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ON THE OPTIMALITY CRITERION IN COMPOUND DECISION PROBLEMS[1]

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This paper shows the asymptotic equivalence of the classical and symmetric optimality criteria for the finite state, arbitrary action compound decision problem.

1. Introduction and notation. We consider a compound decision scheme with its component scheme defined as

$$((\mathscr{X}, \mathscr{B}), \mathscr{P}, X, (\mathscr{A}, \sigma_{\mathscr{A}}), L) \tag{1.1}$$

where: (i) $\mathscr{X}$ is a set and $\mathscr{B}$ is a σ-field of subsets of $\mathscr{X}$; (ii) $\mathscr{P} \equiv \{P_\theta \mid \theta \in \Theta\}$ is a family of probability measures P_θ on $(\mathscr{X}, \mathscr{B})$. The set Θ is called the *parameter* or *state space*; (iii) X is an $\mathscr{X}$-valued random variable which is distributed according to P_θ for some $\theta \in \Theta$; (iv) $\mathscr{A}$ is a set called the *action space* and $\sigma_{\mathscr{A}}$ is a σ-field of subsets of $\mathscr{A}$; (v) $L(x, \theta, a)$, the *loss function*, is a mapping $L: \mathscr{X} \times \Theta \times \mathscr{A} \to R^+$ (nonnegative reals) such that $L(\cdot, \theta, \cdot)$ is a $\mathscr{B} \times \sigma_{\mathscr{A}}$-measurable function for each $\theta \in \Theta$. Then the *compound decision scheme of order* N is denoted

$$((\mathscr{X}^N, \mathscr{B}^N), \mathscr{P}_N, \mathbf{X}_N, (\mathscr{A}, \sigma_{\mathscr{A}}), L) \tag{1.2}$$

where N is a positive integer and (i) $\mathscr{X}^N$ is the N-fold Cartesian product of the space $\mathscr{X}$ and $\mathscr{B}^N$ is the product σ-field in $\mathscr{X}^N$ generated by the σ-field $\mathscr{B}$ in $\mathscr{X}$; (ii) $\mathscr{P}_N \equiv \{P_{\boldsymbol{\theta}_N} \mid \boldsymbol{\theta}_N \in \Theta^N\}$ where Θ^N is the N-fold Cartesian product of Θ, $\boldsymbol{\theta}_N = (\theta_i)_{i=1}^N$, and $P_{\boldsymbol{\theta}_N} \equiv P_{\theta_1} \times \cdots \times P_{\theta_N}$; (iii) $\mathbf{X}_N \equiv (X_1, \cdots, X_N)$ is an $\mathscr{X}^N$-valued random variable which is distributed according to $P_{\boldsymbol{\theta}_N}$ for some $\boldsymbol{\theta}_N \in \Theta^N$; and (iv), (v) $(\mathscr{A}, \sigma_{\mathscr{A}})$ and L are defined as in (1.1).

A *compound decision rule* is an N-dimensional vector function

$$T_N(\mathbf{x}_N) \equiv (T_1(A \mid \mathbf{x}_N), \cdots, T_N(A \mid \mathbf{x}_N)) \tag{1.3}$$

where for each k, $1 \leqq k \leqq N$, $T_k: \sigma_{\mathscr{A}} \times \mathscr{X}^N \to [0, 1]$ is a mapping such that for each $A \in \sigma_{\mathscr{A}}$, $T_k(A \mid \cdot)$ is a measurable function with respect to the usual Borel field on $[0, 1]$ and for each $\mathbf{x}_N \in \mathscr{X}^N$, $T_k(\cdot \mid \mathbf{x}_N)$ is a probability measure on $(\mathscr{A}, \sigma_{\mathscr{A}})$.

