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AN APPROXIMATE INVERSE FOR THE COVARIANCE MATRIX OF MOVING AVERAGE AND AUTOREGRESSIVE PROCESSES[1]

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Let $\mathbf{\Sigma}$ denote the covariance matrix of a vector $\mathbf{x} = (x_1, \cdots, x_T)'$ of T successive observations from a stationary process $\{x_t\}$ with continuous positive spectral density $f(\lambda)$. Let $\mathbf{\Gamma}$ be the $T \times T$ matrix with elements $\gamma(s, t) = (2\pi)^{-2} \int_{-\pi}^{\pi} e^{i\lambda(s-t)} f^{-1}(\lambda)\, d\lambda$. The properties of $\mathbf{\Gamma}$ considered as an approximate inverse of $\mathbf{\Sigma}$ are studied. When $\{x_t\}$ is a(n) moving average (autoregressive) process of order q, rows (columns) $q+1, \cdots, T-q$ of $\mathbf{\Sigma\Gamma} - \mathbf{I}$ are zero vectors. In this case $\mathbf{\Sigma\Gamma} - \mathbf{I}$ has $2q$ positive characteristic roots which approach paired positive limiting values as $T \to \infty$ if the roots of $\sum_{j=0}^{q} \beta_j z^{q-j} = 0$ are less than 1 in absolute value, where $\beta_1, \cdots, \beta_q$ are the coefficients of the process. Statistical properties of $\mathbf{x}'\mathbf{\Gamma x} - \mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x}$ and $\mathbf{x}'\mathbf{\Gamma x}/\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x}$ are also discussed.

1. Introduction. Let $\{x_t\}$, $t = 0, \pm 1, \cdots$, be a real-valued, second-order stationary process with mean zero and a continuous, positive spectral density $f(\lambda)$. The covariance function of the process is

$$Ex_t x_{t+h} = \sigma(h) = \int_{-\pi}^{\pi} e^{i\lambda h} f(\lambda)\, d\lambda\,, \qquad h = 0, \pm 1, \cdots.$$

Let $\mathbf{\Sigma}_T$ denote the covariance matrix of $\mathbf{x} = (x_1, \cdots, x_T)'$, consisting of T consecutive observations from $\{x_t\}$. We study the approximation to $\mathbf{\Sigma}_T^{-1}$ obtained by forming $\mathbf{\Gamma}_T$ with

$$(\mathbf{\Gamma}_T)_{s,t} = \gamma(s, t) = \gamma(s-t) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} e^{i\lambda(s-t)} f^{-1}(\lambda)\, d\lambda\,, \qquad s, t = 1, \cdots, T. \tag{1}$$

Clearly $\mathbf{\Gamma}_T$ is a covariance matrix for all T. The associated spectral density is $\{(2\pi)^2 f(\lambda)\}^{-1}$.

A class of stationary processes of particular interest is the class of autoregressive-moving average (ARMA) processes. Let $\{\varepsilon_t\}$ be a sequence of uncorrelated random variables with mean zero and variance v. The process $\{x_t\}$ defined by

$$\sum_{j=0}^{p} \alpha_j x_{t-j} = \sum_{k=0}^{q} \beta_k \varepsilon_{t-k}\,, \qquad t = 0, \pm 1, \cdots, \tag{2}$$

with $\alpha_0 = \beta_0 = 1$, is an ARMA process of order (p, q). If $q = 0$, $\{x_t\}$ is also called an autoregressive process of order p, and if $p = 0$ it is called a moving average process of order q. Let $A(z) = 1 + \alpha_1 z + \cdots + \alpha_p z^p$ and $B(z) =$

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$1 + \beta_1 z + \cdots + \beta_q z^q$. The spectral density of $\{x_t\}$ defined by (2) is $f(\lambda) = \{v/(2\pi)\}|B(e^{i\lambda})|^2/|A(e^{i\lambda})|^2$, $-\pi \leqq \lambda \leqq \pi$.

Inversion of $\mathbf{\Sigma}_T$ is of interest when $\{x_t\}$ is a Gaussian process and statistical inference is to be based on the observed time series. In the literature inference is essentially based upon a modified likelihood function obtained by approximating $\mathbf{\Sigma}_T^{-1}$.

Whittle (1953), (1954, Sections 2.2, 2.5), Durbin (1959), and Walker (1964) have used the approximation $\mathbf{\Gamma}_T$ defined in (1) to treat estimation in ARMA models and in models where the spectral density depends upon a finite number of unknown parameters. The quadratic form $\mathbf{x}'\mathbf{\Gamma}_T\mathbf{x}$ is

$$\frac{T}{2\pi} \int_{-\pi}^{\pi} I(\lambda) f^{-1}(\lambda)\, d\lambda ,$$

where $I(\lambda)$ is the periodogram. Hannan (1969) modified $\mathbf{\Gamma}_T$ by replacing an integral by an approximating sum. Anderson (1969, 1970, 1971 b) has developed maximum likelihood estimation in a class of models which includes the ARMA $(0, q)$ processes and modifications of ARMA $(p, 0)$ processes. In Anderson (1971 b) $\mathbf{\Sigma}_T^{-1}$ is approximated by $\mathbf{\Gamma}_T$. The sequence $\{(2\pi)^2\gamma(h)\}$ has also been studied by Cleveland (1972).

