S_2 \leqq \hat{s})$
4. $(\gamma_2 < a, S_2 > \hat{s})$, 4′. $(\gamma_2 = a, S_2 > \hat{s})$
5. $(a < \gamma_2 \leqq b, S_2 \geqq \hat{s})$
6. $(\gamma_2 > b, S_2 \geqq \hat{s})$
7. $(\gamma_2 > a, S_2 \leqq b, w_{a,\hat{s}} \geqq 0)$
8. $(a < \gamma_2 \leqq b < S_2 < \hat{s}, w_{a,\hat{s}} \geqq 0)$
9. $(\gamma_2 > b, S_2 < \hat{s}, w_{a,\hat{s}} \geqq 0)$
10. $(S_2 \leqq b, w_{a,\hat{s}} < 0 \leqq w_{a,b})$
11. $(\gamma_2 \leqq b < S_2, w_{a,\hat{s}} < 0)$
12. $(\gamma_2 > b, w_{a,\hat{s}} < 0 \leqq w_{b,\hat{s}})$
13. $(w_{a,b} < 0)$
14. $(w_{b,\hat{s}} < 0)$

TABLE 2*

Cylinder sets	$\Lambda_{1,2}$	$\Lambda_{1,3}$	Λ_2	Λ_3	$\Lambda_{1,1}$	Λ_4	$\Lambda_{5,1}$	$\Lambda_{5,2}$	$\Lambda_{5,3}$
1					0		$\eta_a, 1$		
2	0			$\xi_{a,b}$	$\xi_{a,\hat{s}}$		$\xi_{a,1}, 1$		
2′	0			$\xi_{a,b}$	$\xi_{a,\hat{s}}$			1	
3		0		$\xi_{b,\hat{s}}$	$\xi_{a,\hat{s}}$		$\xi_{a,1}, 1$		
3′		0		$\xi_{b.\hat{s}}$	$\xi_{a,\hat{s}}$			1	
4					0		$\xi_{a,1}, 1$		
4′					0			1	
5					0	η_a		$\xi^*_{a,b}, 1$	
6					0	η_a			$\eta_b, 1$
7	0			$\xi_{a,b}$	$\xi_{a,\hat{s}}$	η_a		$\xi^*_{a,b}, 1$	
8		0		$\xi_{b,\hat{s}}$	$\xi_{a,\hat{s}}$	η_a		$\xi^*_{a,b}, 1$	
9		0		$\xi_{b,\hat{s}}$	$\xi_{a,\hat{s}}$	η_a			$\eta_b, 1$
10	0			$\xi_{a,b}$		$\xi_{a,b,\hat{s}}$		$\xi^*_{a,b}, 1$	
11		0		$\xi_{b,\hat{s}}$		$\xi_{a,b,\hat{s}}$		$\xi^*_{a,b}, 1$	
12		0		$\xi_{b,\hat{s}}$		$\xi_{a,b,\hat{s}}$			$\eta_b, 1$
13	0		η_a					$\eta_b, 1$	
14		0							$\eta_b, 1$

* See explanation in text which follows.

Explanation of Table 2. Numbers in column one designate the cylinder sets (with bases in M_2) illustrated and explicitly defined for Figure 2. The set of entries in any row denote a partition of the cylinder set at the left by indicating limits on p_3 for the intersection of that set with the sets which appear in the column headings. This is done as follows: Let C denote the cylinder set at the left in any row and let Λ denote the set which heads any column except the *last* one for which an entry in that row appears. Then $C \cap \Lambda = \varnothing$, if there is no entry for that row and column. Otherwise,

$$C \cap \Lambda = C \cap (k_1 \leqq p_3 < k_2)$$

where k_1 is the entry for that row and column and k_2 is the next entry to the right in that row. If Λ heads the column in which the last entry or entries for for the row appear, then

$$C \cap \Lambda = C \cap (k \leqq p_3 \leqq 1) \quad \text{or} \quad C \cap (p_3 = 1)$$

according as the entries are k, 1 or the entry is 1.

For example, suppose that

$$C = (a < \gamma_2 \leqq b < S_2 < \hat{s},\, w_{a,\hat{s}} \geqq 0)\,.$$

This is set #8. See Figure 2. C is partitioned by its intersections with the sets $\Lambda_{1,3}$, Λ_3, $\Lambda_{1,1}$, Λ_4, $\Lambda_{5,2}$. The intersections of C with these sets are the same as its respective intersections with the sets

$$(p_3 < \xi_{b,\hat{s}})\,, \qquad (\xi_{b,\hat{s}} \leqq p_3 < \xi_{a,\hat{s}})\,, \qquad (\xi_{a,\hat{s}} \leqq p_3 < \eta_a)\,,$$
$$(\eta_a \leqq p_3 < \xi^*_{a,b})\,, \qquad (p_3 \geqq \xi^*_{a,b})\,.$$

Note that the intersection of certain nonempty *subsets* of C with some of the above sets may be empty. For example, let

$$C' = (\gamma_2 \leqq b < S_2,\, w_{a,\hat{s}} = 0)\,.$$

This is the cylinder set whose base is one of the line segments which bound the base C. Then $\varnothing \neq C' \subset C$, but (see Lemma 9.3)

$$C' \cap \Lambda_{11} = C' \cap (\xi_{a,\hat{s}} \leqq p_3 < \eta_a) = \varnothing\,.$$

Note also, that the cylinder set #8 itself is empty when a, b satisfies not only (2.6) but in addition satisfies the inequality

$$s(a, b) \geqq 1\,.$$

Finally, the table indicates that

$$C \cap [\Lambda_{1,2} + \Lambda_2 + \Lambda_3 + \Lambda_{5,1} + \Lambda_{5,3}] = \varnothing\,.$$

Thus, when $s^*(a, b) \leqq 0$, $U^{(3)}_{a,b}$ takes the form specified in Section 2 for $\Lambda_{1,3}$, Λ_3, $\Lambda_{1,1}$, Λ_4, or $\Lambda_{5,2}$, according as $p_3 < \xi_{b,\hat{s}}$, $\xi_{b,\hat{s}} \leqq p_3 < \xi_{a,\hat{s}}$, $\xi_{a,\hat{s}} \leqq p_3 < \eta_a$, $\eta_a \leqq p_3 < \xi^*_{a,b}$, or $p_3 \geqq \xi^*_{a,b}$, whenever the first two moments belong to C (i.e., to the set #8 of Figure 2).