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Associated with a compound decision rule is the *average risk function*

$$\bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N) \equiv N^{-1} \sum_{k=1}^{N} r_k(\boldsymbol{\theta}_N, \mathbf{T}_N) \tag{1.4}$$

where $r_k(\cdot, \mathbf{T}_N) : \Theta^N \to R^+$ is defined by

$$r_k(\boldsymbol{\theta}_N, \mathbf{T}_N) \equiv \int_{\mathscr{X}^N} \int_{\mathscr{A}} L(x_k, \theta_k, a)\, dT_k(a \mid \mathbf{x}_N)\, dP_{\boldsymbol{\theta}_N}(\mathbf{x}_N) .$$

One important type of compound decision rule is a *simple compound decision rule* (sometimes called a simple symmetric rule) with $T_k(A \mid \mathbf{x}_N) = T(A \mid x_k)$, $k = 1, \cdots, N$. We denote a simple compound decision rule by $\mathbf{T}_N^*(\mathbf{x}_N) \equiv (T(A \mid x_k))_{k=1}^N$.

Given a compound decision scheme (1.2) and a specified compound decision rule (1.3), one asks if the rule is optimal in some sense. The most frequently used optimality criterion is the *classical optimality criterion*, introduced by Robbins (1951). It is

$$B^*(\boldsymbol{\theta}_N, \mathbf{T}_N) \equiv \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N) - r^*(G_{\boldsymbol{\theta}_N}) \to 0 \quad \text{as} \quad N \to \infty \quad \text{uniformly for all} \quad \boldsymbol{\theta} \in \Theta^\infty$$

where Θ^∞ is the countable Cartesian product of Θ, $\boldsymbol{\theta}_N$ is an initial N-section of $\boldsymbol{\theta}$, $G_{\boldsymbol{\theta}_N}$ is a probability measure on Θ which assigns to each $\theta \in \Theta$ mass $1/N$ for each occurrence of θ as a coordinate of the vector $\boldsymbol{\theta}_N$, and $r^*(G_{\boldsymbol{\theta}_N})$ is the Bayes envelope with respect to $G_{\boldsymbol{\theta}_N}$, i.e.,

$$r^*(G_{\boldsymbol{\theta}_N}) \equiv \inf_{T_1} N^{-1} \sum_{k=1}^{N} r(\theta_k, \mathbf{T}_1) = \inf_{\mathbf{T}_N^*} \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N^*) .$$

Note that $r^*(G_{\boldsymbol{\theta}_N})$ depends only on simple compound decision rules.

Another optimality criterion has been formulated in terms of compound decision rules $\mathbf{T}_N$ with components T_k, $1 \leqq k \leqq N$, which may treat the kth observation x_k in any manner, but which treat the other $N - 1$ observations in a symmetric manner. To formulate this notion of symmetry mathematically, define $H_N \equiv \{\pi \mid \pi \text{ is a permutation of the integers } (1, \cdots, N)\}$. For a vector $\mathbf{Y}_N = (Y_1, \cdots, Y_N)$ denote by $\pi\mathbf{Y}_N$ the vector $\pi\mathbf{Y}_N = (Y_{\pi(1)}, \cdots, Y_{\pi(N)})$. In this notation, a compound decision rule (1.3) is a *symmetric compound decision rule* (sometimes called an *invariant* or *equivariant* compound decision rule) if

$$\pi^{-1}\mathbf{T}_N(\pi\mathbf{x}_N) = \mathbf{T}_N(\mathbf{x}_N) \tag{1.5}$$

or equivalently, if $\pi\mathbf{T}_N(\mathbf{x}_N) = \mathbf{T}_N(\pi\mathbf{x}_N)$ for all permutations $\pi \in H_N$, all $\mathbf{x}_N \in \mathscr{X}^N$, and all N.

Let S denote the collection of all symmetric compound decision rules, i.e. $S \equiv \{\mathbf{T}_N \mid \pi^{-1}\mathbf{T}_N(\pi\mathbf{x}_N) = \mathbf{T}_N(\mathbf{x}_N) \text{ for all } \pi \in H_N, \text{ all } \mathbf{x}_N \in \mathscr{X}^N, \text{ all } N\}$. Then the *symmetry standard* is

$$B(\boldsymbol{\theta}_N, \mathbf{T}_N) \equiv \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N) - \inf_{\mathbf{T}_N' \in S} \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N')$$

where $\inf_{\mathbf{T}_N' \in S} \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N')$ is called the *symmetry envelope*.

