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EXACT ROBUSTNESS STUDIES OF TESTS OF TWO MULTIVARIATE HYPOTHESES BASED ON FOUR CRITERIA AND THEIR DISTRIBUTION PROBLEMS UNDER VIOLATIONS[1]

BY K. C. S. PILLAI[2] AND SUDJANA[3]

Purdue University

This paper deals with robustness studies of tests of two hypotheses (A) $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ in $N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$, $i = 1, 2$, and (B) $\boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_l$ in $N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma})$, $i = 1, 2, \cdots, l$, $\boldsymbol{\Sigma}$ unknown, based on four test criteria (a) Hotelling's trace, (b) Pillai's trace, (c) Wilks' Λ and (d) Roy's largest root. The robustness for (A) is against non-normality and for (B) against unequal covariance matrices and is studied in the exact case, unlike the results obtained earlier. In this connection, Pillai's density of the latent roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ under violations is used to derive the distributions or the moments of the criteria. Numerical studies of the tests of the two hypotheses based on the four criteria are made for the two-roots case.

1. Introduction. Consider a $p \times n_1$ matrix variate $\mathbf{U}$ and $p \times n_2$ matrix variate $\mathbf{V}$, $n_1, n_2 \geqq p$, where the columns are all independently normally distributed with covariance matrix $\boldsymbol{\Sigma}_1$ for $\mathbf{U}$ and $\boldsymbol{\Sigma}_2$ for $\mathbf{V}$ while $E(\mathbf{U}) = \mathbf{M}$ and $E(\mathbf{V}) = \mathbf{0}$. It is well known that the density of $\mathbf{S}_1 = \mathbf{U}\mathbf{U}'$ is a non-central Wishart $W(p, n_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Omega})$, $\boldsymbol{\Omega} = \frac{1}{2}\mathbf{M}\mathbf{M}'\boldsymbol{\Sigma}_1^{-1}$ and $\mathbf{S}_2 = VV'$ is a central Wishart $W(p, n_2, \boldsymbol{\Sigma}_2, \mathbf{0})$, where $\mathbf{S}_1$ and $\mathbf{S}_2$ are independently distributed. Under the assumption that $\boldsymbol{\Sigma}_1^{\frac{1}{2}}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}_1^{\frac{1}{2}}$ is "random" (see Section 2 for the definitions of the term), Pillai [10, 11] has obtained the joint density of the latent roots $r_1, \cdots, r_p$ of $\mathbf{S}_1\mathbf{S}_2^{-1}$ in the form

$$(1.1) \qquad C_p e^{-\mathrm{tr}\,\boldsymbol{\Omega}} |\boldsymbol{\Lambda}|^{-\frac{1}{2}n_1} |\mathbf{R}|^m |\mathbf{I} + \lambda\mathbf{R}|^{-\nu} \prod_{i>j} (r_i - r_j) \times \sum_{k=0}^{\infty} \sum_{\kappa} \{(\nu)_\kappa / k!\} C_\kappa\{\lambda\mathbf{R}(\mathbf{I} + \lambda\mathbf{R})^{-1}\} F_p\,,$$

where F_p is defined by

$$(1.2) \qquad F_p = \sum_{d=0}^{k} \sum_{\delta} [\{a_{\kappa,\delta} C_\delta(-\lambda^{-1}\boldsymbol{\Lambda}^{-1}) L_\delta^m(\boldsymbol{\Omega})\} / \{(\tfrac{1}{2}n_1)_\delta C_\delta(\mathbf{I}) C_\delta(\mathbf{I})\}]\,,$$

and C_p is given by

$$(1.3) \qquad C_p = \frac{\pi^{\frac{1}{2}p} \prod_{i=1}^{p} \Gamma(\frac{1}{2}(2m + 2n + p + i + 2))}{\prod_{i=1}^{p} \{\Gamma(\frac{1}{2}(2m + i + 1))\Gamma(\frac{1}{2}(2n + i + 1))\Gamma(\frac{1}{2}i)\}}\,,$$

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$m = (n_1 - p - 1)/2$, $n = (n_2 - p - 1)/2$, $\nu = (n_1 + n_2)/2$, $\lambda > 0$, $\mathbf{R} = \operatorname{diag}(r_1, \cdots, r_p)$, $0 < r_1 < \cdots < r_p < \infty$, $\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \cdots, \lambda_p)$, $0 < \lambda_1 < \cdots < \lambda_p < \infty$ being the latent roots of $\boldsymbol{\Sigma}_1 \boldsymbol{\Sigma}_2^{-1}$, $C_\kappa(\mathbf{A})$ denotes the zonal polynomial expressible in terms of the latent roots of $\mathbf{A}$ (James [6]), $L_\delta^m(\boldsymbol{\Omega})$, the Laguerre polynomial which in turn can be expressed as a zonal polynomial series (Constantine [2]), $a_{\kappa,\delta}$ are constants (Constantine [2], Pillai and Jouris [16]) and $(a)_\kappa = \prod_{i=1}^p (a - \frac{1}{2}(i-1))_{k_i}$, $(a)_k = \Gamma(a+k)/\Gamma(a)$, where k_i are the components of the partition κ of k such that $\kappa = (k_1, \cdots, k_p)$ into not more than p components with $k_1 \geqq \cdots \geqq k_p \geqq 0$ and $k = k_1 + \cdots + k_p$.

Further, let us consider the hypotheses (A) and (B) against respective alternatives as follows:

(A) $H_0\colon \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ or equivalently $H_0\colon \lambda_i = 1$, $i = 1, \cdots, p$, against $H_1\colon \lambda_i \geqq 1$, $\sum_{i=1}^p \lambda_i > p$, and

(B) $H_0\colon \boldsymbol{\Omega} = \mathbf{0}$ given $\boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_l$ (unknown) against $H_1\colon \boldsymbol{\Omega} \neq \mathbf{0}$ given $\boldsymbol{\Sigma}_1 = \cdots = \boldsymbol{\Sigma}_l$ (unknown).

Some studies on the tests of these hypotheses have been carried out by Pillai and Jayachandran [14, 15] for $p = 2$ based on the powers of four criteria (a) the criterion $U^{(p)} = \sum_{i=1}^p r_i$, n_2 times Hotelling's T_0^2, (b) Pillai's trace $V^{(p)} = \sum_{i=1}^p \{r_i/(1 + r_i)\}$, (c) Wilks' likelihood ratio $W^{(p)} = \prod_{i=1}^p (1 + r_i)^{-1}$ and (d) Roy's largest root r_p. For $p = 2$ and $p = 3$, Pillai and Dotson [13] have studied the powers of individual latent roots for test of (B) and Pillai and Al-Ani [12] for test of (A).

In this paper, first the following are derived: (1) the density of $T = U^{(p)}$, under a condition, (2) the density of $W^{(p)}$, (3) the density of Roy's largest root in two forms and (4) the distributions of all the above criteria in the two-roots case in a suitable form for computation. These are achieved by the use of density (1.1) and the method employed by Pillai [10]. Using the distributions of the criteria in the two-roots case, an attempt is made to study the exact robustness of the above four criteria for test of (A) when the assumption of normality is violated and of (B) when that of common covariance matrix is disturbed. Lower or upper tail probabilities of the criteria were computed in view of tests of (A) and (B). A few inferences are drawn on the basis of the tabulations.

2. The density function of T. In this section the density function of $T = \lambda \operatorname{tr} \mathbf{S}_1 \mathbf{S}_2^{-1}$, $\lambda > 0$, is derived, where we assume that $n_1, n_2 \geqq p$ so that we have p nonzero latent roots of $\mathbf{S}_1 \mathbf{S}_2^{-1}$. We further assume that $\boldsymbol{\Omega}$ is partially random (denoted by "random" hereafter) in the sense that it is diagonalized by an orthogonal transformation $\mathbf{H} \in \mathscr{O}(p)$ and integrated with respect to $\mathbf{H}$ over orthogonal group $\mathscr{O}(p)$. This implies putting a Haar prior on $\mathbf{H}$, leaving the characteristic roots non-random.

THEOREM 2.1. *Let* $\mathbf{S}_1(p \times p)$ *and* $\mathbf{S}_2(p \times p)$ *be independently distributed,* $\mathbf{S}_1$ *having* $W(p, n_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Omega})$ *and* $\mathbf{S}_2$ *having* $W(p, n_2, \boldsymbol{\Sigma}_2, \mathbf{0})$. *If* $T = \lambda \operatorname{tr} \mathbf{S}_1 \mathbf{S}_2^{-1}$, $\lambda > 0$, *and* $\boldsymbol{\Omega}$ *is* *"random", then the density function of T is given by*

$$(2.1)\quad f(T) = \{\Gamma_p(\nu)/\Gamma_p(\tfrac{1}{2}n_2)\}|\lambda\Lambda|^{-\frac{1}{2}n_1}e^{-\operatorname{tr}\Omega}T^{\frac{1}{2}pn_1-1} \times \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}[(\nu)_\kappa(-T)^k C_\kappa(\lambda^{-1}\Lambda^{-1})L_\kappa^m(\Omega)/\{k!\, L(\tfrac{1}{2}pn_1+k)C_\kappa(\mathbf{I})\}]\,,$$

where $|T/(\lambda\lambda_1)| < 1$.