5. Numerical examples. Let

$$[a, b] = [\tfrac{1}{4}, \tfrac{1}{3}],$$

and suppose that

$$\mathbf{c} = (c_1, c_2, c_3) = (.55, .42, .35).$$

At $\mathbf{c}$,

$$p_1 = 1 - q_1 = .55, \qquad p_2 = 1 - q_2 = (.42 - (.55)^2)/(.55)(.45) \cong .4747,$$
$$p_3 = 1 - q_3 \cong (.35 - ((.42)^2/.55))/(.55)(.45)(.4747)(.5253) \cong .4743,$$

so that $\mathbf{c}$ does belong to M_3° and is in fact an interior point. Now

$$\hat{s} = \hat{s}(\tfrac{1}{4}, \tfrac{1}{3}) = \tfrac{1}{4} + \tfrac{1}{3} + 1/2(3)^{\frac{1}{2}} \cong .8720,$$

and at $\mathbf{c}$,

$$S_2 = \tfrac{42}{55} \cong .7636, \qquad \gamma_2 = \tfrac{13}{45} \cong .2889,$$

so that in the present case,

$$a < \gamma_2 < b < S_2 < \hat{s}.$$

Moreover, at $\mathbf{c}$,

$$w_{a,\hat{s}} \cong .42 - (.25 + .8720)(.55) + (.25)(.8720) \cong .0209 > 0.$$

It follows that (c_1, c_2) belongs to region #8 of Figure 2. Consulting the row which corresponds to region #8 in Table 2, we calculate that at $\mathbf{c}$,

$$\xi_{b,\hat{s}} \cong (.7636 - \tfrac{1}{3})(.8720 - .7636)/(.45)(.4747)(.5253) \cong .4156,$$
$$\xi_{a,\hat{s}} = (S_2 - a)\xi_{b,\hat{s}}/(S_2 - b) \cong .4961,$$

hence that in the present case,

$$\xi_{b,\hat{s}} < p_3 < \xi_{a,\hat{s}}.$$

It follows that

$$\mathbf{c} \in \Lambda_3.$$

Consulting Section 2, we have finally, since $s(\tfrac{1}{4}, \tfrac{1}{3}) < 1$, and computing

$$d(\tfrac{1}{4}, \tfrac{1}{3}) = (\tfrac{1}{4})(\tfrac{1}{3})(\tfrac{1}{2} + (1/3^{\frac{1}{2}}))^2 \cong .0967,$$
$$\xi_{\hat{s},\hat{s}} \cong -(.7636 - .8720)^2/(.45)(.4747)(.5253) \cong -.1047,$$
$$R_2 = \nu_3^+ - \nu_3^- \cong (.55)(.45)(.4747)(.5253) \cong .0617,$$

that

$$\mathscr{U}^{(3)}_{\frac{1}{4},\frac{1}{3}}(.55, .42, .35) \cong (.0617)(.4743 + .1047)/(.0967) \cong .3694.$$

A distribution on [0, 1] whose first three moments are .55, .42, .35 which assigns this weight to the interval $[\tfrac{1}{4}, \tfrac{1}{3}]$ is now easily found by calculating the barycentric coordinates of (.55, .42, .35) relative to the simplex $L(0, \tfrac{1}{4}, \tfrac{1}{3}, \hat{s}(\tfrac{1}{4}, \tfrac{1}{3}))$. This distribution is given by the following table

support	0	$\frac{1}{4}$	$\frac{1}{3}$	.8720
probability	.1143	.2797	.0897	.5162 .

Keeping the interval the same, let us now find the sharp upper bound corresponding to normalized moment values,

$$p_1 = p_2 = p_3 = \tfrac{1}{2}\,.$$

These determine uniquely the moment point (see (1.6))

$$\mathbf{c} = (\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16})\,;$$

which is for example the first three moments of a Beta distribution with parameters $\frac{1}{2}$, $\frac{1}{2}$; also the first three moments of a Binomial distribution with parameters 2, $\frac{1}{2}$ (after division by 2). These particular distributions respectively assign weights of .0585 and zero to the interval $[\frac{1}{4}, \frac{1}{3}]$. At this moment point,

$$S_2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}\,, \qquad \gamma_2 = \tfrac{1}{4}\,.$$

$\hat{s}$, which depends only on the interval, we have already computed to be .8720. Thus in the case we are now considering

$$\gamma_2 = a \qquad \text{and} \qquad b < S_2 \leqq \hat{s}\,.$$

This mean that (c_1, c_2) belongs to region #3′ of Figure 2. Consulting the corresponding row of Table 2 we calculate that at the moment point which concerns us

$$\xi_{b,\hat{s}} \cong (\tfrac{3}{4} - \tfrac{1}{3})(.8720 - \tfrac{3}{4})/\tfrac{1}{8} \cong .4067\,,$$
$$\xi_{a,\hat{s}} \cong (\tfrac{3}{4} - \tfrac{1}{4})(.8720 - \tfrac{3}{4})/\tfrac{1}{8} \cong .4880\,,$$

so that in the present case, because

$$\xi_{a,\hat{s}} \leqq p_3 < 1\,,$$

it follows that

$$\mathbf{c} \in \Lambda_{1,1}\,.$$

Consulting Section 2; observing that because $\mathbf{c}$ is in $\Lambda_{1,1}$,

$$\lambda = \delta = \tfrac{1}{4}\,;$$

and computing

$$\xi_{\frac{1}{4},\frac{1}{4}} = -(\tfrac{3}{4} - \tfrac{1}{4})^2/\tfrac{1}{8} = -2\,,$$

we have

$$\mathscr{U}^{(3)}_{\frac{1}{4},\frac{1}{3}}(\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16}) = \frac{\frac{1}{2}}{\frac{1}{4}}\left(1 - \frac{2}{\frac{5}{2}}\right) = \tfrac{2}{5}\,.$$

A distribution on [0, 1] having moments $\frac{1}{2}$, $\frac{3}{8}$, $\frac{5}{16}$ which assigns this weight to the interval $[\frac{1}{4}, \frac{1}{3}]$ is found by observing (using the last column of Table 1 and Lemma 9.2) that

$$(\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16}) \in L(0, \tfrac{1}{4}, \tfrac{7}{8}) \subset \mathscr{L}(0, \tfrac{1}{4}, [0, \tfrac{1}{4}] + [\hat{s}, 1])$$

and calculating the barycentric coordinates of $(\frac{1}{2}, \frac{3}{8}, \frac{5}{16})$ relative to this simplex. (It suffices to calculate the barycentric coordinates of $(\frac{1}{2}, \frac{3}{8})$ relative to the projection of this simplex on M_2). This distribution is given by the following table,

support	0	$\frac{1}{4}$	$\frac{7}{8}$
probability	$\frac{2}{35}$	$\frac{2}{5}$	$\frac{16}{35}$.

To illustrate that in using these procedures, we are not restricted to consideration only of the class of all distributions on [0, 1], let us now find the sharp upper bound for the probability that may be assigned to the interval $[\frac{1}{4}, \frac{1}{3}]$ among all distributions on the interval $[-3, 2]$ whose first three moments are $\frac{1}{2}, \frac{3}{8}, \frac{5}{16}$. Using simple calculations (e.g., as in the proof of Theorem 5 in [6] taking $n = 3$) we find that

$$\mathbf{p}_{3,[-3,2]}(\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16}) = (\tfrac{7}{10}, \tfrac{1}{42}, \tfrac{291}{410}) .$$