Several authors have studied how the expression

$$B(\boldsymbol{\theta}_N, \mathbf{T}_N) - B^*(\boldsymbol{\theta}_N, \mathbf{T}_N) = r^*(G_{\boldsymbol{\theta}_N}) - \inf_{\mathbf{T}_N' \in S} \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N')$$

behaves as $N \to \infty$ for each $\boldsymbol{\theta} \in \Theta^\infty$. It has been found that even though

$$\inf_{\mathbf{T}_N' \in S} \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N') \leqq \inf_{\mathbf{T}_N^*} \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N^*) = r^*(G_{\boldsymbol{\theta}_N}) \qquad \text{for all} \quad N,$$

these two functions are asymptotically equal in the following cases.

Define the $r \times s$ *compound decision scheme* to be one with component schemes of the form (1.1) with $\Theta \equiv \{1, \cdots, r\}$, $\mathscr{A} \equiv \{1, \cdots, s\}$, $\sigma_{\mathscr{A}}$ the power set of $\mathscr{A}$, and $L(x, \theta, a)$ a bounded, nonnegative, real function. In the special case of the 2×2 compound decision scheme with zero-one loss function, Hannan and Robbins (1955) showed that for every $\varepsilon > 0$ there exists an integer $N^*(\varepsilon)$ such that

$$r^*(G_{\boldsymbol{\theta}_N}) - \varepsilon \leqq \inf_{\mathbf{T}_N' \in S} \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N') \leqq r^*(G_{\boldsymbol{\theta}_N})$$

for all $N \geqq N^*(\varepsilon)$ uniformly in $\boldsymbol{\theta} \in \Theta^\infty$. This result was partially extended by Horn (1968) to the $r \times s$ compound decision scheme. It was further extended by Hannan and Huang (1972a) to the arbitrary action, finite state compound decision scheme. In this note we establish an alternative to Theorem 1 of Hannan and Huang (1972a) using a simpler measure theoretic lemma.

2. The asymptotic equivalence of the classical and symmetry standards. We consider the compound decision scheme (1.2) with component scheme (1.1) with $\Theta \equiv \{1, \cdots, r\}$. From definition (1.5) it follows that $\mathbf{T}_N \in S$ if and only if there exists a conditional probability measure t on $\sigma_{\mathscr{A}} \times \mathscr{X} \times \mathscr{X}^{N-1}$ which is symmetric on $\mathscr{X}^{N-1}$ and is such that for each $k = 1, \cdots, N$

$$T_k(A \mid \mathbf{x}_N) = t(A \mid x_k, \mathbf{x}_N^k) \tag{2.1}$$

where $\mathbf{x}_N^k \equiv (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_N)$, $1 \leqq k \leqq N$.

The average risk function (1.4) may be written as

$$\begin{aligned}\bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N) &= N^{-1} \textstyle\sum_{k=1}^N \int_{\mathscr{X}^N} \int_{\mathscr{A}} L(x_k, \theta_k, a)\, dT_k(a \mid \mathbf{x}_N)\, dP_{\boldsymbol{\theta}_N}(\mathbf{x}_N) \\ &= N^{-1} \textstyle\sum_{i=1}^r \sum_{k \mid P_{\theta_k} = P_i} \int_{\mathscr{X}} \int_{\mathscr{X}^{N-1}} \int_{\mathscr{A}} L(x_k, i, a)\, dT_k(a \mid \mathbf{x}_N)\, dP_{\boldsymbol{\theta}_N^k}(\mathbf{x}_N^k) \\ &\qquad \times dP_i(x_k)\end{aligned}$$

where $P_{\boldsymbol{\theta}_N^k} \equiv P_{\theta_1} \times \cdots \times P_{\theta_{k-1}} \times P_{\theta_{k+1}} \times \cdots \times P_{\theta_N}$, $k = 1, \cdots, N$.
For a given $\boldsymbol{\theta}_N$ let $N_i \equiv \# \{k \mid P_{\theta_k} = P_i, 1 \leqq k \leqq N\}$ for $i = 1, \cdots, r$ and

$$\begin{aligned} N_{ji} &\equiv N_j - 1 \qquad \text{if} \quad j = i, \\ &\equiv N_j \qquad\qquad \text{if} \quad j \neq i. \end{aligned}$$

Using the above, the average risk function of a symmetric compound decision rule may be written as

$$\begin{aligned}\bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N) & \\ &= N^{-1} \textstyle\sum_{i=1}^r N_i \int_{\mathscr{X}} \int_{\mathscr{A}} \int_{\mathscr{X}^{N-1}} L(x_1, i, a)\, dt(a \mid x_1, \mathbf{x}_N^1) \\ &\qquad \times \textstyle\prod_{j=1}^r dP_j^{N_{ji}}(\mathbf{x}_N^1)\, dP_i(x_1)\,.\end{aligned} \tag{2.2}$$

Since the integrand is symmetric in $\mathbf{x}_N^1$, the order of the P_j in $\prod_{j=1}^r dP_j^{N_{ji}}$ is inessential.

