The Annals of Statistics
1975, Vol. 3, No. 4, 846-861

A GENERALIZED APPROACH TO MAXIMUM LIKELIHOOD PAIRED COMPARISON RANKING

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In many situations one is faced with the task of constructing a linear order, or ranking, of n objects based on data derived from a paired comparison experiment. Alternatively, one may desire to estimate the preference relation on the set of objects. Numerous criteria appear in the literature, and in practice, under which rankings or preferences may be selected. The main emphasis of this paper is on developing a general approach to maximum likelihood estimation of rankings and preferences. Utilizing what we term f criteria, our results unify and extend both the theory of constrained maximum likelihood ranking estimation and the work of Singh and Thompson on preference estimation. In addition, we show that a specific mathematical programming problem subsumes the problem of finding a maximum likelihood ranking and also that an efficient branch search algorithm can be used to find maximum likelihood preferences. Four specific f criteria are selected for illustration and each is applied to three examples from the paired comparison literature.

1. Introduction. In many situations one is faced with the task of constructing a linear order, or ranking, of n objects, $A_1, A_2, \cdots, A_n$, based on data derived from a paired comparison experiment. Alternatively, one may wish to find the preference relationship existing among the objects based on the same data. In a paired comparison experiment distinct pairs of objects are compared: A_i with A_j exactly k_{ij} times. The data are n_{ij}, the number of times A_i has been preferred to A_j, t_{ij}, the number of times they are tied, and n_{ji}, the number of times A_j has been preferred to A_i.

Certain situations naturally produce paired comparisons such as the sporting events, football, basketball, and baseball which involve only two teams. The associated season records of wins, ties, and losses for the teams of interest constitute the data. In other situations, such as food tasting, paired comparisons are desirable because of the difficulty of distinguishing preferences when more than two objects are considered simultaneously.

It may happen that the data suggest a natural ranking or preference relation. On the other hand, due to inconsistencies, ties, or failure to directly compare some objects, a natural choice may not be apparent. The intent of this paper is to extend and unify the statistical theory which underlies both the selection of natural rankings and preferences and selection made in the more complex case of inconsistencies, ties, or incomplete comparisons.

Received September 1973; revised June 1974.

AMS 1970 *subject classifications*. Primary 62F07; Secondary 62G05, 62A10.

Key words and phrases. Maximum likelihood, paired comparison ranking, paired comparisons, preferences for paired comparisons.

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A stochastic model for paired comparison data has been given as: $N = \sum_{1 \leqq i < j \leqq n} k_{ij}$ paired comparisons are assumed to occur independently with π^*_{ij} the probability of A_i being preferred over A_j, γ^*_{ij} the probability of A_i tying A_j, and π^*_{ji} the probability of A_j being preferred over A_i. In other words, N independent comparisons are conducted, three possible outcomes for each, with the probability of each outcome dependent on the objects being compared.

Kendall and Smith (1939) initiated the early statistical work on whether a ranking might reasonably be inferred from the data, dealing with the case where no ties were allowed and all $k_{ij} = 1$. They introduced a statistic d which was based on the number of circular triads present in the data. One of the prime motivations for its use was its ready computability. Slater (1961) argued against the use of d since it gave unequal weights to inconsistencies and proposed a statistic i, the minimum total number of inconsistencies that can be achieved in any ranking. His test for the existence of a ranking was based on the null hypothesis that the model holds with all $\pi^*_{ij} = \frac{1}{2}$, all $\gamma^*_{ij} = 0$, and all $k_{ij} = 1$.

If the data pass Slater's test, a minimum total inconsistency ranking was selected. He called these rankings nearest neighbor rankings. One disadvantage of Slater's statistic is the apparent difficulty in computing it. Remage and Thompson (1966) show that a procedure suggested by Alway fails, and give a dynamic programming algorithm for its computation. Other methods are available for the computation of i, e.g., the linear programming formulation of De Cani (1969), the branch and bound algorithm of De Cani (1972), and the branch search algorithm of Flueck and Korsh (1974). Kendall (1955) mentioned ranking under a nearest neighbor criterion and under a row sum criterion. Ford (1957) also deals with row sum ranking.

Singh and Thompson (1968), in a fundamental paper, study the selection of rankings and preferences in the context of a general theory of statistical inference. The likelihood function, given the observations, serves as the statistical basis for this purpose. The main emphasis of their paper is on the selection of a maximum likelihood preference which need not imply a unique ranking. They achieve a characterization of these preferences using graph theory.

They also consider a number of alternative approaches including the selection of a maximum likelihood weak stochastic ranking. De Cani (1969) deals with weak stochastic rankings for the case of all $k_{ij} = k$, Remage and Thompson for the case $\gamma^*_{ij} = 0$, and Thompson and Remage (1964) for the case of all $\gamma^*_{ij} = 0$ and all $k_{ij} = 1$.

Other authors, including Brunk (1960), Bradley–Terry (1952), and Mosteller (1951), investigate the theory of paired comparisons, but invoke additional assumptions on the π^*_{ij} and γ^*_{ij} of the stochastic model. David (1963) summarizes this work, and we will not pursue these parametric approaches.

In the present paper the emphasis is on developing a general unifying approach to constrained maximum likelihood estimation of rankings and preferences. In Section 3 we give initial results which generalize those of Singh and Thompson to what we call f rankings and to f preferences selected under f criteria. In Section 4 we study maximum likelihood f rankings. Our main conclusion is that the estimation of optimal rankings may be achieved by solving a specific mathemathical programming problem, and any algorithm for its solution will find optimal rankings. Section 5 introduces a branch search algorithm for maximum likelihood (ml) preferences, and for purposes of illustration and clarification Section 6 presents four specific f criteria; weak stochastic, preference, semipreference, and row sum. Three numerical examples are solved under each of the criteria. Some confusion seems to exist in the literature in connection with the selection of rankings or preferences. We resolve this problem.