Thus applying Theorem 1.1 with $n = 3$, we find that

$$\mathscr{U}^{(3)}_{\frac{1}{4},\frac{1}{3},[-3,2]}(\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16}) = \mathscr{U}^{(3)}_{\frac{13}{20},\frac{2}{3}}(\mathbf{c}_3{}^\circ) ,$$

where

$$\mathbf{c}_3{}^\circ = \mathbf{p}_3{}^{-1}(\tfrac{7}{10}, \tfrac{1}{42}, \tfrac{291}{410}) = (\tfrac{7}{10}, \tfrac{99}{200}, \tfrac{707}{2000}) ;$$

that is, the bound we seek is also the sharp upper bound for the probability assignable to the interval $[\frac{13}{20}, \frac{2}{3}]$ among all distributions on [0, 1] whose first three normalized moments (relative to the class of all distributions on [0, 1]) are $\frac{7}{10}, \frac{1}{42}, \frac{291}{410}$. It is unnecessary to find, as we have done above (using (1.6)), the standard moments that correspond to these normalized moments in order to calculate the sharp upper bound. Thus simple calculations as in the previous examples show that

$$\hat{s}(\tfrac{13}{20}, \tfrac{2}{3}) = 1 , \qquad \mathbf{c}_3{}^\circ \in (\gamma_2 > \tfrac{2}{3}) ,$$

so that $(c_1{}^\circ, c_2{}^\circ)$ belongs to region #14 of Figure 2 (regions #6, 9, and 12 are empty). Consulting the last line of Table 2, we calculate

$$\eta_{\frac{2}{3}}(\mathbf{c}^\circ) = \tfrac{34}{41} > \tfrac{291}{410} = p_3(\mathbf{c}^\circ)$$

and this means that

$$\mathbf{c}_3{}^\circ \in \Lambda_{1,3}(\tfrac{13}{20}, \tfrac{2}{3}) .$$

Consulting Section 2, observing that

$$\lambda(\mathbf{c}_3{}^\circ) = \delta(\mathbf{c}_3{}^\circ) = \tfrac{2}{3}$$

and computing

$$\xi_{\frac{2}{3},\frac{2}{3}}(\mathbf{c}_3{}^\circ) = -\tfrac{289}{1230} ,$$

we calculate

$$\mathscr{U}^{(3)}_{\frac{13}{20},\frac{2}{3}}(\mathbf{c}_3{}^\circ) = (\tfrac{21}{20})(\tfrac{873}{1162}) \simeq .7889 ,$$

which is the desired bound. Consulting the last column in Table 1 we find that

$$\mathbf{c}_3{}^\circ \in L(0, \tfrac{2}{3}, t)$$

for some t in $[\frac{2}{3}, 1]$. This means (see Lemma 9.2) that

$$\xi_{\frac{2}{3},t}(\mathbf{c}_3{}^\circ) = \tfrac{291}{410} ,$$

which we may solve to find that

$$t = \tfrac{141}{170} \, .$$

Finding barycentric coordinates of $\mathbf{c}_3{}^\circ$ relative to the above simplex yields the distribution

$$\begin{array}{ccc} 0 & \tfrac{2}{3} & \tfrac{141}{170} \\ .0012 & .7889 & .2099 \end{array}$$

on $[0, 1]$ with moments $\mathbf{c}_3{}^\circ$ whose value at $[\tfrac{13}{20}, \tfrac{2}{3}]$ attains the sharp upper bound. By a simple transformation we obtain the distribution

$$\begin{array}{ccc} -3 & \tfrac{1}{3} & \tfrac{195}{170} \\ .0012 & .7889 & .2099 \end{array}$$

on $[-3, 2]$ with moments $\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16}$ whose value at $[\tfrac{1}{4}, \tfrac{1}{3}]$ attains the same bound.

As a final illustration of the use to which these tables may be put, we take note of the interesting fact that for each interval $[a, b] \subset [0, 1]$ there are values of the first two moments for which only one form for the sharp upper bound will suffice. For example, when $[a, b] = [\tfrac{1}{4}, \tfrac{1}{3}]$ and (c_1, c_2) is in the region $\#4'$ of Figure 2; e.g., when $p_1(\mathbf{c}) = \tfrac{3}{4}$, $p_2(\mathbf{c}) = \tfrac{2}{3}$ (so that $c_1 = \tfrac{3}{4}$, $c_2 = \tfrac{11}{16}$), we find using Table 2 and Section 2 that we may express the sharp upper bound in terms of $p_3 = p_3(\tfrac{3}{4}, \tfrac{11}{16}, c_3)$ without restriction on the third moment. Thus

$$\mathscr{U}^{(3)}_{\frac{1}{4},\frac{1}{3}} = 3p_3(p_3 + 8) \, .$$

The reader should verify that the bound (see Table 2) is in fact continuous at $p_3 = 1$ so that in fact only this one form will suffice.

6. Intervals for which $s^*(a, b)$ is positive. In this section, the definition of the sets in (3.4) is extended to those (a, b) in T which do not satisfy (2.6); i.e., to those (a, b) which belong to the region that lies above the upper curve in Figure 1. The simple modification in form for the sharp upper bound function (namely that which obtains on Λ_3), which is necessitated by this extension, is exhibited.

By Theorem 1.2, taking $n = 3$,

$$\mathscr{U}^{(3)}_{a,b} = \mathscr{U}^{(3)}_{1-b,1-a} \circ \mathbf{K} \, , \qquad (a, b) \in T \, . \tag{6.1}$$

It is moreover easily seen that

$$s^*(a, b) > 0 \quad \text{if and only if} \quad s^*(1 - b, 1 - a) < -((ab)^{\frac{1}{2}} + ((1 - a)(1 - b))^{\frac{1}{2}}) \, ,$$

so that $(1 - b, 1 - a)$ must satisfy (2.6) whenever (a, b) does not. Thus when (2.6) is not satisfied, we need only compose the sharp upper bound for the interval $[1 - b, 1 - a]$ with $\mathbf{K}$ (i.e., change the odd indexed p's and q's in the formulae of Section 2 to q's and p's) to obtain the sharp upper bound for $[a, b]$ in equally explicit detail. Here, we define extensions of the sets in (3.4) to those (a, b) in T for which $s^*(a, b) > 0$, by taking the sets to be the appropriate images under $\mathbf{K}$ of their realizations at $(1-b, 1-a)$. Thus for each (a, b) in T such that $s^*(a, b) > 0$, we define

$$\Lambda_i(a, b) = \mathbf{K}\Lambda_i(1-b, 1-a)\,, \qquad i = 2, 3, 4\,;$$
$$\Lambda_{i,j}(a, b) = \mathbf{K}\Lambda_{6-i,4-j}(1-b, 1-a)\,, \qquad i = 1, 5, j = 1, 2, 3\,.$$

We now describe these extensions in a manner analogous to that used in Section 3. With the exception of $\Lambda_{1,1}$ and $\Lambda_{1,3}$ each of these extensions is of the form

$$(B^- < p_3 \leqq B^+)\,, \tag{6.2}$$

where again, $B^\pm = B^\pm_{a,b}$ are functions on $M_3{}^\circ$, at most dependent on the first two moments. Note that compared with (3.5), weak and strong inequalities have switched sides. The exceptional sets $\Lambda_{1,1}$ and $\Lambda_{1,3}$ are of the form (6.2) intersected respectively with the sets $(S_2 < a)$ and $(S_2 > b)$. Λ_1 is again defined to be the union of the $\Lambda_{1,j}$; Λ_5, of the $\Lambda_{5,j}$. Table 3 is strictly analogous to Table 1. Thus, consulting the first line of Table 3, and taking into account the additional condition cited above which holds for $\Lambda_{1,1}$ we have for example that

$$\Lambda_{1,1}(a, b) = (p_3 \leqq \eta_a, S_2 < a) \qquad \text{when} \quad s^*(a, b) > 0\,.$$

Consulting the third column we find that this set is the simplex union $\mathscr{L}(0, a, [0, a])$ minus a set of 3-dimensional Lebesgue measure zero.