PROOF. From the joint density of $\mathbf{S}_1$ and $\mathbf{S}_2$, after making transformations $\mathbf{A}_1 = \frac{1}{2}\Sigma_1^{-\frac{1}{2}}\mathbf{S}_1\Sigma_1^{-\frac{1}{2}}$, $\mathbf{A}_2 = \frac{1}{2}\Sigma_1^{-\frac{1}{2}}\mathbf{S}_2\Sigma_1^{-\frac{1}{2}}$ and then $\mathbf{B}_1 = \Sigma^{\frac{1}{2}}\mathbf{A}_1\Sigma^{\frac{1}{2}}$, $\mathbf{B}_2 = \Sigma^{\frac{1}{2}}\mathbf{A}_2\Sigma^{\frac{1}{2}}$, where $\Sigma = \Sigma_1^{\frac{1}{2}}\Sigma_2^{-1}\Sigma_1^{\frac{1}{2}}$ ($\Sigma_1^{-\frac{1}{2}}$ being symmetric positive definite like other matrices of the form $\mathbf{A}^{\frac{1}{2}}$ defined later), we have

$$(2.2)\qquad C|\Sigma|^{-\frac{1}{2}n_1}e^{-\operatorname{tr}\mathbf{B}_2}|\mathbf{B}_2|^n|\mathbf{B}_1|^m e^{-\operatorname{tr}\Sigma^{-1}\mathbf{B}_1}{}_0F_1(\tfrac{1}{2}n_1;\, \Sigma^{-\frac{1}{2}}\Omega\Sigma^{-\frac{1}{2}}\mathbf{B}_1)\,,$$

where $C = \{\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)\}^{-1}e^{-\operatorname{tr}\Omega}$. Now, the Laplace transform of T, after integrating $\mathbf{B}_1$ out and then transforming $\mathbf{D} = \Sigma^{\frac{1}{2}}\mathbf{B}_2\Sigma^{\frac{1}{2}}$ has the form

$$(2.3)\quad g(t) = C\Gamma_p(\tfrac{1}{2}n_1)|\Sigma|^{\frac{1}{2}n_2}(t\lambda)^{-\frac{1}{2}pn_1}\int_{\mathbf{D}>0} e^{-\operatorname{tr}\Sigma\mathbf{D}}|\mathbf{D}|^{\nu-\alpha} \times |\mathbf{I} + (t\lambda)^{-1}\mathbf{D}|^{-\frac{1}{2}n_1}{}_0F_0(\Omega(t\lambda)^{-1}\mathbf{D}(\mathbf{I} + (t\lambda)^{-1}\mathbf{D})^{-1})\,d\mathbf{D}\,,$$

where $\alpha = \frac{1}{2}(p+1)$.

Assuming Ω "random" and integrating over $\mathscr{O}(p)$ using Theorem 1 of Constantine [2] we have

$$(2.4)\quad g(t) = C\Gamma_p(\tfrac{1}{2}n_1)|\Sigma|^{\frac{1}{2}n_2}(t\lambda)^{-\frac{1}{2}pn_1}\int_{\mathbf{D}>0} e^{-\operatorname{tr}\Sigma\mathbf{D}}|\mathbf{D}|^{\nu-\alpha} \times \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}[\{L_\kappa^m(\Omega)C_\kappa(-(t\lambda)^{-1}\mathbf{D})\}/\{k!\, C_\kappa(\mathbf{I})\}]\,d\mathbf{D}\,.$$

Upon integrating with respect to t and inverting $g(t)$, see page 222 of Constantine [2], we obtain

$$(2.5)\quad f(T) = C\Gamma_p(\tfrac{1}{2}n_1)|\Sigma|^{\frac{1}{2}n_2}\lambda^{-\frac{1}{2}pn_1}T^{\frac{1}{2}pn_1-1}\int_{\mathbf{D}>0} e^{-\operatorname{tr}\Sigma\mathbf{D}}|\mathbf{D}|^{\nu-\alpha} \times \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}[\{L_\kappa^m(\Omega)C_\kappa(-T\lambda^{-1}\mathbf{D})\}/\{k!\,\Gamma(\tfrac{1}{2}pn_1+k)C_\kappa(\mathbf{I})\}]\,d\mathbf{D}\,.$$

Applying the estimate of $L_\kappa^m(\Omega)$, Theorem 3 of [2], and finally integrating $\mathbf{D}$ out we obtain the result as stated in the theorem.

Special cases of (2.1) are

(a) For $\Omega = \mathbf{0}$ we have formula (9) of Khatri [7], and
(b) For $\Lambda = \mathbf{I}$ and $\lambda = 1$, we obtain Theorem 4 of Constantine [2].

3. The density function of $W^{(p)}$. For the derivation of the density of $W^{(p)}$ we consider $W^{(p)} = |\mathbf{I} - \mathbf{L}|$, i.e. we let $\mathbf{L} = \lambda\mathbf{R}(\mathbf{I} + \lambda\mathbf{R})^{-1}$ in (1.1), where now $\mathbf{L} = \operatorname{diag}(l_1, \cdots, l_p)$. From the theorem of Pillai [10, 11], the joint density of $l_1, l_2, \cdots, l_p$ is given by

$$(3.1)\qquad C|\mathbf{L}|^m|\mathbf{I} - \mathbf{L}|^n \textstyle\prod_{i>j}(l_i - l_j)\sum_{k=0}^{\infty}\sum_{\kappa}\{(\nu)_\kappa C_\kappa(\mathbf{L})/k!\}F_p\,,$$

where $C = C_p e^{-\operatorname{tr}\Omega}|\lambda\Lambda|^{-\frac{1}{2}n_1}$ and F_p is as in (1.2).

The hth moment of $W^{(p)}$ is given by

$$(3.2)\qquad E((W^{(p)})^h) = C\int_{\mathbf{L}>0}^{\mathbf{I}}|\mathbf{L}|^m|\mathbf{I} - \mathbf{L}|^{n+\frac{1}{2}h}\textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{(\nu)_\kappa C_\kappa(\mathbf{L})/k!\}(d\mathbf{L})F_p\,.$$

Now integrate $\mathbf{L}$ out to obtain the hth moment of $W^{(p)}$ in the form

$$(3.3)\qquad E((W^{(p)})^h) = [\Gamma_p(\nu)/\Gamma_p(\tfrac{1}{2}n_2)]e^{-\operatorname{tr}\Omega}|\lambda\mathbf{\Lambda}|^{-\frac{1}{2}n_1} \times \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa(\frac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I})}{k!}\cdot\frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)}F_p,$$

where $r = \frac{1}{2}n_2 + h - \frac{1}{2}(p-1)$, $b_i = \frac{1}{2}(i-1)$ and $a_i = \frac{1}{2}n_1 + k_{p-i+1} + b_i$. Finally, by the inverse Mellin transform we get the density function of $W^{(p)}$

$$(3.4)\qquad \begin{aligned} f(W^{(p)}) &= [\Gamma_p(\nu)/\Gamma_p(\tfrac{1}{2}n_2)]e^{-\operatorname{tr}\Omega}|\lambda\mathbf{\Lambda}|^{-\frac{1}{2}n_1} \\ &\times \sum_{k=0}^{\infty}\sum_{\kappa}\{(\nu)_\kappa(\tfrac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I})/k!\}(W^{(p)})^n \\ &\times (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}(W^{(p)})^{-r}\frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)}\,dr\cdot F_p\,. \end{aligned}$$

But the integral in (3.4) is expressible in terms of Meijer's G-function, [17], so that we obtain the following:

THEOREM 3.1. *Let* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *be as in Theorem* 2.1 *and let* $\mathbf{R} = \operatorname{diag}(r_1, \cdots, r_p)$ *where* $r_1, \cdots, r_p$ *are the latent roots of* $\mathbf{S}_1\mathbf{S}_2^{-1}$. *If* $\mathbf{\Sigma}_1^{\frac{1}{2}}\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1^{\frac{1}{2}}$ *is "random", then the density function of* $W^{(p)} = |\mathbf{I} + \lambda\mathbf{R}|^{-1}$, $\lambda > 0$, *is given by*

$$(3.5)\qquad f(W^{(p)}) = \{\Gamma_p(\nu)/\Gamma_p(\tfrac{1}{2}n_2)\}e^{-\operatorname{tr}\Omega}|\lambda\mathbf{\Lambda}|^{-\frac{1}{2}n_1}\{W^{(p)}\}^n \times \sum_{k=0}^{\infty}\sum_{\kappa}\{(\nu)_\kappa(\tfrac{1}{2}n_1)_\kappa C_\kappa(\mathbf{I})/k!\}G_{p,p}^{p,0}\left(W^{(p)}\,\middle|\,\begin{matrix}a_1\cdots a_p\\ b_1\cdots b_p\end{matrix}\right)F_p\,,$$

where $a_i = \frac{1}{2}n_1 + k_{p-i+1} + b_i$, $b_i = \frac{1}{2}(i-1)$.

The following are special cases of (3.5):

(a) Substituting $\Omega = \mathbf{0}$, $\mathbf{\Lambda} = \mathbf{I}$ and $\lambda = 1$ into (3.3) and (3.5) we have the hth moment and the density of $W^{(p)}$ as were obtained by Consul [3].

(b) For $\Omega = \mathbf{0}$, formula (3.5) gives the result of Pillai, Al-Ani and Jouris [17] formula (4.7) for testing the hypothesis H_0: $\lambda\mathbf{\Lambda} = \mathbf{I}$, $\lambda > 0$ being given.

(c) If in (3.5) we let $\mathbf{\Lambda} = \mathbf{I}$ and $\lambda = 1$, then formula (3.2) of Pillai, Al-Ani and Jouris [17] is obtained.