2. Preliminaries. We take the likelihood function to be:

$$L(\pi) = \prod_{1 \leqq i<j \leqq n} \frac{k_{ij}!}{n_{ij}!\, t_{ij}!\, n_{ji}!} \pi_{ij}^{n_{ij}} \gamma_{ij}^{t_{ij}} \pi_{ji}^{n_{ji}} \tag{1}$$

where the vector $\pi = (\pi_{12}, \gamma_{12}, \pi_{13}, \gamma_{13}, \cdots, \pi_{1n}, \gamma_{1n}, \cdots, \pi_{(n-1)n}, \gamma_{(n-1)n})$ with $\pi_{ij} + \gamma_{ij} + \pi_{ji} = 1,\ 0 \leqq \pi_{ij}, \gamma_{ij}\ \pi_{ji}$.
Since $\pi_{ji} = 1 - \pi_{ij} - \gamma_{ij}$, there are only $2\binom{n}{2}$ independent arguments of L, the π_{ij}'s and the γ_{ij}'s. We may write $\log L(\pi)$ as a constant plus

$$\sum_{1 \leqq i<j \leqq n} k_{ij}(\hat{\pi}_{ij} \log \pi_{ij} + \hat{\gamma}_{ij} \log \gamma_{ij} + \hat{\pi}_{ji} \log \pi_{ji}) \tag{2}$$

where

$$\hat{\pi}_{ij} = n_{ij}/k_{ij}\,, \qquad \hat{\gamma}_{ij} = t_{ij}/k_{ij}\,, \qquad \hat{\pi}_{ji} = n_{ji}/k_{ij} \qquad \text{for } k_{ij} > 0\,,$$

and we will take

$$\hat{\pi}_{ij} = \hat{\gamma}_{ij} = \hat{\pi}_{ji} = \tfrac{1}{3} \qquad \text{if } \cdot k_{ij} = 0\,.$$

To simplify notation define:

$$c_{ij}(\pi) = k_{ij}(\hat{\pi}_{ij} \log \pi_{ij} + \hat{\gamma}_{ij} \log \gamma_{ij} + \hat{\pi}_{ji} \log \pi_{ji})\,;$$

then (2) becomes

$$\sum_{1 \leqq i<j \leqq n} c_{ij}(\pi) = c(\pi)\,. \tag{2a}$$

The $\hat{\pi}$, whose components are given by $(\hat{\pi}_{ij}, \hat{\gamma}_{ij})$ for all i, j, is the usual ml estimator of the population parameter vector π^*. It is well known that $L(\pi)$ has a unique maximum at $\hat{\pi}$. In the following, π will denote either an estimator or estimate of π^*.

Let $f(\pi_{ij}, \gamma_{ij}, \pi_{ji})$ be any function of $(\pi_{ij}, \gamma_{ij}, \pi_{ji})$ for which $S_f = \{\pi \mid f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) \geqq 0 \text{ for all } i, j\}$ is closed and bounded.

A ranking R is an f ranking w.r.t. π iff, whenever A_i precedes A_j in R, $f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) \geqq 0$. Let

$$L(\pi_R) = \max_{\pi \ni R \text{ is an } f \text{ ranking w.r.t. } \pi} L(\pi)$$

and take Γ to be the set of all rankings.

A maximum likelihood f ranking R_0 is an f ranking satisfying:

$$L(\pi_{R_0}) = \max_{R \in \Gamma} L(\pi_R) .$$

The maximizing π_{R_0}'s will be taken as estimators of π^* when selecting rankings under an f criterion. We call such a π an f optimal estimate or *f-optimal*.

The selection of f corresponds to establishing a criterion for ranking, namely A_j must be ranked no higher than A_i if "A_i is at least as good as A_j," i.e. iff $f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) \geqq 0$. Let us say that an inconsistency occurs in an f ranking if A_i precedes A_j in the ranking and $f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) < 0$. We note that with respect to π, all ml f rankings will have zero inconsistencies. We will take such rankings as estimates of the population rankings. Therefore, we are assuming that with respect to π^*, the population rankings have zero inconsistencies.

We will consider three relations $D_f(\pi)$, $C_f(\pi)$, and $T_f(\pi)$ which are intended to capture the idea of "preferred", "equivalent", and "extended preferred" respectively. If $(A_i, A_j) \in D_f(\pi)$, $C_f(\pi)$, or $T_f(\pi)$ we will say that A_i is preferred to A_j, A_i is equivalent to A_j, or A_i is preferred (in an extended manner) to A_j, respectively. These are required before we can introduce the idea of an f preference.

$D_f(\pi)$ is the relation induced by f and π:

$$(A_i, A_j) \in D_f(\pi) \qquad \text{iff} \quad f(\pi_{ji}, \gamma_{ij}, \pi_{ij}) < 0 .$$

Intuitively, if $(A_i, A_j) \in D_f(\pi)$ then, under the criterion f, "A_i is preferred to A_j" iff A_j is not at least as good as A_i. Note that $D_f(\pi)$ need not be transitive or asymmetric.

$C_f(\pi)$ is the relation induced by f and π:

$$(A_i, A_j) \in C_f(\pi) \qquad \text{iff} \quad f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) = 0 \quad \text{and} \quad f(\pi_{ji}, \gamma_{ij}, \pi_{ij}) = 0 .$$

If $(A_i, A_j) \in C_f(\pi)$, then, under the criterion f, "A_i is at least as good as A_j" and "A_j is at least as good as A_i" so that neither is preferred to the other. In this case we term them equivalent.

A natural way of treating the equivalence of two objects is: if A_i and A_j are equivalent then any object that is preferred to one is preferred to the other and any object over which one is preferred the other is preferred over also. This allows us to "extend" the concept of a "preferred" relation to $T_f(\pi)$. That is, $(A_i, A_j) \in T_f(\pi)$ iff either A_i is preferred to A_j, A_i is preferred to some object which is preferred to A_j, or A_i is equivalent to some object which is preferred to A_j.

The relation $C_f \cup D_f(\pi)$ induces a relation $T_f(\pi)$: $(A_i, A_j) \in T_f(\pi)$ iff a directed path exists from A_i to A_j in $C_f \cup D_f(\pi)$, the union of $C_f(\pi)$ and $D_f(\pi)$. That is, iff there exists $A_{i_1}, A_{i_2}, \cdots, A_{i_r}$ such that $(A_{i_j}, A_{i_{j+1}}) \in C_f \cup D_f(\pi)$ for $j = 1, 2, \cdots, r-1$, with at least one $(A_{i_j}, A_{i_{j+1}}) \in D_f(\pi)$ and $A_{i_1} = A_i$, $A_{i_r} = A_j$.

A ranking R is consistent with a relation T if $(A_i, A_j) \in T$ implies A_i precedes A_j

in R. That is, R is consistent with T iff R satisfies the constraints imposed by T. More than one ranking may be consistent with T. In this terminology an f ranking is a ranking consistent with $D_f(\pi)$. Define $T_{D(f)}(\pi)$ as the relation induced by $D_f(\pi)$. A relation will be called a *preference* if it is transitive and asymmetric. A relation $T = T_f(\pi)$ *will be called an f preference* when it is a preference. Let

$$L(\pi_T) = \sup_{\pi \ni T \text{ is an } f \text{ preference w.r.t. } \pi} L(\pi)$$

and take ρ to be the set of all $T_f(\pi_T)$ that are preferences. A *maximum likelihood f preference* T_0 is an f preference satisfying:

$$L(\pi_{T_0}) = \max_{T \in \rho} L(\pi_T) .$$

The maximizing π_{T_0}'s will be taken as estimators of π^* when selecting preferences under an f criterion. We call such a π an fp optimal estimate or *fp-optimal*.