It may now be verified by direct application of (6.1) that with these extended definitions of the sets (3.4), the forms for the sharp upper bound function (with the single exception of the form that holds on Λ_3) *continue to hold precisely as given in Section* 2, for all (a, b) such that $s^*(a, b) > 0$.

On $\Lambda_3(a, b)$,

$$\mathscr{U}^{(3)}_{a,b} = R_2(\xi^*_{s^*,s^*} - p_3)/\hat{d}\,,$$

where

$$\hat{d} = \hat{d}(a, b) = (1-a)(1-b)((1-a)^{\frac{1}{2}} + (1-b)^{\frac{1}{2}})^2\,,$$

whenever $s^*(a, b) > 0$.

TABLE 3*

Set	B^-	B^+	Simplex union
$\Lambda_{1,1}$	-1	η_a	$\mathscr{L}(0, a, [0, a])$
$\Lambda_{1,2}$	-1	$\eta_a h_{a,b} + \xi_{a,b} h'_{a,b}$	$\mathscr{L}(0, a, [a, b])$
$\Lambda_{1,3}$	-1	$\eta_b h_{b,1} + \xi_{b,1} h'_{b,1}$	$\mathscr{L}(0, b, [b, 1])$
Λ_2	$\eta_a h_{a,b} + h'_{a,b}$	η_b	$\mathscr{L}(a, b, [a, b])$
Λ_3	$\xi_{s^*,a,b} h_{s^*,b} + \xi^*_{s^*,b} h'_{s^*,b}$	$\xi^*_{a,b} h_{a,1} + \xi^*_{s^*,a} h'_{a,1}$	$L(s^*, a, b, 1)$
Λ_4	$h'_{0,b} + \xi_{a,b} h_{0,b} h'_{0,a} + \eta_a h_{0,a}$	$\xi^*_{s^*,a,b} h_{s^*,b} + \eta_b h'_{s^*,b}$	$\mathscr{L}(a, b, [0, s^*])$
$\Lambda_{5,1}$	$\eta_a h_{s^*,a} + \xi^*_{s^*,a} h'_{s^*,a}$	1	$\mathscr{L}(1, a, [s^*, a])$
$\Lambda_{5,2}$	$\eta_b h_{a,b} + \xi^*_{a,b} h'_{a,b}$	1	$\mathscr{L}(1, b, [a, b])$
$\Lambda_{5,3}$	$\eta_b h_{0,b,1} + \xi^*_{0,b} h'_{0,b,1}$	$\xi^*_{s^*,b}$	$\mathscr{L}(1, b, [0, s^*] + [b, 1])$

* As a notational convenience, the arguments a, b of the set Λ and of the numbers s^* have been suppressed. $s^*(a, b)$ is defined in (2.1). We are assuming here that $s^*(a, b) > 0$. To apply this table see preceding text, in particular (6.2) and the remarks concerning $\Lambda_{1,1}$ and $\Lambda_{1,3}$.

7. Sharp upper bounds for some special intervals. Considerable simplification in the form of the sharp upper bound is achieved when the index (a, b) is restricted to certain special interval classes.

We consider first, the degenerate intervals (represented by the lower diagonal boundary of T in Figure 1).

THEOREM 7.1. *For each a, $0 < a < 1$,*

$$\mathscr{U}_{a,a}^{(3)} = \frac{\zeta_1}{a}\left(1 + \frac{\xi_{a,a}}{p_3 - \xi_{a,a}}\right) \quad or \quad \frac{\gamma_1}{1-a}\left(1 + \frac{\hat{\xi}_{a,a}}{q_3 - \hat{\xi}_{a,a}}\right), \tag{7.1}$$

according as

$$p_3 \leqq \quad or \quad \geqq \eta_a h_{0,a,1} + \xi_{a,1} h'_{0,a,1}. \tag{7.2}$$

When equality holds in (7.2), *both expressions on the right-hand side of* (7.1) *yield*

$$\mathscr{U}_{a,a}^{(3)} = (\zeta_1 \zeta_2 / w_{a,a}) h_{0,a,1} + (\gamma_1 \gamma_2 / a(1-a)) h'_{0,a,1} = \mathscr{U}_{a,a}^{(2)} \tag{7.3}$$

For the two extreme values of a, we have with no restrictions,

$$\mathscr{U}_{0,0}^{(3)} = q_1 p_2 q_3 / (p_1 q_2 + p_2 q_3), \qquad \mathscr{U}_{1,1}^{(3)} = p_1 p_2 p_3 / (q_1 q_2 + p_2 p_3). \tag{7.4}$$

PROOF. An application of results in Section 3 shows that

$$\Lambda_{1,2}(a, a),\ \Lambda_2(a, a),\ \Lambda_3(a, a),\ \Lambda_4(a, a)$$

all are empty whenever $s^*(a, a) \leqq 0$ (i.e., when $a \leqq \frac{2}{3}$). Moreover

$$\Lambda_1(a, a) = (p_3 < \eta_a h_{0,a,1} + \xi_{a,1} h'_{0,a,1}), \qquad \Lambda_5(a, a) = M_3{}^\circ \backslash \Lambda_1(a, a).$$

Application of the forms in Section 2 will now yield the desired results for $0 \leqq a \leqq \frac{2}{3}$. For $\frac{2}{3} < a \leqq 1$, these formulae may be deduced from the results in Section 6. Proof of (7.3), (7.4) is left to the reader.

The interval $[a, b]$ may be taken dependent upon the moments. As a simple illustration we consider the following corollary to the above theorem.

COROLLARY 7.1.

$$\mathscr{U}_{\nu_1,\nu_1}^{(3)} = q_2 p_3 / (q_1 p_2 + q_2 p_3) \quad or \quad q_2 q_3 / (p_1 p_2 + q_2 q_3), \tag{7.5}$$

according as $p_3 \leqq$ or $\geqq q_1$. Both expressions on the right-hand side of (7.5) *yield*

$$\mathscr{U}_{\nu_1,\nu_1}^{(3)} = q_2 = \mathscr{U}_{\nu_1,\nu_1}^{(2)}, \qquad when \quad p_3 = q_1. \tag{7.6}$$

Note that for each $\mathbf{c} = (c_1, c_2, c_3)$ in $M_3{}^\circ$,

$$\mathscr{U}_{\nu_1,\nu_1}^{(3)}(\mathbf{c}) = \mathscr{U}_{c_1,c_1}^{(3)}(\mathbf{c})$$

is the maximum probability assignable to the degenerate interval $\{c_1\}$ by any distribution on $[0, 1]$ having moments c_1, c_2, c_3.

The following theorem exhibits the sharp upper bound for intervals with left endpoint zero (represented by the left hand, vertical boundary of T in Figure 1).