4. The density function of the largest root. In this section we derive two expressions for the density function of the largest root r_p of $\mathbf{S}_1\mathbf{S}_2^{-1}$. In obtaining this density we start from the joint density of the roots $r_1, \cdots, r_p$ of $\mathbf{S}_1\mathbf{S}_2^{-1}$ which is given in [10, 11] by the formula

$$(4.1)\qquad C_1|\mathbf{R}|^m\prod_{i>j}(r_i - r_j)\int_{\operatorname{Re}\mathbf{Z}>0}e^{\operatorname{tr}\mathbf{Z}}|\mathbf{Z}|^{-\frac{1}{2}n_1} \times 2^{-p}\int_{\mathcal{O}(p)}|\mathbf{I} + \mathbf{HRH}'\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{U}\mathbf{\Sigma}^{-\frac{1}{2}}|^{-\nu}\,d\mathbf{H}\,d\mathbf{Z}\,,$$

where $\mathbf{W} = \Omega^{\frac{1}{2}}\mathbf{Z}^{-1}\Omega^{\frac{1}{2}}$, $\mathbf{U} = \mathbf{I} - \mathbf{W}$ and

$$(4.2)\qquad C_1 = e^{-\operatorname{tr}\Omega}|\mathbf{\Sigma}|^{-\frac{1}{2}n_1}2^f\Gamma_p(\nu)/\{(2\pi_i)^g\Gamma_p(\tfrac{1}{2}n_2)\}\,,$$

and where $f = \frac{1}{2}p(p-1)$, $g = \frac{1}{2}p(p+1)$.
Now use Lemma 2 of Khatri [7] by taking $g(F) = r_p$. Integrating over $\mathcal{O}(p)$,

the second integral in (4.1) becomes

$$[\pi^{\frac{1}{2}p^2}/\Gamma_p(\tfrac{1}{2}p)]|\mathbf{I} + \Sigma^{-\frac{1}{2}}\mathbf{U}\Sigma^{-\frac{1}{2}}r_p|^{-\nu}{}_1F_0(\nu; r_p^{-1}(\mathbf{I} + \Sigma^{-\frac{1}{2}}\mathbf{U}\Sigma^{-\frac{1}{2}}r_p^{-1}), r_p\mathbf{I} - \mathbf{R}) .$$

Let $y_i = r_i/r_p$, $i = 1, 2, \cdots, p-1$ and $\mathbf{Y} = \mathrm{diag}(y_1, \cdots, y_{p-1})$. Further, expand ${}_1F_0$ in terms of zonal polynomials, and for the integration with respect to $\mathbf{Y}$ we apply Lemma 3 of Khatri [7]. Then we obtain the density function of the largest root r_p in the form

$$(4.3)\quad \begin{aligned} &C_2 \cdot r_p^{\frac{1}{2}pn_1-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa \Gamma_{p-1}(\frac{1}{2}p+1, \kappa) C_\kappa(\mathbf{I}_{p-1})}{k!\, \Gamma_{p-1}(\frac{1}{2}(n_1+p+1), \kappa) C_\kappa(\mathbf{I})} \\ &\times \int_{\mathrm{Re}\,\mathbf{Z}>0} e^{\mathrm{tr}\,\mathbf{Z}} |\mathbf{Z}|^{-\frac{1}{2}n_1} |\mathbf{I} + \Sigma^{-\frac{1}{2}}\mathbf{U}\Sigma^{-\frac{1}{2}}r_p|^{-\nu} \\ &\times C_\kappa[\Sigma^{-\frac{1}{2}}\mathbf{U}\Sigma^{-\frac{1}{2}}r_p\{\mathbf{I} + \Sigma^{-\frac{1}{2}}\mathbf{U}\Sigma^{-\frac{1}{2}}r_p\}^{-1}]\, d\mathbf{Z} , \end{aligned}$$

where $\mathbf{I}_{p-1}$ is the identity matrix of order $p-1$, $\mathbf{I}$ is that of order p and $C_2 = \pi^{\frac{1}{2}p^2} C_1 \Gamma_{p-1}(\frac{1}{2}(n_1-1))\Gamma_{p-1}(\frac{1}{2}(p-1))/\{\pi^{\frac{1}{2}(p-1)^2}\Gamma_p(\frac{1}{2}p)\}$. Applying Theorem 1 of Constantine [1] and taking $\mathbf{B} = r_p\Sigma^{-\frac{1}{2}}\mathbf{S}\Sigma^{-\frac{1}{2}}$, the integral in (4.3) becomes

$$(4.4)\quad \begin{aligned} &\{|r_p\Sigma^{-1}|^{-\nu}/\Gamma_p(\nu, \kappa)\} \sum_{t=0}^{\infty} \sum_{\tau} \{(-1)^t/t!\} \sum_{\delta} g_{\kappa,\tau}^{\delta} \int_{\mathrm{Re}\,\mathbf{Z}>0} e^{\mathrm{tr}\,\mathbf{Z}} |\mathbf{Z}|^{-\frac{1}{2}n_1} \\ &\times \int_{\mathbf{B}>0} \exp\{-\mathrm{tr}\, r_p^{-1}\Sigma\mathbf{B}\} |\mathbf{B}|^{\nu-\alpha} C_\delta(\mathbf{UB})\, d\mathbf{B}\, d\mathbf{Z} , \end{aligned}$$

where $\alpha = \frac{1}{2}(p+1)$, $g_{\kappa,\tau}^{\delta}$ are constants and $\sum_{i=1}^{p} \delta_i = k + t$, $\delta = (\delta_1, \cdots, \delta_p)$, $\delta_1 \geqq \cdots \geqq \delta_p \geqq 0$. Now, integrate $\mathbf{B}$ out, then (4.4) reduces to

$$(4.5)\quad \begin{aligned} &\{\Gamma_p(\nu, \kappa)\}^{-1} \sum_{t=0}^{\infty} \sum_{\tau} \{(-1)^t/t!\} \sum_{\delta} g_{\kappa,\tau}^{\delta} \Gamma_p(\nu, \delta) \\ &\times \int_{\mathrm{Re}\,\mathbf{Z}>0} e^{\mathrm{tr}\,\mathbf{Z}} |\mathbf{Z}|^{-\frac{1}{2}n_1} C_\delta\{\mathbf{U}(r_p\Sigma^{-1})\}\, d\mathbf{Z} . \end{aligned}$$

Assuming $(r_p\Sigma^{-1})$ "random" and integrating over $\mathcal{O}(p)$ and finally applying (17) of Constantine [2] we obtain the density function of r_p as stated in the following:

THEOREM 4.1. *Let the hypotheses be as in Theorem* 3.1. *Then the density function of* r_p *is given by*

$$(4.6)\quad \begin{aligned} &C e^{-\mathrm{tr}\,\Omega} |\Lambda|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}p+1)_\kappa (\frac{1}{2}(p-1))_\kappa}{k!\, (\frac{1}{2}(n_1+p+1))_\kappa (\frac{1}{2}p)_\kappa} \\ &\times \sum_{t=0}^{\infty} \sum_{\tau} \{(-1)^t/t!\} \sum_{\delta} g_{\kappa,\tau}^{\delta} [(\nu)_\delta C_\delta(r_p\Lambda^{-1}) L_\delta^m(\Omega)/\{(\tfrac{1}{2}n_1)_\delta C_\delta(\mathbf{I})\}] , \end{aligned}$$

where the constant C *is given by*

$$(4.7)\quad C = \Gamma(\tfrac{1}{2})\Gamma_p(\nu)\Gamma_{p-1}(\tfrac{1}{2}p+1)/\{\Gamma(\tfrac{1}{2}p)\Gamma(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\Gamma_{p-1}(\tfrac{1}{2}(n_1+p+1))\} .$$

The density in (4.6) *may not converge for all* r_p, $0 < r_p < \infty$.

To obtain the second expression of the density of r_p, let us return to formula (4.4). The second integral in (4.4) can be written as

$$(4.8)\quad \int_{\mathbf{B}>0} \exp\{-\mathrm{tr}\,[(\mathbf{I} + r_p^{-1}\Sigma)\mathbf{B}]\} e^{\mathrm{tr}\,\mathbf{B}} |\mathbf{B}|^{\nu-\alpha} C_\delta(\mathbf{UB})\, d\mathbf{B} .$$

Expand the first exponent in (4.8) and assume $(\mathbf{I} + r_p^{-1}\Sigma)$ "random" then integrate over $\mathcal{O}(p)$. Further diagonalize $\mathbf{B}$ and integrate out $\mathbf{H}$. After de-diagonalizing $\mathbf{B}$, the integral in (4.8) now becomes

$$(4.9)\quad \int_{\mathbf{B}>0} \exp\{-\mathrm{tr}\,[\mathbf{I} + r_p^{-1}\Lambda)\mathbf{B}] e^{\mathrm{tr}\,\mathbf{B}}\} |\mathbf{B}|^{\nu-\alpha} \{C_\delta(\mathbf{B}) C_\delta(\mathbf{U})/C_\delta(\mathbf{I})\}\, d\mathbf{B} .$$

Expand the second exponent and use (27) of Constantine [2].

Then (4.4) now becomes

$$\{|\mathbf{I} + r_p\mathbf{\Lambda}^{-1}|^{-\nu}/\Gamma_p(\nu,\kappa)\} \sum_{s=0}^{\infty} \sum_{\sigma} \{(-1)^s/s!\} \sum_{\delta} g_{\kappa,\sigma}^{\delta} \{C_\delta(\mathbf{I})\}^{-1} \times \int_{\mathrm{Re}\,\mathbf{Z}>0} e^{\mathrm{tr}\,\mathbf{Z}} |\mathbf{Z}|^{-\frac{1}{2}n_1} C_\delta(\mathbf{U})\, d\mathbf{Z} \times \sum_{t=0}^{\infty} \sum_{\tau} (t!)^{-1} \sum_{\mu} g_{\delta,\tau}^{\mu} \Gamma_p(\nu,\mu) C_\mu\{(\mathbf{I} + r_p^{-1}\mathbf{\Lambda})^{-1}\}\,. \tag{4.10}$$

Replacing $\mathbf{U}$ by $\mathbf{I} - \mathbf{W}$, $\mathbf{W}$ by $\mathbf{\Omega}^{\frac{1}{2}}\mathbf{Z}^{-1}\mathbf{\Omega}^{\frac{1}{2}}$ and making use of (17) of Constantine [2] we finally obtain the following:

THEOREM 4.2. *Let $\mathbf{S}_1$, $\mathbf{S}_2$ and r_p be as before. If $\mathbf{\Sigma}_1^{\frac{1}{2}}\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1^{\frac{1}{2}}$ is "random", then the density of r_p is given by*

$$C \cdot e^{-\mathrm{tr}\,\mathbf{\Omega}} |\mathbf{\Lambda}|^{-\frac{1}{2}n_1} |\mathbf{I} + r_p\mathbf{\Lambda}^{-1}|^{-\nu} r_p^{\frac{1}{2}pn_1 - 1} \times \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}p+1)_\kappa(\frac{1}{2}(p-1))_\kappa}{k!(\frac{1}{2}(n_1+p+1))_\kappa(\frac{1}{2}p)_\kappa} \sum_{s=0}^{\infty} \sum_{\sigma} \frac{(-1)^s}{s!} \sum_{\delta} \frac{g_{\kappa,\sigma}^{\delta} L_\delta{}^m(\mathbf{\Omega})}{(\frac{1}{2}n_1)_\delta C_\delta(\mathbf{I})} \times \sum_{t=0}^{\infty} \sum_{\tau} (t!)^{-1} \sum_{\mu} g_{\delta,\tau}^{\mu} (\nu)_\mu C_\mu\{(\mathbf{I} + r_p^{-1}\mathbf{\Lambda})^{-1}\}\,. \tag{4.11}$$

For $s = t = 0$ and $\mathbf{\Omega} = \mathbf{0}$, (4.11) *reduces to* (16) *of Khatri* [7].