Suppose π^*, the actual population parameter vector, were known. How would we select a relation, $T(\pi^*)$, so that if $(A_i, A_j) \in T(\pi^*)$, we accept A_i as being preferred to A_j, in some sense? If we assume objects are being compared on the same basis or scale, then it seems reasonable to require that $T(\pi^*)$ be transitive and asymmetric. This is a requirement we might be unwilling to make if more than one basis for comparison is presumed. Note that accepting the requirement that $T(\pi^*)$ be a preference relation still leaves open the possibility that more than one ranking may be consistent with it.

In the following, we attempt to estimate preferences based on statistical considerations. The theory developed here assumes that, with respect to π^*, the population relation $T(\pi^*)$ is a preference. Note that the sense in which A_i is preferred to A_j has been assumed to be dependent only upon $(\pi_{ij}^*, \gamma_{ij}^*, \pi_{ji}^*)$ and expressible as an f criterion. The intriguing question of which, if any, of these criteria meaningfully serve as models for how subjects determine rankings or preferences will not be examined.

3. Initial results.

LEMMA. *$T_f(\pi)$ and $T_{D(f)}(\pi)$ are preferences* iff *at least one ranking is consistent with $D_f(\pi)$.*

PROOF. $T_f($· and $T_{D(f)}(\pi)$ are preferences iff $D_f(\pi)$ is asymmetric. $D_f(\pi)$ is asymmetric i at least one ranking is consistent with it.

In general more rankings will be consistent with $C_f \cup D_f(\pi)$ than $T_f(\pi)$. In the event that $T_f(\pi)$ is a preference, a ranking is consistent with $C_f \cup D_f(\pi)$ iff it is consistent with $T_f(\pi)$. A ranking is consistent with $T_{D(f)}(\pi)$ iff it is consistent with $D_f(\pi)$.

Suppose we are given a ranking R and find a π, π_R, which maximizes the likelihood, subject to the constraint that the ranking be an f ranking. Such a π_R must exist since $c(\pi)$ is a concave function, by a lemma of Singh and Thompson, and is being maximized over the closed and bounded set S_f. Let $L(\pi_{R_0}) = \max_{R \in \Gamma} L(\pi_R)$. Then $T_{D(f)}(\pi_{R_0})$ is a preference and R_0 is consistent with $D_f(\pi_{R_0})$.

Thus π_{R_0} must be a π which maximizes the likelihood, subject to the constraint that $T_{D(f)}(\pi_{R_0})$ be a preference. Thus π_{R_0} is *f-optimal*. We have proved the following theorem.

THEOREM 1. *Given an f-optimal π, only maximum likelihood f rankings are consistent with $D_f(\pi)$, there must be at least one such ranking, and each such ranking is consistent with one and only one $D_f(\pi)$.*

Consequently, whether we select an R whose π_R yields maximal $L(\pi)$ subject to the constraint that R be an f ranking w.r.t. π, or we find an f optimal π which maximizes $L(\pi)$ subject to the constraint that $T_{D(f)}(\pi)$ be a preference and then select an R consistent with $T_{D(f)}(\pi)$, we are selecting from exactly the same set of rankings. In short, the ranking and the preference viewpoint are dual.

To summarize: The selection of a ml f ranking may be viewed in two ways.

(i) Either given a ranking R, find π which maximizes the likelihood subject to the constraint that the given ranking be an f ranking w.r.t. π; those R that maximize the likelihood are ml f rankings and the corresponding π's are f-optimal.

(ii) Or find π which maximizes the likelihood subject to the constraint that $T_{D(f)}(\pi)$ be a preference; all rankings consistent with $T_{D(f)}(\pi)$ are ml f rankings and there may be more than one π which is f-optimal.

Given a ranking R, suppose that A_i precedes A_j in R and $f(\hat{\pi}_{ij}, \hat{\gamma}_{ij}, \hat{\pi}_{ji}) \geqq 0$. Then the (i, j)th component of π_R, $(\pi^R_{ij}, \gamma^R_{ij}, \pi^R_{ji})$, may be taken to be $(\hat{\pi}_{ij}, \hat{\gamma}_{ij}, \hat{\pi}_{ji})$. If $f(\hat{\pi}_{ij}, \hat{\gamma}_{ij}, \hat{\pi}_{ji}) > 0$, then $(\pi^R_{ij}, \gamma^R_{ij}, \pi^R_{ji})$ will yield a $c_{ij}(\pi_R)$ which is strictly less than $c_{ij}(\hat{\pi})$, assuming $k_{ij} > 0$. This follows from a well-known information theory inequality (Rao, page 17, 1965). Hence a ml f ranking must correspond to a subset of $D_f(\hat{\pi})$ which is asymmetric and which is not contained in any other asymmetric subset of $D(\hat{\pi})$. Such a subset is called a maximal circuit free subbigraph for $D_f(\hat{\pi})$ in the terminology of Singh and Thompson.

Since $T_f(\pi)$ contains $T_{D(f)}(\pi)$, $T_{D(f)}(\pi)$ is also a preference for fp-optimal π's. Hence $L(\pi_D) \geqq L(\pi_T)$ when π_D is f-optimal and π_T is fp-optimal. Also, when $T_f(\pi)$ is a preference, a ranking will be consistent with $T_f(\pi)$ iff it is consistent with $T_{D(f)}(\pi)$. If $L(\pi_T) = L(\pi_D)$ then any ranking consistent with $T_f(\pi_T)$ is a ml f ranking.

Singh and Thompson emphasize the selection of a maximum likelihood preference. At one extreme this might be the null preference with which all rankings are consistent, and at the other extreme this might be a complete relation, with which a unique ranking is consistent.

Consider a fp-optimal π and the corresponding ml preferences $T_f(\pi)$. Suppose that $(A_i, A_j) \notin T_f(\pi)$, but $(A_i, A_j) \in C_f(\hat{\pi})$ or $(A_i, A_j) \in D_f(\hat{\pi})$. Let π' be the same as π, except $(\pi'_{ij}, \gamma'_{ij}, \pi'_{ji}) = (\hat{\pi}_{ij}, \hat{\gamma}_{ij}, \hat{\pi}_{ji}) \; (\neq (\pi_{ij}, \gamma_{ij}, \pi_{ji}))$. If $T_f(\pi')$ is also a preference, then $c(\pi')$ must be strictly greater than $c(\pi)$ since $c_{ij}(\pi') > c_{ij}(\pi)$, Rao (1965). The contradicts the assumption that $T_f(\pi)$ is a ml preference.