THEOREM 7.2. *For each b, $0 < b \leqq 1$,*

$$\mathscr{U}_{0,b}^{(3)} = 1 - \frac{\zeta_1^3(S_2 - b)^3 h'_{0,b}}{R_2^2(p_3 - \xi_{a,b})(p_3 - \xi_{b,b})} \quad \text{or} \quad \frac{\gamma_1}{1-b}\left(1 + \frac{\hat{\xi}_{b,b}}{q_3 - \hat{\xi}_{b,bh_{b,1}+h'_{b,1}}}\right), \tag{7.7}$$

according as

$$p_3 \leqq \quad \text{or} \quad \geqq \eta_b h_{0,b,1} + \xi_{b,1} h'_{0,b,1} \,. \tag{7.8}$$

When equality holds in (7.8), *both expressions on the right-hand side of* (7.7) *yield*

$$\mathscr{U}_{0,b}^{(3)} = 1 - \frac{(\zeta_1 - b)^2}{w_{b,b}} h_{b,1} - \frac{w_{0,b}}{1-b} h'_{0,b,1} = \mathscr{U}_{0,b}^{(2)} \,. \tag{7.9}$$

PROOF. Since

$$s^*(0, b) = b - 1 - (1 - b)^{\frac{1}{2}} \leqq 0 \,, \quad 0 < b \leqq 1 \,,$$

results in Sections 2 and 3 apply. We find that

$$\Lambda_1(0, b),\ \Lambda_3(0, b),\ \Lambda_{5,1}(0, b)$$

are all empty. Moreover

$$\Lambda_2(0, b) = (p_3 < \eta_b h_{0,b}) \,, \qquad \Lambda_4(0, b) = (p_3 < \eta_b h_{b,1} + \xi_{b,1} h'_{b,1}) \,,$$
$$\Lambda_5(0, b) = M_3^\circ \backslash (\Lambda_2(0, b) + \Lambda_4(0, b)) = (p_3 \geqq \eta_b h_{0,b,1} + \xi_{b,1} h'_{0,b,1}) \,.$$

The theorem now follows by application of the forms in Section 2. Proof of (7.9) is left to the reader.

If we apply (6.1) to Theorem 7.2 or the extended definition of the sets Λ in Section 6, we obtain the sharp upper bound for intervals with right endpoints one (top, horizontal boundary of T, Figure 1).

COROLLARY. *For each a, $0 \leqq a < 1$,*

$$\mathscr{U}_{a,1}^{(3)} = 1 - \frac{\gamma_1^3(a - \gamma_2)^3 h'_{a,1}}{R_2^2(q_3 - \hat{\xi}_{a,a})(q_3 - \hat{\xi}_{a,1})} \quad \text{or} \quad \frac{\zeta_1}{a}\left(1 + \frac{\xi_{a,a}}{p_3 - \xi_{ah_{0,a},a}}\right), \tag{7.10}$$

according as

$$q_3 \leqq \quad \text{or} \quad \geqq (1 - \eta_a) h_{0,a,1} + \hat{\xi}_{0,a} h'_{0,a,1} \,. \tag{7.11}$$

When equality holds in (7.11), *both expressions on the right-hand side of* (7.10) *yield*

$$\mathscr{U}_{a,1}^{(3)} = 1 - \frac{(a - \zeta_1)^2}{w_{a,a}} h_{0,a} - \frac{w_{a,1}}{a} h'_{0,a,1} = \mathscr{U}_{a,1}^{(2)} \,.$$

8. Further numerical examples. We consider first a straightforward application of Theorem 7.1. Let us suppose that $a = \frac{1}{3}$ and that $\mathbf{c} = (c_1, c_2, c_3) \in M_3^\circ$ is such that at $\mathbf{c}$, $p_1 = p_2 = \frac{1}{2}$; i.e., that $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{8}$. Then, of course,

$\zeta_1 = \gamma_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $\gamma_2 = \frac{1}{4}$, so that at any such $\mathbf{c}$,

$$\xi_{\frac{1}{3},\frac{1}{3}} = -\tfrac{25}{18}\,, \qquad \hat{\xi}_{\frac{1}{3},\frac{1}{3}} = -\tfrac{1}{18}\,, \qquad \xi_{\frac{1}{3},1} = \tfrac{5}{6}\,,$$
$$h_{0,\frac{1}{3},1} = 1 - h'_{0,\frac{1}{3},1} = 0\,.$$

Thus the maximum probability assignable to the singleton $\{\frac{1}{3}\}$ by any distribution on the unit interval with first 3 moments $\frac{1}{2}$, $\frac{3}{8}$, c_3, expressed in terms of its dependence upon $p_3(\frac{1}{2}, \frac{3}{8}, c_3) = 1 - q_3(\frac{1}{2}, \frac{3}{8}, c_3)$, is given by

$$\mathscr{U}^{(3)}_{\frac{1}{3},\frac{1}{3}} = \frac{3}{2}\left(1 - \frac{25}{25 + 18p_3}\right) \quad \text{or} \quad \frac{3}{4}\left(1 - \frac{1}{1 + 18q_3}\right), \tag{8.1}$$

according as $p_3 \leqq$ or $\geqq \frac{5}{6}$. It follows for example that if $p_3 = \frac{1}{2}$ (i.e., $c_3 = \frac{5}{16}$), the sharp upper bound is $\frac{27}{68}$; if $p_3 = .9$ (i.e., $c_3 = \frac{27}{80}$), the sharp upper bound is $\frac{27}{56}$. If $p_3 = \frac{5}{6}$ (i.e., $c_3 = \frac{1}{3}$), both expressions on the right-hand side of (8.1), as well as (7.3) with appropriate substitutions, yield the sharp upper bound value $\frac{9}{16}$. This latter value is also the maximum weight that any distribution on the unit interval having first two moments $\frac{1}{2}$, $\frac{3}{8}$, can assign to the singleton $\{\frac{1}{3}\}$.

To find distributions with the specified moments that assign the maximum weight to $\{\frac{1}{3}\}$, we note, using the proof of Theorem 7.1 together with Table 1 of Section 3 that

$$(\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16}) \in L(0, \tfrac{1}{3}, \tfrac{9}{10})\,, \qquad (\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{27}{80}) \in L(1, \tfrac{1}{3}, \tfrac{1}{10})\,,$$
$$(\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{1}{3}) \in L(0, \tfrac{1}{3}, 1)\,.$$

The third vertex of each simplex is found by using the fact (see Lemma 9.2) that

$$\mathbf{c} \in L(a, b, t) \Longrightarrow p_3(\mathbf{c}) = \xi_{a,b,t}(\mathbf{c})\,,$$

solving the right hand equality for t. The barycentric coordinates of each point relative to the simplex that contains it yield the desired distributions. These are exhibited below

0	$\frac{1}{3}$	$\frac{9}{10}$
$\frac{7}{36}$	$\frac{27}{68}$	$\frac{125}{306}$

$\frac{1}{10}$	$\frac{1}{3}$	1
$\frac{25}{126}$	$\frac{27}{56}$	$\frac{23}{72}$

0	$\frac{1}{3}$	1
$\frac{1}{8}$	$\frac{9}{16}$	$\frac{5}{16}$

.