5. Non-central distribution of the criteria for $p = 2$. In this section the non-central distribution of the criteria suitable for computation will be derived for $p = 2$. The method of derivation is analogous to the one used in Pillai and Jayachandran [14] except that the results obtained here are more general.

I. *Non-central distribution of $U^{(2)}$.* Putting $p = 2$ and $\lambda = 1$ in (1.1) and expressing the zonal polynomials in terms of the latent roots of the matrix involved there, we have the joint density of r_1 and r_2 in the form

$$C_2 e^{-(w_1+w_2)}(\lambda_1\lambda_2)^{-\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \{(\nu)_\kappa/k!\} \times (r_1r_2)^m\{(1+r_1)(1+r_2)\}^{-q}(r_2 - r_1) \sum b_\kappa(r,s) a_1{}^r a_2{}^s F_2\,, \tag{5.1}$$

where $q = m + n + 3$, w_1, w_2 are the latent roots of $\mathbf{\Omega}$, λ_1, λ_2 are those of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$, F_2 and C_2 are obtained from (1.2) and (1.3) for $p = 2$ respectively, the last summation is such that $r + 2s = k$, $b_\kappa(r, s)$ are constants whose values up to $k = 6$ are tabulated in Sudjana [18] and given in Appendix A, and a_i $(i = 1, 2)$ is the ith elementary symmetric function in $r_i/(1 + r_i)$. Taking $x = U^{(2)} = r_1 + r_2$ and $y = r_1r_2$ and integrating y from 0 to $x^2/4$, the density of $U^{(2)}$ is expressible in terms of the integral $h_{a,b}(x) = \int_0^{x^2/4} \{y^{m+a}/(1 + x + y)^{q+b}\, dy$, where a and b depend on k.

The cdf of $U^{(2)}$ involves the double integral

$$H_{rs}(U^{(2)}) = \sum_{i=0}^{r} \binom{r}{i} 2^{r-i} \int_0^{U^{(2)}} \int_0^{x^2/4} \frac{x^i y^{m+j}}{(1+x+y)^{q+l}}\, dy\, dx\,, \tag{5.2}$$

where $j = r + s - i$, $l = r + s$ and the above double integral is expressible in terms of the expression

$$B_{a,b}(u) = (n + b - a + 1)^{-1}\{2B_w(2(m+a+1), 2(n+b-a+1)+1) - (1+u)^{a-n-b-1}B_z(m+a+1, n+b-a+2)\}\,, \tag{5.3}$$

where the incomplete beta function $B_x(p, q) = \int_0^x y^{p-1}(1-y)^{q-1}\,dy$, and where $w = u/(2+u)$, $z = w^2$ and a and b depend on k. Summarizing the above, we have the following:

THEOREM 5.1. *Let the assumptions be as stated in Theorem* 3.1. *Then the exact non-central distribution of* $U^{(2)}$ *is given by*

$$(5.4)\qquad C\sum_{k=0}^{\infty}\sum_{\kappa}\{(\nu)_\kappa/k!\}\sum b_\kappa(r,s)H_{rs}(U^{(2)})F_2\,,\qquad 0<U^{(2)}<\infty\,,$$

where

(5.5) $C = C_2 e^{-(w_1+w_2)}(\lambda_1\lambda_2)^{-\frac{1}{2}n_1}$, *with* C_2 *as obtained from* (1.3) *for* $p = 2$, *the last summation is such that* $r + 2s = k$ *and the meanings of other symbols are as explained above.*

The expression up to the sixth order has the form

$$(5.5)\qquad F(U^{(2)}) = C\sum_{j=0}^{6}\sum_{i=0}^{j}(-1)^{i+j}D_{ij}H_{ij}(U^{(2)})\,,$$

where $H_{ij}(U^{(2)})$ is as described in (5.2), and the coefficients D_{ij}'s are available in an unpublished thesis by Sudjana [18] and given in Appendix B.

II. *Non-central distribution of* $V^{(2)}$. From (3.1), we have the joint density of l_1 and l_2 as

$$(5.6)\qquad C\sum_{k=0}^{\infty}\sum_{\kappa}\{(\nu)_\kappa/k!\}(l_1l_2)^m\prod_{i=1}^{2}(1-l_i)^n(l_2-l_1)\sum b_\kappa(r,s)a_1{}^r a_2{}^s F_2\,,$$

where the summation is such that $r + 2s = k$, a_1 and a_2 are the first and the second esf in l_i $(i = 1, 2)$ and F_2 is as discussed earlier. As in I, starting from the joint density of $x = V^{(2)} = l_1 + l_2$ and $y = l_1l_2$, the cdf of $V^{(2)}$ will involve the integral

$$(5.7)\qquad F_{rs}(V^{(2)}) = \int_0^{V^{(2)}}\int_0^{x^2/4} x^r y^{m+s}(1-x+y)^n\,dy\,dx\,.$$

This integral is expressible in terms of incomplete beta function as follows: for $0 < V^{(2)} < 1$,

$$(5.8)\qquad F_{rs}(V^{(2)}) = \{2^{r+1}/(m+s+1)\}\sum_{i=0}^{n}(-1)^i R_i B_v(a+4i+2,\, b-4i-2)\,,$$

and for $1 \leqq V^{(2)} \leqq 2$,

$$(5.9)\qquad F_{rs}(V^{(2)}) = \frac{2^{r+1}}{m+s+1}\sum_{i=0}^{n}(-1)^i R_i B_{0.5}(a+4i+2,\, b-4i-2) + \frac{2^{r+1}}{n+1}\sum_{i=0}^{m+s}(-1)^i P_i[B_v(a,b) - B_{0.5}(a,b)]\,,$$

where $v = \frac{1}{2}V^{(2)}$, $a = 2m + 2s - 2i + r + 1$, $b = 2n + 2i + 3$,

$$(5.10)\qquad \begin{aligned} R_i &= \prod_{j=1}^{i}\{(n+1+j)/(m+s+1+j)\}\,, & R_0 &= 1 \quad \text{and}\\ P_i &= \prod_{j=1}^{i}\{(m+s+1-j)/(n+1+j)\}\,, & P_0 &= 1\,. \end{aligned}$$

In view of the above, we state the following:

THEOREM 5.2. *Let* $\mathbf{S}_1$, $\mathbf{S}_2$ *and* $\mathbf{\Sigma}_1^{\frac{1}{2}}\mathbf{\Sigma}_2^{-1}\mathbf{\Sigma}_1^{\frac{1}{2}}$ *be as before and* $V^{(2)}$ *be as defined above. Then the exact non-central distribution of* $V^{(2)}$ *is expressible in terms of incomplete* *beta functions and is given by*

$$C \sum_{k=0}^{\infty} \sum_{\kappa} \{(\nu)_\kappa/k!\} \sum b_\kappa(r, s) F_{rs}(V^{(2)}) F_2 , \tag{5.11}$$

where the meanings of all symbols are as before, and $F_{rs}(V^{(2)})$ *is as in* (5.8) *and* (5.9).

Upon expanding the series in (5.11) up to the sixth order and combining the like terms, we obtain the cdf of $V^{(2)}$ in the form

$$F(V^{(2)}) = C \cdot \sum_{i+2j=k=0}^{6} E_{ij} F_{ij}(V^{(2)}) , \tag{5.12}$$

where in the summation, only integral solutions (i, j) of $i + 2j = k$ are taken and the coefficients E_{ij}'s are available in Sudjana [18] and given in Appendix C.

III. *Non-central distribution of* $W^{(2)}$. Using l_1 and l_2, the Wilks' criterion $W^{(2)}$ is given by $W^{(2)} = (1 - l_1)(1 - l_2)$. As in the previous sections, to obtain the cdf of $W^{(2)}$, we shall use the joint density of $x = W^{(2)} = (1 - l_1)(1 - l_2)$ and $y = l_1 l_2$. After making substitutions in (5.6) and integrating y out first and then x, the cdf of $W^{(2)}$ involves integrals of the form

$$G_{rs}(W^{(2)}) = \int_0^{W^{(2)}} \int_0^{(1-x^{\frac{1}{2}})^2} x^n y^{m+s} (1 - x + y)^r \, dy \, dx , \tag{5.13}$$

which is expressible in terms of incomplete beta function as

$$G_{rs}(W^{(2)}) = \sum_{i=0}^{r} \frac{(-1)^i 2^{r-i+1}}{m + s + 1} Q_i B_z(2n + 2, 2m + 2s + r + i + 3) , \tag{5.14}$$

where $Q_i = \prod_{j=1}^{i} \{(r + 1 - j)/(m + s + 1 + j)\}$, $Q_0 = 1$ and $z = (W^{(2)})^{\frac{1}{2}}$. Note that (5.14) is obtained easily from (5.13) by integration by parts. Now we have the following:

THEOREM 5.3. *Let* $\mathbf{S}_1$, $\mathbf{S}_2$ *and* $\Sigma_1^{\frac{1}{2}}\Sigma_2^{-1}\Sigma_1^{\frac{1}{2}}$ *be as in Theorem* 5.2 *and* $W^{(2)}$ *be as defined above. Then the exact non-central* cdf *of* $W^{(2)}$ *is expressible in terms of incomplete beta function and is given by*

$$C \sum_{k=0}^{\infty} \sum_{\kappa} \{(\nu)_\kappa/k!\} \sum b_\kappa(r, s) G_{rs}(W^{(2)}) F_2 , \qquad 0 < W^{(2)} < 1 , \tag{5.15}$$

where the meanings of all symbols are as explained above.