A ml preference $T_f(\pi)$ is induced by $C_f \cup D_f(\pi)$.
Take

$$C_f(\pi, \hat{\pi}) = C_f(\pi) - \{(A_i, A_j) \mid (A_i, A_j) \notin C_f(\hat{\pi})\}$$
$$D_f(\pi, \hat{\pi}) = D_f(\pi) - \{(A_i, A_j) \mid (A_i, A_j) \notin D_f(\hat{\pi})\}$$
$$A(\pi, \hat{\pi}) = C_f(\pi, \hat{\pi}) \cup D_f(\pi, \hat{\pi}) \qquad \text{and}$$
$$B(\pi, \hat{\pi}) = [C_f(\pi) - C_f(\hat{\pi})] \cup [D_f(\pi) - D_f(\hat{\pi})] .$$

Note that $A(\pi, \hat{\pi})$ is a subbigraph of $C_f \cup D_f(\hat{\pi})$, and $B(\pi, \hat{\pi})$ is disjoint from $C_f \cup D_f(\hat{\pi})$ and from $A(\pi, \hat{\pi})$. Clearly $C_f \cup D_f(\pi) = A(\pi, \hat{\pi}) \cup B(\pi, \hat{\pi})$. By Theorem 4 of Singh and Thompson such a union induces a preference iff it is circuit free. We conclude that no (A_i, A_j) could be removed from $B(\pi, \hat{\pi})$ and added to $A(\pi, \hat{\pi})$ with the resultant union remaining circuit free. Hence $A(\pi,\hat{\pi})$ is a maximal subbigraph of $C_f \cup D_f(\hat{\pi})$. $B(\pi, \hat{\pi})$ represents those (A_i, A_j) that are in $C_f(\pi)$ but not in $C_f(\hat{\pi})$, or in $D_f(\pi)$ but not in $D_f(\hat{\pi})$. Under some f's $B(\pi, \hat{\pi})$ will always be null, and under others it need not be null. We thus have the following result.

THEOREM 2. *Let π be fp-optimal and $B(\pi, \hat{\pi}) = C_f \cup D_f(\pi) - A(\pi, \hat{\pi})$. Then any* ml *preference is induced by $A(\pi, \hat{\pi}) \cup B(\pi, \hat{\pi})$ where $A(\pi, \hat{\pi})$ is a maximal subbigraph of $C_f \cup D_f(\hat{\pi})$.*

COROLLARY. *Any (A_i, A_j) not in $C_f \cup D_f(\hat{\pi})$ will not be in $C_f \cup D_f(\pi)$ when π is fp-optimal.*

This generalizes Theorem 12 of Singh and Thompson. Under their specific criterion $B(\pi, \hat{\pi})$ is always null. In this case $A(\pi, \hat{\pi})$ is called a maximal circuit free subbigraph of $C_f \cup D_f(\hat{\pi})$.

We note that $c_{ij}(\pi)$, for π f-optimal or π fp-optimal, may be determined in principle from $\hat{\pi}$ and f. Consequently $\hat{\pi}$ may be regarded as a sufficient statistic w.r.t. selecting ml f rankings or ml preferences.

4. Maximum likelihood rankings. We know that $c(\pi_R) \geqq c(\pi)$, for all π w.r.t. which R is an f ranking. The $c_{ij}(\pi_R)$ are really functions of only $(\pi_{ij}^R, \gamma_{ij}^R, \pi_{ji}^R)$ which are in turn determined directly from $\hat{\pi}$ and f. The contribution of the k_{ij} paired comparisons of A_i with A_j is either $c_{ij}(\pi_R)$ or $c_{ij}(\pi_{R'})$; depending on whether A_i precedes A_j in R or not, where R' is the same as R, except i and j are interchanged. The problem of finding a ml f ranking then becomes:

$$\text{(3)} \qquad \text{maximize}_{R \in \Gamma} \sum_{1 \leqq i < j \leqq n} c_{ij}(\pi_R) .$$

Let $H_f(\hat{\pi})$ be the set of all R such that if A_i immediately precedes A_j in R, then $f(\hat{\pi}_{ij}, \hat{\gamma}_{ij}, \hat{\pi}_{ji}) \geqq 0$.

THEOREM 3. *All* ml *f rankings belong to $H_f(\hat{\pi})$.*

PROOF. Suppose some ml f ranking $R \notin H_f(\hat{\pi})$. Then some A_i immediately precedes A_j, but $c_{ij}(\pi_R) < c_{ij}(\pi_{R'})$. If we interchange A_i and A_j to obtain R' then $c(\pi_{R'}) = c(\pi_R) + c_{ij}(\pi_{R'}) - c_{ij}(\pi_R) > c(\pi_R)$, because no other c_{ij}'s are affected by the interchange. This contradicts the assumption that R is a ml f ranking.

The results of De Cani (1969) show that (3) may be solved as a linear programming problem with $(\frac{1}{6})n(n-1)(n-2)$ constraints and $(\frac{1}{6})(n+1)(n)(n-1)$ variables. De Cani (1972) has also presented a branch and bound algorithm as an alternative method for finding solutions. This algorithm appears to be a significant improvement over the dynamic programming algorithm of Remage and Thompson in terms of average behavior, although its worst case is an exhaustive search through all rankings.

Remage and Thompson do not mention the possibility of incorporating the fact that a solution must belong to $H_f(\hat{\pi})$, in order to reduce the search. It does not appear to be readily incorporated into De Cani's branch and bound algorithm, except at the last stage. Singh and Thompson would, in effect, search through all rankings in $H_f(\hat{\pi})$. In Flueck and Korsh (1974) we present a branch search algorithm which searches through only $H_f(\hat{\pi})$, but allows significant pruning to occur.

5. Maximum likelihood preferences. For the problem of selecting a preference, under an f criterion, consider the estimator π of π^*. Initially we take $\hat{\pi}$ as the estimator of π^*. It is determined by the basic comparisons of the (A_i, A_j). If $T_f(\hat{\pi})$ is a preference then it is a ml preference and we would consider it a natural preference. Otherwise, the comparisons between the A_i and A_j which are estimated by $\hat{\pi}$ are inconsistent, and no natural preference is apparent. Consequently, under the assumption of no inconsistencies in the population, the estimate of one or more of these basic comparisons must be altered. We discuss these alterations.