The essence of the proof of our theorem is contained in the following measure theoretic lemma due to Horn and Schach (1970). A product probability measure $\mu = \prod \mu_i$ is said to be *recurring* if for each $i = 1, 2, \cdots$ there is some $j > i$ such that $\mu_j = \mu_i$, i.e., each factor of μ occurs infinitely often. We denote by U_N the σ-field of sets in $\mathscr{B}^\infty$ which are invariant under all permutations of the first N coordinates.

LEMMA. *Let μ be a recurring product probability measure on $(\mathscr{X}^\infty, \mathscr{B}^\infty)$. If probability measures λ and ν are absolutely continuous with respect to μ, then*

$$\sup_{B \in U_N} |\lambda(B) - \nu(B)| \to 0 \qquad as \quad N \to \infty .$$

The result which we shall employ is an immediate corollary of the lemma. For any $M > 0$ we define $F_N(M) \equiv \{f \mid f$ is a measurable function on $\mathscr{X}^\infty$, $0 \leqq f \leqq M$, and f is symmetric on $\mathscr{X}^N\}$.

COROLLARY 1. *Under the same hypotheses on λ and ν as in the lemma, for each $N = 1, 2, \cdots$ and any $M > 0$ we have*

$$|\int_{\mathscr{X}^\infty} f_N \, d\lambda - \int_{\mathscr{X}^\infty} f_N \, d\nu| \leqq M \sup_{B \in U_N} |\lambda(B) - \nu(B)| \tag{2.3}$$

for all $f_N \in F_N(M)$, and hence

$$|\int_{\mathscr{X}^\infty} f_N \, d\lambda - \int_{\mathscr{X}^\infty} f_N \, d\nu| \to 0 \qquad as \quad N \to \infty \tag{2.4}$$

uniformly for all $f_N \in F_N(M)$.

PROOF. It is clear from the lemma that (2.4) follows from (2.3), so this is what we must prove. But it suffices to prove (2.3) for simple functions in $F_N(M)$, so let $f_N = \sum_{i=1}^r c_i 1_{B_i}$, where $0 \leqq c_i \leqq M$ and the sets $B_i \in U_N$ are disjoint. Define $J_+ \equiv \{i \mid 1 \leqq i \leqq r$ and $[\lambda(B_i) - \nu(B_i)] \geqq 0\}$ and calculate

$$\begin{aligned}\int f_N \, d\lambda - \int f_N \, d\nu &= \sum_{i=1}^r c_i[\lambda(B_i) - \nu(B_i)] \leqq \sum_{i \in J_+} c_i[\lambda(B_i) - \nu(B_i)] \\ &\leqq M \sum_{i \in J_+} [\lambda(B_i) - \nu(B_i)] = M[\lambda(\bigcup_{i \in J_+} B_i) - \nu(\bigcup_{i \in J_+} B_i)] \\ &\leqq M \sup_{B \in U_N} [\lambda(B) - \nu(B)] .\end{aligned}$$

The calculation for the lower bound is similar.

COROLLARY 2. *Let λ, ν, and $F_N(M)$ be as defined in Corollary 1, and define sequences of product probability measures λ_N and ν_N, $N = 1, 2, 3, \cdots$ as follows: for each N, λ_N and ν_N are formed from λ and v respectively by (separate) permutations of the first N factors only. Then*

$$|\int_{\mathscr{X}^\infty} f_N \, d\lambda_N - \int_{\mathscr{X}^\infty} f_N \, d\nu_N| \to 0 \qquad as \quad N \to \infty$$

uniformly for all $f_N \in F_N(M)$.