The cdf of $W^{(2)}$ using up to $k = 6$ is given by

$$F(W^{(2)}) = C \cdot \sum_{i+2j=k=0}^{6} E_{ij} G_{ij}(W^{(2)}) , \tag{5.16}$$

where E_{ij}'s are as discussed in II.

REMARK. If $w_1 = w_2 = 0$, the distributions (5.5), (5.12) and (5.16) reduce to the results of Pillai and Jayachandran [15].

IV. *Non-central distribution of the individual root.* To obtain the distribution of the largest root l_2 we will start from the joint density of l_1 and l_2, $0 < l_1 < l_2 < 1$, which is described in (5.6). Integrating l_1 out from $0 < l_1 < l_2$, we easily obtain the density of the largest root l_2 in the form

$$C \sum_{k=0}^{\infty} \sum_{\kappa} \{(\nu)_\kappa/k!\} \sum b_\kappa(r, s) p_{rs}(l_2) F_2 , \qquad 0 < l_2 < 1 , \tag{5.17}$$

where

$$p_{rs}(l_2) = \sum_{t=0}^{r} \binom{r}{t} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \{a(a+1)\}^{-1} l_2^{\,b} (1-l_2)^n \,, \tag{5.18}$$

with $a = m + i + 1 + s + t$ and $b = 2(m + s + 1) + r + i$.

To get the cdf of the largest root, we integrate (5.17) with respect to l_2. From (5.18) we have

$$P_{rs}(l_2) = \sum_{t=0}^{r} \binom{r}{t} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \{a(a+1)\}^{-1} B_{l_2}(b+1, n+1) \,, \tag{5.19}$$

where $B_x(p, q)$ is the incomplete beta function. From the above we have

THEOREM 5.4. *Let* S_1, S_2 *and* $\Sigma_1^{\frac{1}{2}}\Sigma_2^{-1}\Sigma_1^{\frac{1}{2}}$ *be as stated in Theorem* 5.2 *and* l_p *be the largest root of* $S_1(\mathrm{S}_1 + \mathrm{S}_2)^{-1}$. *Then the non-central distribution of* l_2 *is expressible in terms of incomplete beta function and has the form*

$$C \sum_{k=0}^{\infty} \sum_{\kappa} \{(\nu)_\kappa / k!\} \sum b_\kappa(r, s) P_{rs}(l_2) F_2 \,, \qquad 0 < l_2 < 1 \,, \tag{5.20}$$

where $P_{rs}(l_2)$ *is as stated in* (5.19) *and the other symbols are interpreted as before.*

Again, expanding the series up to the sixth order we have the cdf of l_2 as follows:

$$F(l_2) = C \cdot \sum_{i+2j=k=0}^{6} E_{ij} P_{ij}(l_2) \,, \tag{5.21}$$

where E_{ij}'s are as explained earlier.

Note that for $w_1 = w_2 = 0$, the cdf of l_2 up to the sixth order has been obtained by Pillai and Al-Ani [12] using a different method. Their expression is not as simple as the one that can be obtained from (5.21) with $w_1 = w_2 = 0$.

Using an approach similar to the above we can obtain the following:

THEOREM 5.5. *Under the assumptions of Theorem* 5.4, *the non-central* cdf *of smallest root* l_1 *is given by*

$$C \sum_{k=0}^{\infty} \sum_{\kappa} \{(\nu)_\kappa / k!\} \sum b_\kappa(r, s) Q_{rs}(l_1) F_2 \,, \qquad 0 < l_1 < l_2 \,, \tag{5.22}$$

where the meaning of the symbols are as before, and

$$Q_{rs}(l_1) = \sum_{t=0}^{r} \binom{r}{t} \sum_{i=0}^{q} (-1)^i \binom{q}{i} (n+i+1)^{-1} (n+i+2)^{-1} B_{l_1}(a, b) \,, \tag{5.23}$$

with $a = m + r + s + 1$, $b = 2n + i + 3$ and $q = m + r + s - t$.

Its expansion up to the sixth order for the cdf of l_1 is

$$F(l_1) = C \cdot \sum_{i+2j=k=0}^{6} E_{ij} Q_{ij}(l_1) \,. \tag{5.24}$$

It may be pointed out here that (5.22) cannot be obtained from (5.20) by transformations. However, if $w_1 = w_2 = 0$ and $\lambda_1 = \lambda_2 = 1$, i.e. in the central case, we can obtain (5.22) from (5.20) by performing transformation $l_2 \to (1 - l_1)$ and $m \to n$ as was done in Pillai [8].

6. Numerical study of robustness based on four criteria. Let us now use the distributions obtained in Section 5 to study the robustness of the tests concerning the two hypotheses stated earlier, namely: (A) equality of covariance matrices in two p-variate normal populations, and (B) equality of p-dimensional mean vectors in l p-variate normal populations having common unknown covariance matrix. First observe that in case (A), if $\mathbf{\Omega} = \mathbf{0}$ we are testing $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$ assuming the two populations are normal. However, for $\mathbf{\Omega} \neq \mathbf{0}$ implies the violation of normality for the test of (A). For (B) when $\mathbf{\Lambda} \neq \mathbf{I}$ the violation occurs. Some numerical values of upper tail probabilities of $U^{(2)}$, $V^{(2)}$ and the largest root r_2 and lower tail probabilities of $W^{(2)}$ have been calculated. The tail probabilities of $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and the largest root are computed using respectively (5.5), (5.12), (5.16) and (5.21). They involve zonal polynomials of degree 0 to 6. In these calculations, the upper/lower 5 percent points of the respective criteria under the null hypotheses of (A) and (B) have been used and were taken from the tables prepared by Pillai and Jayachandran [14] and Pillai and Al-Ani [12]. All computations were carried out on the CDC 6500 Computer at the Purdue University Computing Center. Before computing the tail probability for specific values of the parameters, the total probability in that case over the whole range of the respective statistics for all the terms included in the formula was calculated and the number of decimals included in the tables was determined depending on the number of places of accuracy obtained in the total probability, at least as many decimal places as in the table. Moreover, the total probability was computed by cumulating successively the probability contribution for each term for $k = 0$ to 6 and noting the successive reduction in the contribution for each term. It may be pointed out that the convergence of (1.1) has been discussed in [10]. In that paper it was shown that (1.1) is dominated termwise by the series given by Khatri [7] for the joint density of the latent roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ which was used by Pillai and Jayachandran [15] for power comparisons of tests of equality of two covariance matrices based on four criteria.

For various values of w_1, w_2 (i.e. the latent roots of $\mathbf{\Omega}$ for $p = 2$) and λ_1, λ_2 (i.e. the latent roots of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ for $p = 2$) these upper/lower tail probabilities are tabulated using different values of $m = (n_1 - p - 1)/2$ and $n = (n_2 - p - 1)/2$. Table 1 presents those probabilities for $m = 0$ and Table 2 for $m = 2$. In both cases the values of $n = 5$, 15 and 40. In both tables the powers of the test of (A) assuming $\mathbf{\Omega} = \mathbf{0}$ and those of the test of (B) assuming $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$ are also presented.

From the tabulations it appears that

(1) For hypothesis (A), the powers of tests based on all four criteria show considerable change even for small deviations of (w_1, w_2) from (0, 0) and the difference in the respective powers remains approximately of the same magnitude irrespective of the values of (λ_1, λ_2) for a given (m, n). The changes of powers become larger for bigger deviations of (w_1, w_2). This probably is indicative that the tests are not robust against non-normality.

(2) For hypothesis (B), the powers of tests based on all four criteria show modest changes for small deviations of (λ_1, λ_2) from (1, 1) but changes become pronounced as (λ_1, λ_2) deviate more from (1, 1).

(3) Tabulations do not reveal any advantage of one test statistic over the

others in regard to either hypothesis from the point of view of robustness. It is likely that tabulations for larger deviations may bring more light on this problem.

It may be pointed out that Itô [4] and Itô and Schull [5] have also studied similar cases but their results are based on the large sample theory. The results obtained here are of an exact nature except for some of the assumptions made in the model.