(8.1) may be used to answer questions of the following sort. For what values of c_3 is it the case that there exists no distribution on the unit interval having moments $\frac{1}{2}$, $\frac{3}{8}$, c_3 which assigns more than $\frac{1}{4}$ to the singleton $\{\frac{1}{3}\}$? The answer is of course those c_3 compatible with $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{8}$, for which

$$\mathscr{U}^{(3)}_{\frac{1}{3},\frac{1}{3}}(\tfrac{1}{2}, \tfrac{3}{8}, c_3) \leqq \tfrac{1}{4}\,.$$

That is, by (8.1), those c_3 for which

$$p_3(\tfrac{1}{2}, \tfrac{3}{8}, c_3) \leqq \tfrac{5}{18} \quad \text{or} \quad \geqq \tfrac{35}{36}\,,$$

i.e., those c_3 which satisfy

$$\tfrac{9}{32} \leqq c_3 \leqq \tfrac{43}{144} \qquad \text{or} \qquad \tfrac{197}{576} \leqq c_3 \leqq \tfrac{11}{32}\,.$$

The following is an indirect application of Theorem 7.1. Let θ be a given number between zero and one. For which points a (or more specifically, for which singleton subsets $\{a\}$ of the unit interval) can there exist no distribution on $[0, 1]$ having moments $\frac{1}{2}, \frac{3}{8}, \frac{5}{16}$ (normalized moments $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$) that assigns probability greater than θ to $\{a\}$? The answer: those a for which

$$\mathscr{U}^{(3)}_{a,a}(\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16}) \leqq \theta \, . \tag{8.2}$$

Applying the formulae of Theorem 7.1, we find after some elementary manipulation that

$$\mathscr{U}^{(3)}_{a,a}(\tfrac{1}{2}, \tfrac{3}{8}, \tfrac{5}{16}) = f(a) \quad \text{or} \quad f(1-a)\, , \tag{8.3}$$

where

$$f(a) = 1/2a[1 + (3 - 4a)^2]\, ,$$

according as

$$a \in ([(2 - 2^{1/2})/4, \tfrac{1}{2}] + [(2 + 2^{1/2})/4, 1]) \quad \text{or not.}$$

The graph of (8.3) is sketched in Figure 3. The maximum value of (8.3) is $\frac{1}{2}$, attained at $(2 \pm 2^{1/2})/4$ and $\frac{1}{2}$. Hence there is for no singleton subset of $[0, 1]$ a distribution on $[0, 1]$ having moments $\frac{1}{2}, \frac{3}{8}, \frac{5}{16}$ which can assign to that subset probability greater than $\frac{1}{2}$. On the other hand, its minimum value is $\frac{1}{4}$, attained at zero and one. (A relative minimum equal to $9/2(9 + 6^{1/2})$ is attained at $(6 \pm 6^{1/2})/12$). Hence for every singleton subset there exists a distribution with these moments which can assign to that subset a probability greater than $\frac{1}{4}$. To consider a case in between, suppose that θ were equal to $\frac{2}{5}$. Applying (8.3), we

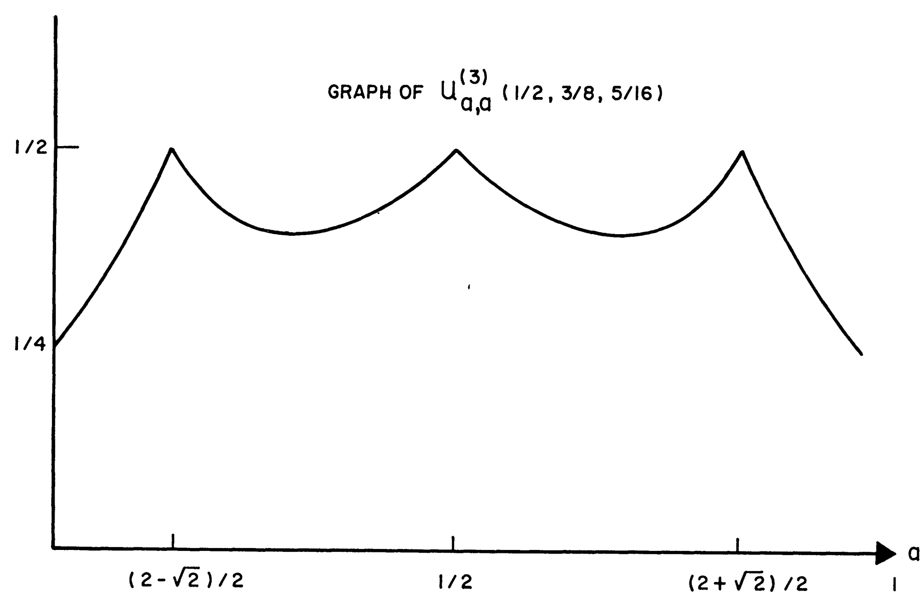

FIG. 3.

find that the set of all points a in $[0, 1]$ which satisfy (8.2) with $\theta = \frac{2}{5}$ is

$$(8.4)\qquad \left[0, \frac{3-5^{\frac{1}{2}}}{8}\right] + \left[\tfrac{1}{4}, \frac{5-5^{\frac{1}{2}}}{8}\right] + \left[\frac{3+5^{\frac{1}{2}}}{8}, \tfrac{3}{4}\right] + \left[\frac{5+5^{\frac{1}{2}}}{8}, 1\right].$$

Thus for each a in this set there exists no distribution on $[0, 1]$ having moments $\frac{1}{2}, \frac{3}{8}, \frac{5}{16}$ that assigns probability greater than $\frac{2}{5}$ to $\{a\}$. Of course for a in the complement of (8.4) relative to $[0, 1]$ such a distribution can be found.

As a final example, we consider a straightforward application of Theorem 7.2. Let us suppose that $b = \frac{1}{3}$ and that again $\mathbf{c} \in M_3{}^\circ$ is such that at $\mathbf{c}$, $p_1 = p_2 = \frac{1}{2}$. If we take advantage of calculations already made for the first example of this section and make several more that are equally innocuous, we find that the maximum probability assignable to the interval $[0, \frac{1}{3}]$ by any distribution on the unit interval with first 3 moments $\frac{1}{2}, \frac{3}{8}, c_3$, expressed in terms of its dependence upon $p_3(\frac{1}{2}, \frac{3}{8}, c_3) = 1 - q_3(\frac{1}{2}, \frac{3}{8}, c_3)$ is given by

$$\mathscr{U}^{(3)}_{0,\frac{1}{3}} = 1 - \frac{250}{3(2p_3+5)(18p_3+25)} \quad \text{or} \quad \frac{3}{4}\left(1 - \frac{1}{9(2q_3+1)}\right),$$

according as $p_3 \leqq$ or $\geqq \frac{5}{6}$. Thus for p_3 respectively equal to $\frac{1}{2}, \frac{9}{10}$, and $\frac{5}{6}$, this bound is $\frac{181}{306}, \frac{49}{54}$, and $\frac{11}{16}$. The latter bound may be obtained as well, with appropriate substitutions, from (7.9) and is also the maximum probability that any distribution on the unit interval having first two moments $\frac{1}{2}, \frac{3}{8}$, can assign to $[0, \frac{1}{3}]$.

It is clear that other applications of Theorem 7.2, analogous to those considered for Theorem 7.1 may be developed.

9. Proof for one form of the sharp upper bound. In this section we shall sketch a derivation for the form which the sharp upper bound assumes on the set $\Lambda_4(a, b)$ whenever $s^*(a, b) \leq 0$. Similar proofs are given in somewhat greater detail in [8] for each of the forms which the sharp upper bounds assumes.