As shown in Section 3, selecting a ml preference under an f criterion amounts to selecting a preference with a maximal $A(\pi, \hat{\pi})$ component of $C_f \cup D_f(\hat{\pi})$ which yields a maximum likelihood $C_f \cup D_f(\pi)$. This preference may be thought of as arising from $C_f \cup D_f(\hat{\pi})$ by performing operations on each of its (A_i, A_j), and there may be an infinite number of π's which induce it. Suppose this preference is induced by $C_f \cup D_f(\pi)$. Let us say that an undirected line $(-)$ occurs between A_i and A_j in $C_f \cup D_f(\hat{\pi})$ if $(A_i, A_j) \in C_f(\hat{\pi})$, a directed line $(\rightarrow)$ occurs from A_i to A_j if $(A_i, A_j) \in D_f(\hat{\pi})$, but $(A_j, A_i) \notin D_f(\hat{\pi})$, two directed lines $(\rightleftarrows)$ occur between A_i and A_j if both (A_i, A_j) and (A_j, A_i) are in $D_f(\hat{\pi})$, and a blank line occurs between A_i and A_j otherwise. The operations on $C_f \cup D_f(\hat{\pi})$ that yield $C_f \cup D_f(\pi)$ may, in general, change any one of these four types of lines between A_i and A_j to any one of the others except the two directed lines. Under some f criteria this choice of operations may be further limited. To find a candidate for a ml preference a number of these operations must be performed on $C_f \cup D_f(\hat{\pi})$. As indicated, this amounts to selecting both a line type for each

A_i and A_j pair and a (π_{ij}, γ_{ij}) which will best induce the line and yield a maximal $c_{ij}(\pi)$ component of $c(\pi)$. A constraint on the selection is that the resultant $C_f \cup D_f(\pi)$ must have a maximal $A(\pi, \hat{\pi})$ component of $C_f \cup D_f(\hat{\pi})$. Finally, all the candidates which maximize the likelihood are ml preferences and their π's are fp-optimal.

In Singh and Thompson it was shown that only maximal circuit free (mcf) subbigraphs of $C_f \cup D_f(\hat{\pi})$ need be considered when searching for a ml preference under their specific f criterion, and our Theorem 2 extends this result. Still, the number of such subbigraphs may be very large. We now present an algorithm for finding ml preferences, under general f criteria, which appears to be much more efficient, on the average, than searching through all mcf subbigraphs.

For each $(A_{i_k}, A_{i_{k+1}})$ let $(\bar{\pi}_{i_k i_{k+1}}, \bar{\gamma}_{i_k i_{k+1}})$ maximize $c_{i_k i_{k+1}}$ subject to $f(\bar{\pi}_{ij}, \bar{\gamma}_{ij}, \bar{\pi}_{ji}) \geqq 0$. We emphasize that if $(A_{i_k}, A_{i_{k+1}}) \in C_f \cup D_f(\hat{\pi})$ then $c_{i_k i_{k+1}}$ is maximized by $(\hat{\pi}_{i_k i_{k+1}}, \hat{\gamma}_{i_k i_{k+1}})$. We now assign a unique $\pi(R)$ to some rankings $R = A_{i_1}, A_{i_2}, \cdots, A_{i_n}$. Take $(\pi_{i_k i_{k+1}}(R), \gamma_{i_k i_{k+1}}(R))$ to be $(\bar{\pi}_{i_k i_{k+1}}, \bar{\gamma}_{i_k i_{k+1}})$ for $k = 1, 2, \cdots, n-1$. Now starting with $k = 3$ and progressing sequentially to $k = n$, consider (A_{i_j}, A_{i_k}), starting with $j = k - 2$ and regressing sequentially to $j = 1$. Take $(\pi_{i_j i_k}(R), \gamma_{i_j i_k}(R))$ to be such that it maximizes $c_{i_j i_k}$ subject to the constraint that $C_f \cup D_f(\pi(R, j, k,))$ be circuit free. If no such maximum exists, this R will be ignored. $C_f \cup D_f(\pi(R, j, k))$ denotes the $\{(A_{i_r}, A_{i_s}) \in C_f \cup D_f(\pi(R))$: $j < r < s \leqq k$ or $s = r + 1$, $1 \leqq r \leqq k - 2\}$. We then associate the unique preference induced by $C_f \cup D_f(\pi(R))$ with the ranking R. We call $C_f \cup D_f(\pi(R))$ the relation generated by R.

THEOREM 4. *Let $C_f \cup D_f(\pi)$ induce a* ml *preference. At least one ranking R that is consistent with this* ml *preference will generate $C_f \cup D_f(\pi)$.*

PROOF. Let $D_0 = \{R \mid R$ is consistent with the ml preference induced by $C_f \cup D_f(\pi)$ but A_{i_s} precedes A_{i_r} in R for all (A_{i_r}, A_{i_s}) belonging to $D_f(\hat{\pi})$ but not to $D_f(\pi)\}$. Any $R \in D_0$ cannot yield an (A_{i_r}, A_{i_s}) in $D_f(\pi(R))$ and not in $D_f(\pi)$. Suppose (A_{i_r}, A_{i_s}) belongs to $C_f(\hat{\pi})$ but not to $C_f(\pi)$. Then a directed path must exist from either A_{i_r} to A_{i_s} or from A_{i_s} to A_{i_r} in $C_f \cup D_f(\pi)$. Assume it is from A_{i_r} to A_{i_s}. Let C_0 denote the set of all rankings R such that for any $(A_{i_r}, A_{i_s}) \in C_f(\hat{\pi})$ but not to $C_f(\pi)$, $R = A_{i_1} \cdots A_{i_r} A_{i_{r+1}} \cdots A_{i_s} \cdots A_{i_n}$ and $A_{i_r} \cdots A_{i_s}$ represents a directed path from A_{i_r} to A_{i_s} in $C_f \cup D_f(\pi)$. Then $D_0 \cap C_0$ contains all rankings R which generate $C_f \cup D_f(\pi(R))$'s that induce $C_f \cup D_f(\pi)$. The significance of this theorem is that it allows a sequential, systematic generation of all possible ml preferences and only ml preferences. The branch search algorithm of Flueck and Korsh may be modified to yield a branch search algorithm for preferences based on the above theorem.

6. Examples. We will consider four distinct f criteria in order to illustrate our results and clarify some previous misconceptions. All but one are taken from the paired comparison literature. The criteria are:

weak stochastic—$f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) = \pi_{ij} - \pi_{ji}$,

preference—$f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) < 0$ if $\pi_{ji} > \max(\pi_{ij}, \gamma_{ij})$
$= 0$ if $\gamma_{ij} > \max(\pi_{ij}, \pi_{ji})$
> 0 otherwise.

semi-preference—$f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) = \pi_{ij} - \max(\pi_{ji}, \gamma_{ij})$.

row-sum—$f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) = \pi_{ij} - \pi_{ji}$. In this case, the π_{ij} of our model are interpreted as $\sum_r \pi'_{ir}/(\sum_r \pi'_{ir} + \sum_r \pi'_{jr})$, where $\pi'_{ir} = n_{ir}/k_{ir}$.

Note that other f functions may yield the same criteria.

Hence,

a ranking is a weak stochastic ranking if, whenever A_i precedes A_j in the ranking, $\pi_{ij} \geqq \pi_{ji}$,

a ranking is a preference ranking if, whenever A_i precedes A_j in the ranking, either $\pi_{ij} \geqq \max(\pi_{ji}, \gamma_{ij})$ or $\gamma_{ij} \geqq \max(\pi_{ij}, \pi_{ji})$,

a ranking is a semi-preference ranking if, whenever A_i precedes A_j in the ranking, $\pi_{ij} \geqq \max(\pi_{ji}, \gamma_{ij})$, and

a ranking is a row sum ranking if, whenever A_i precedes A_j in the ranking $\sum_r \pi'_{ir} \geqq \sum_r \pi'_{jr}$.