TABLE 1

Upper/lower tail probabilities of four criteria, $m = 0$ and $\alpha = 0.05$

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$n = 5$			
0	0.001	1	1.001	0.050106	0.050112	0.050109	0.050100
				0.050079	0.050084	0.050082	0.050073
				0.050026	0.050028	0.050027	0.050026
		1	1.01	0.050823	0.050869	0.050852	0.050769
				0.050796	0.050840	0.050824	0.050744
		1.025	1.025	0.054066	0.054301	0.054213	0.053783
				0.054038	0.054272	0.054184	0.053757
		1	1.1	0.058328	0.058646	0.058571	0.057811
				0.058298	0.058614	0.058538	0.05778
				$n = 15$			
0	0.001	1	1.001	0.050133	0.050134	0.050134	0.050121
				0.050099	0.050100	0.050100	0.050093
				0.050033	0.050033	0.050033	0.050027
		1	1.01	0.051035	0.051043	0.051041	0.050967
				0.051001	0.051009	0.051007	0.050935
		1.025	1.025	0.055132	0.055179	0.055165	0.054795
				0.055096	0.055143	0.055130	0.054762
		1	1.1	0.060585	0.060604	0.060619	0.060013
				0.060546	0.060565	0.060580	0.05998
				$n = 40$			
0	0.001	1	1.001	0.050145	0.050145	0.050145	0.050132
				0.050108	0.050109	0.050109	0.050102
				0.050036	0.050036	0.050036	0.050029
		1	1.01	0.051131	0.051132	0.051132	0.051063
				0.051094	0.051095	0.051095	0.051028
		1.025	1.025	0.055617	0.055625	0.055623	0.055285
				0.055577	0.055586	0.055584	0.055248
		1	1.1	0.061617	0.061601	0.061613	0.061087
				0.061574	0.061558	0.061571	0.06105
				$n = 5$			
0	0.01	1	1.001	0.050344	0.050364	0.050356	0.050322
				0.050079	0.050084	0.050082	0.050073
				0.050264	0.050280	0.050274	0.050248
		1	1.01	0.051064	0.051123	0.051101	0.050993
		1.025	1.025	0.054319	0.054569	0.054475	0.054018
		1	1.1	0.058596	0.058930	0.058849	0.058060

TABLE 1—Continued

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$n = 15$			
0	0.01	1	1.001	0.050432	0.050436	0.050435	0.050401
				0.050099	0.050100	0.050100	0.050093
				0.050332	0.050335	0.050334	0.050307
		1	1.01	0.051338	0.051349	0.051346	0.051251
		1.025	1.025	0.055453	0.055503	0.055489	0.055095
		1	1.1	0.050928	0.060951	0.060965	0.060333
				$n = 40$			
0	0.01	1	1.001	0.050472	0.050473	0.050472	0.050440
				0.050108	0.050109	0.050109	0.050102
				0.050363	0.050363	0.050363	0.050337
		1	1.01	0.051462	0.051464	0.051464	0.051376
		1.025	1.025	0.055969	0.055978	0.055975	0.055616
		1	1.1	0.061995	0.061980	0.061993	0.061442
				$n = 5$			
0	0.1	1	1.001	0.052750	0.052897	0.052844	0.052567
				0.050079	0.050084	0.050082	0.050073
				0.052667	0.052809	0.052758	0.052490
		1	1.01	0.053497	0.053686	0.053618	0.053262
		1.025	1.025	0.056874	0.057262	0.057120	0.056394
		1	1.1	0.06131	0.06178	0.06166	0.06057
				$n = 15$			
0	0.1	1	1.001	0.053462	0.053488	0.053482	0.053246
				0.050099	0.050100	0.050100	0.050093
				0.053357	0.053383	0.053377	0.053149
		1	1.01	0.054408	0.054442	0.054434	0.054132
		1.025	1.025	0.058703	0.058778	0.058757	0.058135
		1	1.1	0.06440	0.06445	0.06446	0.06358
				$n = 40$			
0	0.1	1	1.001	0.053785	0.053788	0.053788	0.053575
				0.050108	0.050109	0.050109	0.050102
				0.053670	0.053673	0.053673	0.053467
		1	1.01	0.054821	0.054826	0.054826	0.054552
		1.025	1.025	0.059536	0.059548	0.059546	0.058980
		1	1.1	0.06582	0.06581	0.06582	0.06504
				$n = 5$			
0	1	1	1.001	0.07899	0.07958	0.07967	0.07733
				0.050079	0.050084	0.050082	0.050073
				0.078873	0.079463	0.079538	0.077229
		1	1.01	0.08001	0.08066	0.08074	0.07828
		1.025	1.025	0.0846	0.0855	0.0855	0.0825
		1	1.1	0.091	0.091	0.091	0.088
				$n = 15$			
0	1	1	1.01	0.08723	0.08706	0.08725	0.08554
				0.050099	0.050100	0.050100	0.050093
				0.087084	0.086913	0.087101	0.085403
		1	1.01	0.08859	0.08843	0.08862	0.08679
		1.025	1.025	0.0947	0.0946	0.0948	0.0924
		1	1.1	0.103	0.103	0.103	0.100

TABLE 1—Continued

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$n = 40$			
0	1	1	1.001	0.09104	0.09088	0.09098	0.08958
				0.050108	0.050109	0.050109	0.050102
				0.090873	0.090717	0.090823	0.089426
		1	1.01	0.09255	0.09239	0.09249	0.09099
		1.025	1.025	0.0994	0.0992	0.0993	0.0973
		1	1.1	0.108	0.108	0.108	0.106
				$n = 5$			
1	1	1	1.001	0.1124	0.1203	0.1164	0.1049
				0.0501	0.0501	0.0501	0.0501
				0.1122	0.1201	0.1164	0.1048
		1	1.01	0.1138	0.1217	0.1178	0.1062
		1.025	1.025	0.120	0.128	0.124	0.112
		1	1.1	0.128	0.138	0.132	0.119
				$n = 15$			
1	1	1	1.001	0.1314	0.1341	0.1327	0.1214
				0.0501	0.0501	0.0501	0.0501
				0.1312	0.1339	0.1326	0.1213
		1	1.01	0.1332	0.1359	0.1345	0.1231
		1.025	1.025	0.141	0.144	0.143	0.130
		1	1.1	0.152	0.153	0.153	0.140
				$n = 40$			
1	1	1	1.001	0.1402	0.1412	0.1406	0.1295
				0.0501	0.0501	0.0501	0.0501
				0.1400	0.1409	0.1405	0.1293
		1	1.01	0.1422	0.1432	0.1426	0.1314
		1.025	1.025	0.151	0.152	0.152	0.140
		1	1.1	0.163	0.163	0.163	0.151

Entries in 2nd row denote powers of the test $H_0: \Sigma_1 = \Sigma_2$ assuming $\mathbf{\Omega} = \mathbf{0}$
Entries in 3rd row denote powers of the test $H_0: \mathbf{\Omega} = \mathbf{0}$ assuming $\Sigma_1 = \Sigma_2$

TABLE 2
Upper/lower tail probabilities of four criteria, $m = 2$, $\alpha = 0.05$

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$n = 5$			
0	0.001	1	1.001	0.050109	0.050122	0.050118	0.050100
				0.050095	0.050107	0.050103	0.050084
				0.050014	0.050015	0.050015	0.050012
		1	1.01	0.050970	0.051089	0.051048	0.050862
				0.050956	0.051074	0.051033	0.050850
		1.025	1.025	0.054894	0.055526	0.055306	0.054310
				0.054879	0.055510	0.055290	0.054297
		1	1.1	0.06012	0.06106	0.06082	0.05904
				0.060100	0.061036	0.060793	0.05901

TABLE 2—Continued

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$n = 15$			
0	0.001	1	1.001	0.050150	0.050153	0.050152	0.050137
				0.050130	0.050134	0.050133	0.050115
				0.050019	0.050019	0.050019	0.050017
		1	1.01	0.051339	0.051368	0.051360	0.051192
				0.051320	0.051349	0.051341	0.051175
		1.025	1.025	0.056812	0.056980	0.056931	0.056005
				0.056791	0.056958	0.056910	0.055987
		1	1.1	0.06424	0.06433	0.06437	0.06287
				0.064218	0.064302	0.064347	0.06284
				$n = 40$			
0	0.001	1	1.001	0.050171	0.050172	0.050171	0.050160
				0.050149	0.050150	0.050150	0.050134
				0.050021	0.050021	0.050021	0.050019
		1	1.01	0.051530	0.051535	0.051534	0.051383
				0.051508	0.051513	0.051512	0.051363
		1.025	1.025	0.057816	0.057850	0.057841	0.056983
				0.057792	0.057826	0.057817	0.056962
		1	1.1	0.06643	0.06636	0.06641	0.06513
				0.066400	0.066336	0.066388	0.06509
				$n = 5$			
0	0.01	1	0.001	0.050231	0.050260	0.050250	0.050208
				0.050095	0.050107	0.050103	0.050084
				0.050136	0.050153	0.050147	0.050124
		1	1.01	0.051094	0.051229	0.051183	0.050971
		1.025	1.025	0.055026	0.05567	0.055449	0.054426
		1	1.1	0.06026	0.06122	0.06097	0.05916
				$n = 15$			
0	0.01	1	1.001	0.050318	0.050326	0.050324	0.050287
				0.050130	0.050134	0.050133	0.050115
				0.050187	0.050192	0.050190	0.050170
		1	1.01	0.051510	0.051544	0.051535	0.051344
		1.025	1.025	0.056998	0.057170	0.057120	0.056168
		1	1.1	0.06445	0.06454	0.06458	0.06305
				$n = 40$			
0	0.01	1	1.001	0.050363	0.050365	0.050356	0.050333
				0.050149	0.050150	0.050150	0.050134
				0.050214	0.050214	0.050214	0.050199
		1	1.01	0.051726	0.051732	0.051731	0.051559
		1.025	1.025	0.058030	0.058065	0.058056	0.057173
		1	1.1	0.06664	0.06660	0.06665	0.06534
				$n = 5$			
0	0.1	1	1.001	0.051464	0.051644	0.051582	0.051299
				0.050095	0.050107	0.050103	0.050084
				0.051367	0.051534	0.051477	0.051213
		1	1.01	0.052345	0.052634	0.052535	0.052076
		1.025	1.025	0.05636	0.05718	0.05689	0.05559
		1	1.1	0.0616	0.0628	0.0625	0.0604