PROOF. For each $f_N \in F_N(M)$, $\int_{\mathscr{X}^\infty} f_N \, d\lambda_N = \int_{\mathscr{X}^\infty} f_N \, d\lambda$ and likewise for ν.

THEOREM. *Let a compound decision scheme be made up of N independent component schemes of the type (1.1) with $\Theta \equiv \{1, \cdots, r\}$. Assume that:* (i) *the probability measures P_θ, $\theta \in \Theta$, are mutually absolutely continuous and distinct, and*

(ii) *the loss function* $L(x, \theta, a)$ *is bounded. Then for each* $\varepsilon > 0$ *and each* $\boldsymbol{\theta} \in \Theta^\infty$ *there exists an integer* $N(\varepsilon, \boldsymbol{\theta})$ *such that for all* $N > N(\varepsilon, \boldsymbol{\theta})$,

$$r^*(G_{\theta_N}) - \varepsilon \leqq \inf_{T_N' \in S} \bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N') \leqq r^*(G_{\theta_N}) . \tag{2.5}$$

PROOF. We shall prove the theorem by using Corollary 2 to construct for each $\boldsymbol{\theta} \in \Theta^\infty$ and for each symmetric rule $\mathbf{T}_N$ an associated simple rule $\mathbf{T}_N^*$ whose risk at $\boldsymbol{\theta}_N$ is close to the risk of $\mathbf{T}_N$ at $\boldsymbol{\theta}_N$. Let N and $\mathbf{T}_N \in S$ be given and let t be the associated conditional probability measure defined in (2.1).

Let $\boldsymbol{\theta} \in \Theta^\infty$ be given and let K be any element of Θ which occurs infinitely often in $\boldsymbol{\theta}$. Let R be the smallest positive integer such that $\boldsymbol{\theta}_R' \equiv (\theta_{R+1}, \theta_{R+2}, \cdots)$, the R-tail of the $\boldsymbol{\theta}$ sequence, is recurring; i.e., each element in $\boldsymbol{\theta}_R'$ occurs infinitely often. Take $\mu \equiv P_K{}^R \times P_{\theta_R'}$; then the measure μ is recurring.

For any integer l define $\boldsymbol{\theta}^l \equiv (\theta_1, \cdots, \theta_{l-1}, \theta_{l+1}, \theta_{l+2}, \cdots,)$. Let k and k^* be integers in $\{1, \cdots, N\}$ and suppose $\theta_k = i$ and $\theta_{k^*} = J$ with $i, J \in \Theta$. Define $\lambda \equiv P_{\theta k}$ and $\nu \equiv P_{\theta k^*}$. Then λ and ν are absolutely continuous with respect to μ. Let $\lambda_N \equiv \prod_{j=1}^r P_j{}^{N_{ji}} \times P_{\theta_{N-1}'}$ and $\nu_N \equiv \prod_{j=1}^r P_j{}^{N_{jJ}} \times P_{\theta_{N-1}'}$. Observe that λ_N and ν_N are formed from λ and ν, respectively, by permutations of the first $N-1$ factors. Hence Corollary 2 guarantees that given $\varepsilon > 0$ there exists $\mathscr{N}_{ij}(\varepsilon, \boldsymbol{\theta})$ such that for all $N > \mathscr{N}_{iJ}(\varepsilon, \boldsymbol{\theta})$

$$\int_{\mathscr{A}} \int_{\mathscr{X}^{N-1}} L(x_1, i, a)\, dt(a \mid x_1, \mathbf{x}_N{}^1) \prod_{j=1}^r dP_j{}^{N_{ji}}(\mathbf{x}_N{}^1) \geqq \int_{\mathscr{A}} \int_{\mathscr{X}^{N-1}} L(x_1, i, a)\, dt(a \mid x_1, \mathbf{x}_N{}^1) \prod_{j=1}^r dP_j{}^{N_{jJ}}(\mathbf{x}_N{}^1) - \varepsilon .$$

Notice that the $\mathscr{N}$ above depends on $\boldsymbol{\theta}$ since μ depends on $\boldsymbol{\theta}$; it also depends on i and J, the values of the components in $\boldsymbol{\theta}_N$ that were omitted, but not on the indices k and k^*. This is because we integrate λ and ν against functions symmetric in the first $N-1$ components and hence we may rearrange the order of the first $N-1$ measures in λ and ν without changing the value of the integrals. If we were not restricted to symmetric functions, then $\mathscr{N}$ could depend on k and k^* and $\max_{k,k^*} \mathscr{N}_{kk^*}$ could approach infinity as $N \to \infty$. Also note that the precise definition of J is inessential; all we need is some fixed element of Θ that occurs in $\boldsymbol{\theta}_N$ and does not depend on i. We can, for example, let $J = \theta_1$ in $\boldsymbol{\theta}_N$.