We emphasize that we are not arguing that only these criteria should be used, nor that they are equally appropriate. In some circumstances one or more may be inappropriate. For example, unless we are willing to assume that $D_{sp}(\pi^*)$ is asymmetric the semi-preference criterion would not be appropriate.

The corresponding $D_{ws}(\pi)$, $D_p(\pi)$, $D_{sp}(\pi)$, and $D_{rs}(\pi)$ are given by:

$$(5) \qquad \begin{aligned} &(A_i, A_j) \in D_{ws}(\pi) && \text{iff} \quad \pi_{ij} > \pi_{ji}, \\ &(A_i, A_j) \in D_p(\pi) && \text{iff} \quad \pi_{ij} > \max(\pi_{ji}, \gamma_{ij}), \\ &(A_i, A_j) \in D_{sp}(\pi) && \text{iff} \quad \max(\pi_{ij}, \gamma_{ij}) > \pi_{ji}, \qquad \text{and} \\ &(A_i, A_j) \in D_{rs}(\pi) && \text{iff} \quad \textstyle\sum_r \pi'_{ir} > \sum_r \pi'_{jr}. \end{aligned}$$

To indicate the derivation of the above statements we consider preference ranking. By definition $(A_i, A_j) \in D_p(\pi)$ iff $f(\pi_{ji}, \gamma_{ij}, \pi_{ij}) < 0$, and R is a preference ranking iff $f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) \geqq 0$. Consequently $f(\pi_{ij}, \gamma_{ij}, \pi_{ji}) \geqq 0$ iff $\pi_{ij} \geqq \max(\pi_{ji}, \gamma_{ij})$ or $\gamma_{ij} \geqq \max(\pi_{ij}, \pi_{ji})$. So $f(\pi_{ji}, \gamma_{ij}, \pi_{ij}) \geqq 0$ iff $\pi_{ji} \geqq \max(\pi_{ij}, \gamma_{ji})$ or $\gamma_{ji} \geqq \max(\pi_{ji}, \pi_{ij})$. Thus $f(\pi_{ji}, \gamma_{ij}, \pi_{ij}) < 0$ iff $\pi_{ji} < \max(\pi_{ij}, \gamma_{ji})$ and $\gamma_{ji} < \max(\pi_{ji}, \pi_{ij})$. That is, iff $\pi_{ij} > \max(\pi_{ji}, \gamma_{ij})$.

Under each of the above criteria, we now give the estimator π_R (defined in Section 3) of π^* used in selecting a ranking. It has been shown by De Cani (1969) that for weak stochastic ranking, π_R, is given by:

$$(6) \qquad \begin{aligned} (\pi^R_{ij}, \gamma^R_{ij}) &= (\hat{\pi}_{ij}, \hat{\gamma}_{ij}) && \text{if} \quad \pi_{ij} \geqq \pi_{ji} \\ &= \left(\frac{\hat{\pi}_{ij} + \hat{\pi}_{ji}}{2}, \hat{\gamma}_{ij}\right) && \text{otherwise,} \end{aligned}$$

when A_i precedes A_j in R.

Preference rankings select those rankings that also would be chosen under the criterion suggested by Singh and Thompson. From Theorem 10 of Singh and Thompson we know that π_R is:

$$(7)\qquad \begin{aligned}(\pi_{ij}^R, \gamma_{ij}^R) &= (\hat{\pi}_{ij}, \hat{\gamma}_{ij}) && \text{if } \hat{\pi}_{ij} \geqq \max(\hat{\pi}_{ji}, \hat{\gamma}_{ij}) \text{ or} \\ & && \hat{\gamma}_{ij} \geqq \max(\hat{\pi}_{ij}, \hat{\pi}_{ji}) \\ &= (\pi_{ij}^P, \gamma_{ij}^P) && \text{otherwise,}\end{aligned}$$

when A_i precedes A_j in R.

The $(\pi_{ij}^P, \gamma_{ij}^P, \pi_{ji}^P)$ are the pooled estimates obtained from $(\hat{\pi}_{ij}, \hat{\gamma}_{ij}, \hat{\pi}_{ji})$ by replacing each of its two largest components by one-half of $\{1 - \min(\hat{\pi}_{ij}, \hat{\gamma}_{ij}, \hat{\pi}_{ji})\}$.

For semi-preference rankings it is easy to show that π_R is

$$(8)\qquad \begin{aligned}(\pi_{ij}^R, \gamma_{ij}^R) &= (\hat{\pi}_{ij}, \hat{\gamma}_{ij}) && \text{if } \hat{\pi}_{ij} \geqq \max(\hat{\pi}_{ji}, \hat{\gamma}_{ij}) \\ &= (\pi_{ij}^P, \gamma_{ij}^P) && \text{if } \min(\hat{\pi}_{ji}, \hat{\gamma}_{ij}) \leqq \hat{\pi}_{ij} < \max(\hat{\pi}_{ji}, \hat{\gamma}_{ij}) \\ &= \left(\frac{\hat{\pi}_{ij} + \hat{\pi}_{ji}}{2}, \hat{\gamma}_{ij}\right) && \text{if } \hat{\pi}_{ij} < \hat{\gamma}_{ij} \leqq \hat{\pi}_{ji} \text{ and } \hat{\gamma}_{ij} \leqq \tfrac{1}{3} \\ &= (\tfrac{1}{3}, \tfrac{1}{3}) && \text{otherwise,}\end{aligned}$$

when A_i precedes A_j in R.

Finally, for row sum rankings, the row sum-optimal π_R is not known in closed form but can be found by an iterative procedure (Ford (1957)). However, any ranking in $H_{rs}(\hat{\pi})$ will be a ml row sum ranking since its π_R may always be taken as $\hat{\pi}$.

Note that for the row sum criteria used here, ties have been given zero weight. Ranking under the weak stochastic or row sum criteria ignores ties only in the sense that they do not enter into the definition of $D_{ws}(\pi)$ or $D_{rs}(\pi)$. If we wish them to have nonzero weights, then our definition of row sum ranking may be extended and may result in different rankings and preferences.

For each of the four f criteria we now give (π_{ij}, γ_{ij}) and their corresponding operations on $C_f \cup D_f(\hat{\pi})$. These yield only these π's which can induce possible ml preferences.