TABLE 2—Continued

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$n = 15$			
0	0.1	1	1.001	0.052022	0.052066	0.052054	0.051798
				0.050130	0.050134	0.050133	0.050115
				0.051887	0.051928	0.051917	0.051679
		1	1.01	0.053245	0.053317	0.053298	0.052881
		1.025	1.025	0.05887	0.05909	0.05903	0.05782
		1	1.1	0.0665	0.0666	0.0667	0.0648
				$n = 40$			
0	0.1	1	1.001	0.052311	0.052318	0.052317	0.052085
				0.050149	0.050150	0.050150	0.050134
				0.052157	0.052164	0.052162	0.051947
		1	1.01	0.053713	0.053725	0.053724	0.053343
		1.025	1.025	0.05019	0.06024	0.06023	0.05910
		1	1.1	0.0691	0.0690	0.0690	0.0675
				$n = 5$			
0	1	1	1.001	0.06464	0.06596	0.06564	0.06309
				0.05010	0.05011	0.05010	0.05008
				0.064523	0.065823	0.065504	0.06299
		1	1.01	0.06570	0.06716	0.06680	0.06401
		1.025	1.025	0.0705	0.0727	0.0721	0.0682
		1	1.1	0.077	0.080	0.079	0.074
				$n = 15$			
0	1	1	1.001	0.07067	0.07075	0.07084	0.06870
				0.05013	0.05013	0.05013	0.05012
				0.070496	0.070579	0.070666	0.06855
		1	1.01	0.07221	0.07233	0.07241	0.07004
		1.025	1.025	0.0793	0.0796	0.0796	0.0763
		1	1.1	0.089	0.089	0.089	0.085
				$n = 40$			
0	1	1	1.001	0.07387	0.07375	0.07384	0.07203
				0.05012	0.05015	0.05015	0.05013
				0.073667	0.073556	0.073641	0.07185
		1	1.01	0.07568	0.07556	0.07565	0.07362
		1.025	1.025	0.0840	0.0839	0.0840	0.0809
		1	1.1	0.095	0.095	0.095	0.091
				$n = 5$			
1	1	1	1.001	0.0809	0.0861	0.0840	0.0764
				0.0501	0.0501	0.0501	0.0501
				0.0808	0.0858	0.0839	0.0763
		1	1.01	0.0822	0.0875	0.0854	0.0775
		1.025	1.025	0.088	0.094	0.092	0.083
		1	1.05	0.088	0.094	0.092	0.083
		1	1.1	0.096	0.099	0.099	0.090
				$n = 15$			
1	1	1	1.001	0.0947	0.0964	0.0957	0.0878
				0.0501	0.0501	0.0501	0.0501
				0.0944	0.0961	0.0955	0.0876
		1	1.01	0.0966	0.0984	0.0976	0.0895
		1.025	1.025	0.105	0.107	0.106	0.097
		1	1.1	0.117	0.117	0.117	0.108

TABLE 2—Continued

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$n = 40$			
1	1	1	1.001	0.1021	0.1026	0.1023	0.0946
				0.0501	0.0502	0.0502	0.0501
				0.1018	0.1023	0.1021	0.0944
		1	1.01	0.1044	0.1049	0.1046	0.0966
		1.025	1.025	0.115	0.115	0.115	0.106
		1	1.1	0.129	0.129	0.120	0.118

Entries in 2nd row denote powers of the test $H_0: \Sigma_1 = \Sigma_2$ assuming $\Omega = 0$
Entries in 3rd row denote powers of the test $H_0: \Omega = 0$ assuming $\Sigma_1 = \Sigma_2$

APPENDIX A

The constants $b_\kappa(r, s)$ up to $k = 6$ for the cdf of the criteria in the two-roots case:

TABLE 3
$b_k(r, s)$ constants

κ	r	s = 0	1	2
(1)	1	1		
(2)	2	1		
	0		$-\frac{4}{3}$	
(1^2)	0		$\frac{4}{3}$	
(3)	3	1		
	1		$-\frac{12}{5}$	
(21)	1		$\frac{12}{5}$	
(4)	4	1		
	2		$-\frac{24}{7}$	
	0			$\frac{48}{35}$
(31)	2		$\frac{24}{7}$	
	0			$-\frac{32}{7}$
(2^2)	0			$\frac{16}{5}$

κ	r	s = 0	1	2	3
(5)	5	1			
	3		$-\frac{40}{9}$		
	1			$\frac{80}{21}$	
(41)	3		$\frac{40}{9}$		
	1			$-\frac{32}{3}$	
(32)	1			$\frac{48}{7}$	
(6)	6	1			
	4		$-\frac{60}{11}$		
	2			$\frac{80}{11}$	
	0				$-\frac{320}{231}$
(51)	4		$\frac{60}{11}$		
	2			$-\frac{1440}{77}$	
	0				$\frac{576}{77}$
(42)	2			$\frac{80}{7}$	
	0				$-\frac{320}{21}$
(3^2)	0				$\frac{64}{7}$

APPENDIX B

The coefficients D_{ij} for the cdf of the criteria in the two-roots case:

$$D_{00} = 1 + A_{11} + 3A_{21} + 5A_{31} + 35A_{41} + 63A_{51} + 231A_{61};$$

$$D_{01} = A_{11} + 6A_{21} + 15A_{31} + 140A_{41} + 315A_{51} + 1386A_{61};$$

$$D_{11} = A_{11} + 2A_{21} + A_{22} + 3A_{31} + A_{32} + 20A_{41} + 3A_{42} + 35A_{51} + 5A_{52} + 126A_{61} + 35A_{62};$$

$$D_{02} = 3A_{21} + 15A_{31} + 210A_{41} + 630A_{51} + 3465A_{61};$$

$$D_{12} = 6A_{21} + 18A_{31} + A_{32} + 180A_{41} + 6A_{42} + 420A_{51} + 15A_{52} + 1890A_{61} + 140A_{62};$$

$$D_{22} = 3A_{21} + 3A_{31} + A_{32} + 18A_{41} + 2A_{42} + A_{43} + 30A_{51} + 3A_{52} + A_{53} + 105A_{61} + 20A_{62} + 3A_{63};$$

$$D_{03} = 5A_{31} + 140A_{41} + 630A_{51} + 4620A_{61};$$

$$D_{13} = 15A_{31} + 300A_{41} + 3A_{42} + 1050A_{51} + 15A_{52} + 6300A_{61} + 210A_{62};$$

$$D_{23} = 15A_{31} + 180A_{41} + 6A_{42} + 450A_{51} + 18A_{52} + A_{53} + 2100A_{61} + 180A_{62} + 6A_{63};$$

$$D_{33} = 5A_{31} + 20A_{41} + 3A_{42} + 30A_{51} + 3A_{52} + A_{53} + 100A_{61} + 18A_{62} + 2A_{63} + A_{64};$$

$$D_{04} = 35A_{41} + 315A_{51} + 3465A_{61};$$

$$D_{14} = 140A_{41} + 980A_{51} + 5A_{52} + 8820A_{61} + 140A_{62};$$

$$D_{24} = 210A_{41} + 1050A_{51} + 15A_{52} + 7350A_{61} + 300A_{62} + 3A_{63};$$

$$D_{34} = 140A_{41} + 420A_{51} + 15A_{52} + 2100A_{61} + 180A_{62} + 6A_{63};$$

$$D_{44} = 35A_{41} + 35A_{51} + 5A_{52} + 105A_{61} + 20A_{62} + 3A_{63};$$

$$D_{05} = 63A_{51} + 1386A_{61}; \qquad D_{15} = 315A_{51} + 5670A_{61} + 35A_{62};$$

$$D_{25} = 630A_{51} + 8820A_{61} + 140A_{62}; \qquad D_{06} = D_{66} = 231A_{61};$$

$$D_{35} = 630A_{51} + 6300A_{61} + 210A_{62}; \qquad D_{16} = D_{56} = 1386A_{61};$$

$$D_{45} = 315A_{51} + 1890A_{61} + 140A_{62}; \qquad D_{26} = D_{46} = 3465A_{61};$$

$$D_{55} = 63A_{51} + 126A_{61} + 35A_{62}; \qquad D_{36} = 4620A_{61};$$

where the coefficients A_{ij} are given by (after letting $m_i = \nu + 2i$ and $n_i' = \nu - i$)

$$A_{11} = \tfrac{1}{2}m_0(1 + B_{11}); \qquad A_{21} = \frac{m_0 m_1}{24}(1 + 2B_{11} + B_{22})$$

$$A_{22} = \frac{m_0 n_1'}{6}(1 + 2B_{11} + B_{21}); \qquad A_{31} = \frac{m_0 m_1 m_2}{240}(1 + 3B_{11} + 3B_{22} + B_{33})$$

$$A_{32} = \frac{m_0 m_1 n_1'}{20}(1 + 3B_{11} + \tfrac{4}{3}B_{22} + \tfrac{5}{3}B_{21} + B_{32});$$

$$A_{41} = \frac{m_0 m_1 m_2 m_3}{13440}(1 + 4B_{11} + 6B_{22} + 4B_{33} + B_{44});$$

$$A_{42} = \frac{m_0 m_1 m_2 n_1'}{336}(1 + 4B_{11} + \tfrac{11}{3}B_{22} + \tfrac{7}{3}B_{21} + \tfrac{6}{5}B_{33} + \tfrac{14}{5}B_{32} + B_{43});$$

$$A_{43} = \frac{m_0 m_1 n_1'(n_1' + 2)}{120}(1 + 4B_{11} + \tfrac{8}{3}B_{22} + \tfrac{10}{3}B_{21} + 4B_{32} + B_{42});$$

$$A_{51} = \frac{m_0 m_1 m_2 m_3 m_4}{241920}(1 + 5B_{11} + 10B_{22} + 10B_{33} + 5B_{44} + B_{55});$$

$$A_{52} = \frac{m_0 m_1 m_2 m_3 n_1'}{4320}(1 + 5B_{11} + 7B_{22} + 3B_{21} + \tfrac{23}{5}B_{33} + \tfrac{27}{5}B_{32} + \tfrac{8}{7}B_{44} + \tfrac{27}{7}B_{43} + B_{54});$$

$$A_{53} = \frac{m_0 m_1 m_2 n_1'(n_1'+2)}{560}(1 + 5B_{11} + \tfrac{16}{3}B_{22} + \tfrac{14}{3}B_{21} + \tfrac{8}{5}B_{33} + \tfrac{42}{5}B_{32} + \tfrac{8}{3}B_{43} + \tfrac{7}{3}B_{42} + B_{53})\,;$$

$$A_{61} = \frac{m_0 m_1 \cdots m_5}{(14784)(720)}(1 + 6B_{11} + 15B_{22} + 20B_{33} + 15B_{44} + 6B_{55} + B_{66})\,;$$