Therefore the average risk function (2.2) of a symmetric compound decision rule has the lower bound

$$\bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N) \geqq N^{-1} \sum_{i=1}^r N_i \int_{\mathscr{X}} dP_i(x_1) \int_{\mathscr{A}} \int_{\mathscr{X}^{N-1}} L(x_1, i, a) \times dt(a \mid x_1, \mathbf{x}_N{}^1) \prod_{j=1}^r dP_j{}^{N_{jJ}}(\mathbf{x}_N{}^1) - \varepsilon \quad \text{for} \quad N > \max_{i,j \in \Theta} \mathscr{N}_{ij}(\varepsilon, \boldsymbol{\theta}) . \tag{2.6}$$

Note that the inner measure on the right-hand side in (2.6) depends on x_1 but not on i. Also for each conditional probability measure t defined by (2.1), for each fixed $J \in \Theta$, and for all $N = 1, 2, 3, \cdots$ the set function $\int_{\mathscr{X}^{N-1}} t(A \mid x_1, \mathbf{x}_N{}^1) \times \prod_{j=1}^r dP_j{}^{N_{jJ}}(\mathbf{x}_N{}^1)$ is a conditional probability measure given (x_1). So we may rewrite (2.6) as

$$\bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N) \geqq N^{-1} \sum_{i=1}^r N_i \int_{\mathscr{X}} dP_i(x_1) \int_{\mathscr{A}} L(x_1, i, a)\, dt^*(a \mid x_1) - \varepsilon . \tag{2.7}$$

Taking the infimum with respect to all simple rules on the right-hand side in (2.7) gives

$$\bar{r}(\boldsymbol{\theta}_N, \mathbf{T}_N) \geqq r^*(G_{\boldsymbol{\theta}_N}) - \varepsilon . \tag{2.8}$$

The conclusion (2.5) of the theorem follows since (2.8) holds for all $\mathbf{T}_N \in S$.

A comparison of our theorem with Theorem 1 of Hannan and Huang (1972a) shows that we have a stronger hypothesis, i.e. assumption (i) in our theorem is stronger than their assumption of pairwise non-orthogonality of $\mathscr{P}$. We do not obtain the full strength of their conclusions; for example, they found that (2.5) holds uniformly for all $\boldsymbol{\theta} \in \Theta^\infty$ and they found a rate of convergence. The major difference lies in the respective measure theoretic lemmata used and compared in the addendum to Hannan and Huang (1972b). We have a much shorter lemma, but it obtains a weaker result.

REFERENCES

[1] HANNAN, J. F. and HUANG, J. S. (1972a). Equivariant procedures in the compound decision problem with finite state component problem. *Ann. Math. Statist.* **43** 102–112.

[2] HANNAN, J. F. and HUANG, J. S. (1972b). A stability of symmetrization of product measures with few distinct factors. *Ann. Math. Statist.* **43** 308–319.

[3] HANNAN, J. F. and ROBBINS, H. (1955). Asymptotic solutions of the compound decision problem for two completely specified distributions. *Ann. Math. Statist.* **26** 37–51.

[4] HORN, S. D. (1968). The optimality criterion for compound decision problems. Technical Report No. 10, Department of Statistics, Stanford Univ.

[5] HORN, S. D. and SCHACH, S. (1970). An extension of the Hewitt-Savage zero-one law. *Ann. Math. Statist.* **41** 2130–2131.

[6] ROBBINS, H. (1951). Asymptotically subminimax solutions of compound statistical decision problems. *Proc. Second Berkeley Symp. Math. Statist. Prob.* Univ. of California Press 131–148.

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