Under a weak stochastic criterion Singh and Thompson, Section 5, have shown:

$$(9)\qquad \begin{aligned}(\pi_{ij}, \gamma_{ij}) &= (\hat{\pi}_{ij}, \hat{\gamma}_{ij}) && \text{if line type between } A_i \text{ and } A_j \text{ is} \\ & && \text{not to be changed,} \\ &= \left(\frac{\hat{\pi}_{ij} + \hat{\pi}_{ji}}{2}, \hat{\gamma}_{ij}\right) && \text{otherwise.}\end{aligned}$$

This only allows some $(A_i, A_j) \in D_{ws}(\hat{\pi})$ not to appear in $D_{ws}(\pi)$ in which case they must appear in $C_{ws}(\pi)$. In other words, a directed line may be changed to an undirected line, but to no other type. If in the definition of directed path $i = j$, $r = n + 1$, and the A_{i_j} are all distinct for $i \leqq j \leqq n$, then we call the path an *elementary circuit*. Hence, as pointed out by Singh and Thompson, an elementary circuit in $C_{ws} \cup D_{ws}(\hat{\pi})$ will result in the ml preference being null.

This is because all $(A_i, A_j) \in D_{ws}(\hat{\pi})$ must then be removed, if a preference is to be obtained. This results in no directed paths and consequently a ml preference which is null.

Under a preference criterion, Theorem 10 of Singh and Thompson gives:

$$(10) \qquad \begin{aligned} (\pi_{ij}, \gamma_{ij}) &= (\hat{\pi}_{ij}, \hat{\gamma}_{ij}) && \text{if line type between } A_i \text{ and } A_j \text{ is not to be changed,} \\ &= (\pi_{ij}^P, \gamma_{ij}^P) && \text{otherwise.} \end{aligned}$$

This only allows some $(A_i, A_j) \in D_p(\hat{\pi})$ not to appear in $D_p(\pi)$ and some $(A_i, A_j) \in C_p(\hat{\pi})$ not to appear in $C_p(\pi)$. In other words, a directed line may be changed to a blank line and an undirected line may be changed to a blank line. Although π allows only these changes to be made, they are all that are needed. Theorem 13 of Singh and Thompson shows that when $C_f \cup D_f(\hat{\pi})$ is a complete relation, then if a unique ranking is consistent with a ml preference that ranking is a ml preference ranking. This result can be generalized to f ranking criteria. This criterion selects those preferences that also would be chosen under the criterion suggested by Singh and Thompson.

Under a semi-preference criterion:

$$(11) \qquad (\pi_{ij}, \gamma_{ij}) = (\hat{\pi}_{ij}, \hat{\gamma}_{ij}) \qquad \text{if line type between } A_i \text{ and } A_j \text{ is not to be changed.}$$

An $A_i \rightarrow A_j$ can be changed only to $A_i - A_j$ and this is accomplished by:

$$\begin{aligned} (\pi_{ij}, \gamma_{ij}) &= (\pi_{ij}^P, \gamma_{ij}^P) && \text{if } \pi_{ij} > \pi_{ji} \geqq \gamma_{ij} \\ &= \left(\frac{\hat{\pi}_{ij} + \hat{\pi}_{ji}}{2}, \hat{\gamma}_{ij}\right) && \text{if } \hat{\pi}_{ij} > \hat{\gamma}_{ij} \geqq \hat{\pi}_{ji} \text{ and } \hat{\gamma}_{ij} \leqq \tfrac{1}{3} \\ &= (\tfrac{1}{3}, \tfrac{1}{3}) && \text{if } \hat{\pi}_{ij} > \hat{\gamma}_{ij} \geqq \hat{\pi}_{ji} \text{ and } \hat{\gamma}_{ij} > \tfrac{1}{3} \text{ or if } \hat{\pi}_{ij} = \hat{\gamma}_{ij} > \hat{\pi}_{ji}\,. \end{aligned}$$

An $A_i \rightleftarrows A_j$ can be changed to,
$A_i \rightarrow A_j$ by:

$$(\pi_{ij}, \gamma_{ij}) = (\pi_{ij}^P, \gamma_{ij}^P) \qquad \text{if } \hat{\gamma}_{ij} > \hat{\pi}_{ij} \geqq \hat{\pi}_{ji}$$

or to $A_i - A_j$ by:

$$(\pi_{ij}, \gamma_{ij}) = (\tfrac{1}{3}, \tfrac{1}{3})\,,$$

An $A_i - A_j$ cannot be changed if $\hat{\pi}_{ij} = \hat{\pi}_{ji} = \hat{\gamma}_{ij}$ but can be changed to,
$A_i \rightarrow A_j$ by:

$$(\pi_{ij}, \gamma_{ij}) = \left(\pi_{ij}\,, \frac{\hat{\pi}_{ji} + \hat{\gamma}_{ji}}{2}\right)$$

or to $A_j \rightarrow A_i$ by:

$$(\pi_{ij}, \gamma_{ij}) = \left(\frac{\hat{\pi}_{ij} + \hat{\gamma}_{ij}}{2}\,, \frac{\hat{\pi}_{ij} + \hat{\gamma}_{ij}}{2}\right) \qquad \text{whenever } \hat{\pi}_{ij} = \hat{\pi}_{ji} > \hat{\gamma}_{ij}$$

It is not difficult to see that if an elementary path exists in $C_{sp} \cup D_{sp}(\hat{\pi})$ then the ml preference will be null, similar to the situation for the weak stochastic criterion.

Under a row sum criterion:

(12) $(\pi_{ij}, \gamma_{ij}) = (\hat{\pi}_{ij}, \hat{\gamma}_{ij})$ if line type between A_i and A_j is not to be changed,

$= \left(\frac{\hat{\pi}_{ij} + \hat{\pi}_{ji}}{2}, \hat{\gamma}_{ij}\right)$ otherwise.

We conclude by applying each of the four f criteria to three numerical examples.

Example one has the following paired comparison data:

$$n_{12} = 2\,, \quad t_{12} = 3\,, \quad n_{21} = 1$$
$$n_{13} = 0\,, \quad t_{13} = 3\,, \quad n_{31} = 3$$
$$n_{23} = 2\,, \quad t_{23} = 2\,, \quad n_{32} = 2\,.$$

The relations $C_f \cup D_f(\hat{\pi})$ and $D_f(\hat{\pi})$, from which the estimators may be constructed, are shown graphically in Figure 1 for each of the four criteria.

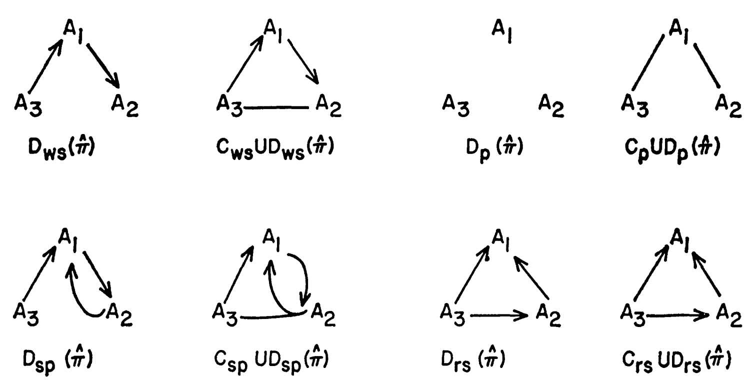

FIG. 1.