By (3.5) and Table 1 in Section 3, $\Lambda_4(a, b)$ is defined for $s^*(a, b) \leqq 0$ to be the $M_3{}^\circ$ subset

$$(9.1)\qquad (\xi_{a,b,\hat{s}}\, h_{a,\hat{s}} + \eta_a h'_{a,\hat{s}} \leqq p_3 < \xi^*_{a,b}\, h_{a,1} h'_{b,1} + \eta_b h_{b,1}) .$$

The arguments a, b of $\hat{s}$ have been suppressed and we will continue this practice (also for s and s^*) where convenient below. The three lemmas which follow may be obtained via straightforward algebraic manipulations.

LEMMA 9.1. *For each pair of numbers a, b, $0 \leqq a, b \leqq 1$,*

$$\xi^*_{a,b} - \xi_{a,b} = w_{a,b}/R_2 .$$

LEMMA 9.2. *For each number triple a, b, t, $0 \leqq a, b, t \leqq 1$*

$$L(a, b, t) \subset (p_3 = \xi_{a,b,t}) .$$

In particular

$$L(0, a, b) \subset (p_3 = \xi_{a,b}) , \qquad L(a, b, 1) \subset (p_3 = \xi^*_{a,b})$$

and

$$L(0, a, 1) \subset (p_3 = \xi_{a,1}) = (p_3 = \xi^*_{0,a}) .$$

LEMMA 9.3. *For each number triple a, b, t, $0 \leqq a, b, t \leqq 1$,*

$$\eta_b - \xi_{a,b,t} = \frac{w_{a,b}\, w_{b,t}}{(\zeta_1 - b)R_2} \qquad on \quad (\zeta_1 \neq b) .$$

At $M_3{}^\circ$ points $\mathbf{c}$ on the boundary of M_3 (i.e., points $\mathbf{c}$ in $M_3{}^\circ$ at which $p_3(\mathbf{c}) = 0$ or 1), $\mathscr{U}^{(3)}_{a,b}(\mathbf{c})$ is the value which the single probability measure contained in $V_3(\mathbf{c})$ (e.g., See [1], Theorem 20.1, page 64) assigns to $[a, b]$. In a straightforward way, using this fact we obtain

LEMMA 9.4.

$$\begin{aligned}
\mathscr{U}^{(3)}_{a,b} &= 1 - (\zeta_1/S_2)I_{(S_2>b)} , && a = 0 \leqq b \leqq 1 \\
&= (\zeta_1/S_2)I_{(a\leqq S_2\leqq b)} , && 0 < a \leqq b \leqq 1 \quad on \quad (p_3 = 0) , \\
&= 1 - (\gamma_1/(1-\gamma_2))I_{(\gamma_2<a)} , && 0 \leqq a \leqq 1 = b \\
&= (\gamma_1/(1-\gamma_2))I_{(a\leqq\gamma_2\leqq b)} , && 0 \leqq a \leqq b < 1 \quad on \quad (p_3 = 1) .
\end{aligned}$$

LEMMA 9.5. *For all (a, b) in T such that $s^*(a, b) \leqq 0$,*

(i) $$\Lambda_4(a, b) \subset (w_{a,1} < 0 < w_{a,b}\, w_{b,\hat{s}}) .$$

(ii) $$\Lambda_4(a, b) = \emptyset \qquad when \quad s(a, b) \geqq 1 .$$

(iii) $$\Lambda_4(a, b) \cap \Lambda_i(a, b) = \emptyset , \quad i = 1, 2, 3, 5 .$$

(iv) $$\Lambda_4(a, a) = \emptyset .$$

(v) $$\Lambda_4(0, b) = (p_3 < \xi_{b,1} h'_{b,1} + \eta_b h_{b,1}) .$$

(vi) $$\begin{aligned}
\Lambda_4(a, b) \cap (p_3 = 0) &= (S_2 > b, p_3 = 0) , && a = 0 \\
&= \emptyset , && a > 0 \\
\Lambda_4(a, b) \cap (p_3 = 1) &= \emptyset .
\end{aligned}$$

PROOF. (i) $\Lambda_4(a, b) \cap (w_{a,1} \geqq 0) = \emptyset$ because the right-hand side of the relationship which defines $\Lambda_4(a, b)$ in (9.1) is equal to zero on $(w_{a,1} \geqq 0)$. On the other hand

$$(w_{a,b}\, w_{b,\hat{s}} \leqq 0) = (w_{a,b} \leqq 0) + (w_{b,\hat{s}} \leqq 0) .$$

But

$$\Lambda_4(a, b) \cap (w_{a,b} \leqq 0) = (\xi_{a,b,\hat{s}} \leqq p_3 < \xi^*_{a,b}) \cap (w_{a,b} \leqq 0)$$

which must be empty by Lemma 9.1, and

$$\Lambda_4(a, b) \cap (w_{b,\hat{s}} \leqq 0) \subset (\xi_{a,b,\hat{s}} \leqq p_3 < \eta_b) \cap (w_{b,\hat{s}} \leqq 0)$$

which must in turn be empty by Lemma 9.3.

(ii) Since

$$\xi_{a,b,1} = \xi^*_{a,b} \qquad \text{and} \qquad \hat{s}(a, b) = 1 , \quad \text{when} \quad s(a, b) \geqq 1 ,$$
$$\Lambda_4(a, b) = (\xi^*_{a,b} \leqq p_3 < \eta_b) \cap (w_{b,1} < 0) = \emptyset , \quad \text{by Lemma 9.3.}$$

(iii) Proof is similar to proof of part (i).

(iv) $$\Lambda_4(a, a) = (\xi_{a,a,3a} \leqq p_3 < \eta_a) \cap (w_{a,3a} < 0) .$$

This is empty, again by Lemma 9.3.

(v) and (vi) Follow via appropriate substitutions.

LEMMA 9.6. *For all* (a, b) *in* T *such that* $s^*(a, b) \leqq 0$,

$$\Lambda_4(a, b) \subset \mathscr{L}(a, b, [\hat{s}, 1]) .$$

PROOF. By parts (ii) and (iv) of Lemma 9.5, the assertion is vacuously true when $s(a, b) \geqq 1$ or $a = b$. Assume that

$$0 \leqq a < b \quad \text{and} \quad s(a, b) < 1 .$$

By (3.7)

$$\mathscr{L}(a, b, [s, 1]) = \bigcup_{s \leqq t \leqq 1} L(a, b, t) . \tag{9.2}$$

We will show that this set is essentially (9.1) with the right hand inequality of its defining relationship weak rather than strong. The $M_3°$ cylinder set whose base is the projection of (9.2) on M_2 is easily seen to be

$$(w_{a,1} \leqq 0 \leqq w_{a,b} w_{b,s}) . \tag{9.3}$$

On the other hand, the $M_3°$ cylinder set whose base is the projection of the simplex $L(a, b, t)$ is the set

$$(w_{a,t} \leqq 0 \leqq w_{a,b} w_{b,t}) . \tag{9.4}$$

A little geometric perspective or some algebraic manipulation will show that for each t such that $s \leqq t \leqq 1$, the set (9.4) is equal to

$$(t_0 \leqq t \leqq t_1) \cap (w_{a,1} \leqq 0 \leqq w_{a,b} w_{b,s}) , \tag{9.5}$$

where t_0, t_1 denote the *functions* on $M_3°$ defined by

$$t_0 = sh_{a,s} + (\gamma_2 \eta_a / a) h'_{a,s} , \qquad t_1 = h'_{b,1} + (\gamma_2 \eta_b / b) h_{b,1} . \tag{9.6}$$