$$A_{62} = \frac{m_0 m_1 \cdots m_4 n_1'}{295680}(1 + 6B_{11} + \tfrac{34}{3}B_{22} + \tfrac{11}{3}B_{21} + \tfrac{56}{5}B_{33} + \tfrac{44}{5}B_{32} + \tfrac{39}{7}B_{44} + \tfrac{66}{7}B_{43} + \tfrac{10}{9}B_{55} + \tfrac{44}{9}B_{54} + B_{65})$$

$$A_{63} = \frac{m_0 m_1 m_2 m_3 n_1'(n_1'+2)}{12096}(1 + 6B_{11} + 9B_{22} + 6B_{21} + \tfrac{28}{5}B_{33} + \tfrac{72}{5}B_{32} + \tfrac{48}{35}B_{44} + \tfrac{66}{7}B_{43} + \tfrac{21}{5}B_{42} + \tfrac{12}{5}B_{54} + \tfrac{18}{5}B_{53} + B_{64})\,;$$

$$A_{64} = \frac{m_0 m_1 m_2 n_1'(n_1'+2)(n_1'+4)}{5040}(1 + 6B_{11} + 8B_{22} + 7B_{21} + \tfrac{16}{5}B_{33} + \tfrac{84}{5}B_{32} + 8B_{43} + 7B_{42} + 6B_{53} + B_{63})\,;$$

while the values of B_{ij} are given by (after letting $r_i = n_1 + 2i$ and $s_i = n_1 - i$)

$$B_{11} = -\tfrac{1}{2}a_1\left(1 - \frac{b_1}{r_0}\right); \qquad B_{22} = \frac{a_{21}}{8}\left(1 - \frac{2b_1}{r_0} + \frac{b_{21}}{2r_0r_1}\right);$$

$$B_{21} = a_{22}\left(1 - \frac{2b_1}{r_0} + \frac{4b_{22}}{r_0s_1}\right); \qquad B_{33} = \frac{-a_{31}}{16}\left(1 - \frac{3b_1}{r_0} + \frac{3b_{21}}{2r_0r_1} - \frac{b_{31}}{2r_0r_1r_2}\right);$$

$$B_{32} = \frac{-a_{32}}{2}\left(1 - \frac{3b_1}{r_0} + \frac{2b_{21}}{3r_0r_1} + \frac{20b_{22}}{3r_0s_1} - \frac{4b_{32}}{r_0r_1s_1}\right);$$

$$B_{44} = \frac{a_{41}}{128}\left(1 - \frac{4b_1}{r_0} + \frac{3b_{21}}{r_0r_1} - \frac{2b_{31}}{r_0r_1r_2} + \frac{b_{41}}{8r_0r_1r_2r_3}\right);$$

$$B_{43} = \frac{a_{42}}{8}\left(1 - \frac{4b_1}{r_0} + \frac{11b_{21}}{6r_0r_1} + \frac{28b_{22}}{3r_0s_1} - \frac{3b_{31}}{5r_0r_1r_2} - \frac{56b_{32}}{5r_0r_1s_2} + \frac{2b_{42}}{r_0r_1r_2s_1}\right);$$

$$B_{42} = a_{43}\left(1 - \frac{4b_1}{r_0} + \frac{4b_{21}}{3r_0s_1} + \frac{40b_{22}}{3r_0s_1} - \frac{16b_{32}}{r_0r_1s_1} + \frac{16b_{43}}{r_0r_1s_1(s_1+2)}\right);$$

$$B_{55} = \frac{-a_{51}}{256}\left(1 - \frac{5b_1}{r_0} + \frac{5b_{21}}{r_0r_1} - \frac{5b_{31}}{r_0r_1r_2} + \frac{5b_{41}}{8r_0r_1r_2r_3} - \frac{b_{51}}{8r_0r_1\cdots r_4}\right);$$

$$B_{54} = \frac{-a_{52}}{16}\left(1 - \frac{5b_1}{r_0} + \frac{7b_{21}}{2r_0r_1} + \frac{12b_{22}}{r_0s_1} - \frac{23b_{31}}{10r_0r_1r_2} - \frac{108b_{32}}{5r_0r_1s_1} + \frac{b_{41}}{7r_0r_1r_2r_3} + \frac{54b_{42}}{7r_0r_1r_2s_1} - \frac{2b_{52}}{r_0r_1r_2r_3s_1}\right);$$

$$B_{53} = \frac{-a_{53}}{2}\left(1 - \frac{5b_1}{r_0} + \frac{8b_{21}}{3r_0r_1} + \frac{56b_{22}}{3r_0s_1} - \frac{4b_{31}}{5r_0r_1r_2} - \frac{168b_{32}}{5r_0r_1s_1} + \frac{16b_{42}}{3r_0r_1r_2s_1} + \frac{112b_{43}}{3r_0r_1s_1(s_1+2)} - \frac{16b_{53}}{r_0r_1r_2s_1(s_1+2)}\right);$$

$$B_{66} = \frac{a_{61}}{1024}\left(1 - \frac{6b_1}{r_0} + \frac{15b_{21}}{2r_0r_1} - \frac{10b_{31}}{r_0r_1r_2} + \frac{15b_{41}}{8r_0r_1r_2r_3} - \frac{3b_{51}}{4r_0r_1\cdots r_4} + \frac{b_{61}}{16r_0\cdots r_5}\right);$$

$$B_{65} = \frac{a_{62}}{128}\left(1 - \frac{6b_1}{r_0} + \frac{17b_{21}}{3r_0r_1} + \frac{44b_{22}}{3r_0s_1} - \frac{28b_{31}}{5r_0r_1r_2} - \frac{176b_{32}}{5r_0r_1s_1} + \frac{39b_{41}}{56r_0r_1r_2r_3} + \frac{132b_{42}}{7r_0r_1r_2s_1} - \frac{5b_{51}}{36r_0r_1\cdots r_4} - \frac{88b_{52}}{9r_0r_1r_2r_3s_1} + \frac{b_{62}}{2r_0r_1r_2r_3r_4s_1}\right);$$

$$B_{64} = \frac{a_{63}}{8}\left(1 - \frac{6b_1}{r_0} + \frac{9b_{21}}{2r_0r_1} + \frac{24b_{22}}{r_0s_1} - \frac{14b_{31}}{5r_0r_1r_2} - \frac{288b_{32}}{5r_0r_1s_1} + \frac{6b_{41}}{35r_0r_1r_2r_3} + \frac{132b_{42}}{7r_0r_1r_2s_1} + \frac{336b_{43}}{5r_0r_1s_1(s_1+2)} - \frac{24b_{52}}{5r_0r_1r_2r_3s_1} - \frac{288b_{53}}{5r_0r_1r_2s_1(s_1+2)} + \frac{8b_{63}}{r_0r_1r_2r_3s_1(s_1+2)}\right);$$

$$B_{63} = a_{64}\left(1 - \frac{6b_1}{r_0} + \frac{4b_{21}}{r_0r_1} + \frac{28b_{22}}{r_0s_1} - \frac{8b_{31}}{5r_0r_1r_2} - \frac{336b_{32}}{5r_0r_1s_1} + \frac{16b_{42}}{r_0r_1r_2s_1} + \frac{112b_{43}}{r_0r_1s_1(s_1+2)} - \frac{96b_{53}}{r_0r_1r_2s_1(s_1+2)} + \frac{64b_{64}}{r_0r_1r_2s_1(s_1+2)(s_1+4)}\right);$$

with $a_1 = \lambda_1^{-1} + \lambda_2^{-1}$, $a_2 = (\lambda_1\lambda_2)^{-1}$, and

$a_{21} = 3a_1^2 - 4a_2$; $\quad a_{22} = a_2$; $\quad a_{31} = 5a_1^3 - 12a_1a_2$; $\quad a_{32} = a_1a_2$;

$a_{41} = 35a_1^4 - 120a_1^2a_2 + 48a_2^2$; $\quad a_{42} = 3a_1^2a_2 - 4a_2^2$; $\quad a_{43} = a_2^2$;

$a_{51} = 63a_1^5 - 280a_1^3a_2 + 240a_1a_2^2$; $\quad a_{52} = 5a_1^3a_2 - 12a_1a_2^2$; $\quad a_{53} = a_1a_2^2$;

$a_{61} = 231a_1^6 - 1260a_1^4a_2 + 1680a_1^2a_2^2 - 320a_2^3$;

$a_{62} = 35a_1^4a_2 - 120a_1^2a_2^2 + 48a_2^3$; $\quad a_{63} = 3a_1^2a_2^2 - 4a_2^3$; $\quad a_{64} = a_2^3$;

and $b_1 = w_1 + w_2$, $b_2 = w_1w_2$ and b_{ij}'s are obtained from corresponding a_{ij}'s by replacing a_1 and a_2 by b_1 and b_2 respectively.

APPENDIX C

E_{ij} coefficients:

$E_{00} = 1$ $\quad\quad E_{01} = A_{22} - 4A_{21}$

$E_{10} = A_{11}$ $\quad\quad E_{11} = A_{32} - 12A_{31}$

$E_{20} = 3A_{21}$ $\quad\quad E_{21} = 3A_{42} - 120A_{41}$

$E_{30} = 5A_{31}$ $\quad\quad E_{31} = 5A_{52} - 280A_{51}$

$E_{40} = 35A_{41}$ $\quad\quad E_{41} = 35A_{62} - 1260A_{61}$

$E_{50} = 63A_{51}$ $\quad\quad E_{02} = A_{43} - 4A_{42} + 48A_{41}$

$E_{60} = 231A_{61}$ $\quad\quad E_{12} = A_{53} - 12A_{52} + 240A_{51}$

$E_{22} = 3A_{63} - 120A_{62} + 1680A_{61}$ and $E_{03} = A_{64} - 4A_{63} + 48A_{62} - 320A_{61}$,

where the A_{ij}'s are given in Appendix B.

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DEPARTMENT OF STATISTICS
PURDUE UNIVERSITY
LAFAYETTE, INDIANA 47907

DEPARTMENT OF STATISTICS
PAJAJARAN UNIVERSITY
BANDUNG, INDONESIA