The weak stochastic-optimal $\pi_R = (\frac{1}{3}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ and the ml weak stochastic ranking is $A_3 A_1 A_2$. As expected, we see that $T_{ws}(\pi)$ is null for π weak stochastic p-optimal since $C_{ws} \cup D_{ws}(\hat{\pi})$ has at least one elementary directed path (A_3 to A_3 for example).

The preference-optimal $\pi_R = (\frac{5}{12}, \frac{5}{12}, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{3})$, and all six rankings are ml preference rankings (that is all rankings are indistinguishable). The ml preference $T_p(\pi)$ is null. The semi-preference-optimal $\pi_R = (\frac{5}{12}, \frac{5}{12}, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ and the ml semi-preference ranking is $A_3 A_1 A_2$. The ml preference $T_{sp}(\pi)$ is null as we expected. For row sum ranking, $\pi_R = (\frac{1}{3}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{3})$, and the ml row sum ranking is $A_3 A_2 A_1$. This unique ranking is consistent with the ml row sum preference.

Example two, from Singh and Thompson, has the following paired comparison data;

$$\begin{aligned} n_{12} &= 2\,, & t_{12} &= 1\,, & n_{21} &= 3 \\ n_{13} &= 4\,, & t_{13} &= 1\,, & n_{31} &= 1 \\ n_{14} &= 0\,, & t_{14} &= 4\,, & n_{41} &= 2 \\ n_{23} &= 1\,, & t_{23} &= 3\,, & n_{32} &= 2 \\ n_{24} &= 1\,, & t_{24} &= 2\,, & n_{42} &= 3 \\ n_{34} &= 4\,, & t_{34} &= 0\,, & n_{43} &= 2\,. \end{aligned}$$

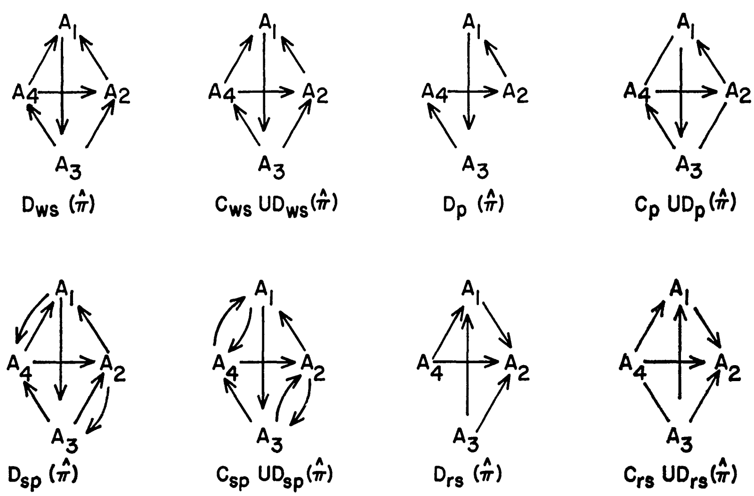

FIG. 2.

Figure 2 graphically presents the relations $D_f(\hat{\pi})$ and $C_f \cup D_f(\hat{\pi})$ for this example under each of the four criteria. De Cani has investigated this example under the weak stochastic ranking criterion and found the unique ml weak stochastic ranking to be $A_4A_1A_3A_2$. We note that the ml preference for the weak stochastic criterion must be null since an elementary directed path exists from A_4 to A_4, namely $A_4A_2A_1A_3A_4$. For this same data, Singh and Thompson found the ml preference rankings to be $A_1A_3A_4A_2$ and $A_2A_1A_3A_4$, and the ml preference, under the preference criterion, to be $\{(A_1, A_2), (A_1, A_3), (A_4, A_2), (A_4, A_3)\}$. Under the semi-preference ranking criterion, the ml semi-preference ranking is $A_4A_1A_3A_2$ and the ml preference is null. Lastly, the ml row sum rankings are $A_4A_3A_1A_2$ and $A_3A_4A_1A_2$. The ml preference under the row sum criterion is $\{(A_3, A_1), (A_3, A_2), (A_4, A_1), (A_4, A_2), (A_1, A_2)\}$.

In his examination of this example, De Cani (1969) was apparently surprised to find that Singh and Thompson discarded the $A_4A_1A_3A_2$ ranking, when as he says, "Singh and Thompson obtained this ranking as the ranking with the least uncertainty." It should be clear from the development presented in this paper that $A_4A_1A_3A_2$ is not properly referred to as "the ranking with the least uncertainty." Rather, it is the preference associated with this ranking that has the least uncertainty (greatest likelihood), and $A_4 A_1 A_3 A_2$ is only one of the four rankings consistent with this preference.

It also should be evident that different ranking criteria may produce vastly different rankings; a result that has apparently been a source of some confusion tn the literature (De Cani (1969)). In particular, although the (1, 4) component of the f-optimal π is $(\pi_{14}, \gamma_{14}, \pi_{41}) = (0, \frac{1}{2}, \frac{1}{2})$, and thus A_4 has a zero estimated probability of losing to A_1, both of the preference rankings placed A_1 above A_4 while all other criteria placed A_4 above A_1. In short, with respect to the four criteria, $A_4 A_1 A_3 A_2$, $A_1 A_3 A_4 A_2$ and $A_2 A_1 A_3 A_4$, $A_4 A_2 A_1 A_3$, and $A_4 A_3 A_1 A_2$ and $A_3 A_4 A_1 A_2$, each have zero total inconsistencies, respectively.

It should be noted that whenever $\hat{\gamma}_{ij} = \min(\hat{\pi}_{ij}, \hat{\gamma}_{ij}, \hat{\pi}_{ji})$ for all i, j, then the weak stochastic, preference, and semi-preference ranking criteria all yield the same solution. In particular, if no ties are present and all $k_{ij} = 1$, then all these criteria simply become the minimum total inconsistency criterion.

As the third example, consider the circular triad, David (1971); $k_{12} = k_{23} = k_{31} = n_{12} = n_{23} = n_{31} = 1$. All but the row sum criterion reduce to the minimum total inconsistency criterion. Under this criterion there are three ml preferences: $\{(A_1, A_2), (A_1, A_3), (A_2, A_3)\}$, $\{(A_2, A_1), (A_2, A_3), (A_3, A_1)\}$, and $\{(A_3, A_1), (A_3, A_2), (A_1, A_2)\}$. There are also three ml rankings: $A_1 A_2 A_3$, $A_2 A_3 A_1$, and $A_3 A_1 A_2$. Each of these rankings is consistent with exactly one of the three ml preferences. However, under the row sum criterion the ml preference is null and each of the six possible rankings is a ml row sum ranking.

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