By (2.4) and Lemma 9.1

$$\xi_{a,b,t} = \xi_{a,b} + (w_{a,b}/R_2) t . \tag{9.7}$$

It is clear that

$$w_{a,b}/R_2 \geqq 0$$

everywhere on the set (9.3). It follows by (9.7) that $\xi_{a,b,t}$ is nondecreasing in t at each point of (9.3). By (9.2) to (9.5) and Lemma 9.2 we have consequently that

$$\mathscr{L}(a, b, [s, 1]) = (\xi_{a,b,t_0} \leqq p_3 \leqq \xi_{a,b,t_1}) \cap (w_{a,1} \leqq 0 \leqq w_{a,b} w_{b,s}) . \tag{9.8}$$

It is an easy matter algebraically to show that

$$\xi_{a,b,\gamma_2 \eta_a / a} = \eta_a ,$$

from which it follows that

(9.9) $\xi_{a,b,t_0} = \xi_{a,b,s} h_{a,s} + \eta_a h'_{a,s}\,, \qquad \xi_{a,b,t_1} = \xi^*_{a,b} h'_{b,1} + \eta_b h_{b,1}\,.$

Note that by (9.1), ξ_{a,b,t_0} and $\xi_{a,b,t_1} h_{a,1}$ are left- and right-hand sides respectively of the inequality which defines $\Lambda_4(a, b)$. To complete the proof it suffices by (9.8) to show that

$$(\xi_{a,b,t_0} \leqq p_3 < \xi_{a,b,t_1} h_{a,1}) \subset (w_{a,1} \leqq 0 \leqq w_{a,b} w_{b,s})\,.$$

But part (i) of Lemma 9.5 is a stronger statement even than this. This completes the proof of Lemma 9.6.

THEOREM 9.1. *For each* (a, b) *in* T

$$\mathscr{U}^{(3)}_{a,b} = 1 - \frac{(w_{a,b})^3}{R_2^{\,2}(p_3 - \xi_{a,a,b})(p_3 - \xi_{a,b,b})} \qquad \textit{on} \quad \Lambda_4(a, b)\,.$$

PROOF. We prove this theorem here only for $s^*(a, b) \leqq 0$. The proof for $s^*(a, b) > 0$ then follows from results in Section 6. By parts (ii) and (iv) of Lemma 9.5, we may also assume that

$$0 \leqq a < b \qquad \text{and} \qquad s(a, b) < 1\,.$$

By part (vi) of Lemma 9.5, the theorem holds on

$$\Lambda_4(a, b) \cap [(p_3 = 0) + (p_3 = 1)]\,,$$

vacuously when $a > 0$, and with appropriate substitution by the first equation of Lemma 9.4 when $a = 0$. It remains to show that the theorem holds on

$$\Lambda_4(a, b) \cap (0 < p_3 < 1)\,.$$

To show this, it will suffice by Lemma 9.6 to show that it holds on

(9.10) $$\mathscr{L}(a, b, [s, 1]) \cap (0 < p_3 < 1)\,.$$

When $a = 0$, we may and will replace

$$[s(0, b), 1] = [b, 1]$$

by the half open interval $(b, 1]$ without altering the intersection (9.10) because

$$L(0, b, b) = L(0, b) \subset (p_3 = 0)\,.$$

Let t be an arbitrary number such that

(9.11) $$s(a, b) \leqq t \leqq 1\,.$$

(When $a = 0$, however, we do not allow that $t = s(0, b) = b$). It is easy to verify that the parallel planes having respective equations

$$t(t + 2\rho)x - (2t + \rho)y + z = t^2\rho\,, \qquad t^2\rho + \frac{(a - t)^2(b - t)^2}{2t - a - b}\,,$$

where

$$\rho = \rho(a, b, t) = a - \frac{(t - b)^2}{2t - a - b} = b - \frac{(t - a)^2}{2t - a - b}\,,$$

constitute an admissible pair. (See discussion following Theorem 1.3.) The first plane supports and intersects with M_3 at the single point (t, t^2, t^3) when $t > s(a, b)$, and in the closed chord joining $(0, 0, 0)$ and (t, t^2, t^3) when $t = s(a, b)$. Let us refer to this intersection as the set B_0 of Kemperman's theorem (Theorem 1.3, taking $n = 3$). The second plane supports and intersects with $M_3^{[a,b]}$ in the closed chord joining (a, a^2, a^3), (b, b^2, b^3) (take this to be the set B_1 of Kemperman's theorem), and its separates $M_3^{[a,b]}$ from the first plane. It is clear that

$$L(a, b, t) \subset \operatorname{conv}(B_0 + B_1) .$$

By the theorem cited it must therefore be the case that everywhere on the set

$$L(a, b, t) \cap (0 < p_3 < 1) ,$$

$$\mathscr{U}_{a,b}^{(3)} = \frac{t(t + 2\rho)\nu_1 - (2t + \rho)\nu_2 + \nu_3 - t^2\rho}{(a - t)^2(b - t)^2/(2t - a - b)} .$$

If we convert to normalized moments using the inverse mapping (1.6), the numerator of the expression on the right-hand side simplifies to

$$R_2(p_3 - \xi_{t,t,\rho}) .$$

But on $L(a, b, t)$, by Lemma 9.2, this equals

$$R_2(\xi_{a,b,t} - \xi_{t,t,\rho}) ,$$

which using (9.7) simplifies in turn to

$$(t - a)(t - b)[(t - a)(t - b) - w_{a,b}]/(2t - a - b) .$$

Thus

$$\mathscr{U}_{a,b}^{(3)} = 1 - \frac{w_{a,b}}{(t - a)(t - b)} \quad \text{on} \quad L(a, b, t) \cap (0 < p_3 < 1) . \tag{9.12}$$

Clearly,

$$L(a, b) \subset L(a, b, t) .$$

But

$$w_{a,b} = 0 \quad \text{on} \quad L(a, b) .$$

Hence by (9.12)

$$\mathscr{U}_{a,b}^{(3)} = 1 \quad \text{on} \quad L(a, b)$$

(as of course it must by (1.4) since $L(a, b) \subset M_3^{[a,b]}$). On the other hand

$$w_{a,b} > 0 \quad \text{on} \quad L(a, b, t)\backslash L(a, b) .$$

Hence by Lemma 9.2 and (9.7),

$$R_2(p_3 - \xi_{a,b,a})/w_{a,b} = t - a , \qquad R_2(p_3 - \xi_{a,b,b})/w_{a,b} = t - b ,$$

everywhere on $L(a, b, t)\backslash L(a, b)$. But then by (9.12),

$$\mathscr{U}_{a,b}^{(3)} = 1 - \frac{w_{a,b}^3}{R_2^2(p_3 - \xi_{a,b,a})(p_3 - \xi_{a,b,b})} \tag{9.13}$$

on $L(a, b, t) \cap (0 < p_3 < 1)$. But t is arbitrary subject only to the restriction

(9.11) if $a > 0$ ($s(0, b) = b < t \leqq 1$, if $a = 0$). It follows that (9.13) holds everywhere on (9.10). This completes the proof.

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