The exact inverse when $\{x_t\}$ is an autoregressive process was given implicitly by Champernowne (1948, page 206) and explicitly by Siddiqui (1958). See also Wise (1955). For a first-order moving average process $\mathbf{\Sigma}_T^{-1}$ was given by Uppuluri and Carpenter (1969) and Shaman (1969). (See also Lovass-Nagy and Powers (1969) and Kershaw (1969).) Tiao and Ali (1971) give the inverse for an ARMA (1, 1) process, and Mentz (1972) and Shaman (1973) give techniques which can be used to construct the inverse for processes of order $(0, q)$.

2. Preliminaries. In this section we motivate the use of (1) to approximate $\mathbf{\Sigma}_T^{-1}$. We shall drop the subscript T from $\mathbf{\Sigma}_T$ and $\mathbf{\Gamma}_T$.

If both $\mathbf{\Sigma}$ and $\mathbf{\Gamma}$ are taken to be infinite dimensional matrices, then $\mathbf{\Sigma}^{-1} = \mathbf{\Gamma}$; that is,

$$\sum_{r=-\infty}^{\infty} \sigma(s-r)\gamma(r-t) = \delta(s-t) , \qquad s, t = \cdots, -1, 0, 1, \cdots , \tag{3}$$

where $\delta(0) = 1$ and $\delta(r) = 0$, $r = \pm 1, \pm 2, \cdots$. This follows because the left side of (3) gives the convolution of two covariance sequences. The Fourier transform of this convolution is 2π times the product of the corresponding spectral densities. Therefore the sum in (3) is the covariance function of uncorrelated random variables with variance 1.

The element in row s and column t of $\mathbf{\Sigma\Gamma}$ is, by (3),

$$\begin{aligned} \sum_{r=1}^{T} \sigma(s-r)\gamma(r-t) &= \delta(s-t) - \sum_{r=-\infty}^{0} \sigma(s-r)\gamma(r-t) - \sum_{r=T+1}^{\infty} \sigma(s-r)\gamma(r-t) \\ &= \delta(s-t) - \sum_{j=s}^{\infty} \sigma(j)\gamma(s-t-j) - \sum_{j=-\infty}^{s-T-1} \sigma(j)\gamma(s-t-j) \\ &= \delta(s-t) - \sum_{j=-\infty}^{-t} \sigma(s-t-j)\gamma(j) - \sum_{j=T+1-t}^{\infty} \sigma(s-t-j)\gamma(j) , \\ & \qquad s, t = 1, \cdots, T . \end{aligned} \tag{4}$$

Therefore, if $\{x_t\}$ is a(n) moving average (autoregressive) process of order q rows (columns) $q+1, \cdots, T-q$ of $\boldsymbol{\Sigma\Gamma}$ have elements $\delta(s-t)$. This result was noted in Shaman (1969) for the moving average case with $q = 1$.

We consider

$$d = \mathbf{x}'\boldsymbol{\Gamma}\mathbf{x} - \mathbf{x}'\boldsymbol{\Sigma}^{-1}\mathbf{x} \tag{5}$$

and

$$r = \mathbf{x}'\boldsymbol{\Gamma}\mathbf{x}/\mathbf{x}'\boldsymbol{\Sigma}^{-1}\mathbf{x} \tag{6}$$

and their moments when $\{x_t\}$ is Gaussian.

The difference d can be written in the canonical form

$$d = \sum_{t=1}^{T} \nu_t z_t^2 , \tag{7}$$

where $\nu_1, \cdots, \nu_T$ are the characteristic roots of $\boldsymbol{\Sigma\Gamma} - \mathbf{I}$ and $z_1, \cdots, z_T$ are independent and $N(0, 1)$. The ratio r in canonical form is

$$r = \frac{\sum_{t=1}^{T} (1 + \nu_t) z_t^2}{\sum_{t=1}^{T} z_t^2} . \tag{8}$$

Since $\boldsymbol{\Gamma}$ and $\boldsymbol{\Sigma}$ are positive definite, $\nu_t > -1$, $t = 1, \cdots, T$.

3. The approximate inverse for moving average and autoregressive processes. Comparison of $\boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{\Gamma}$ is of particular interest when $\{x_t\}$ is a moving average or an autoregressive process of order q. Let $\boldsymbol{\Sigma}_{\mathrm{MA}} = (\sigma_{\mathrm{MA}}(s, t))$ denote the $T \times T$ covariance matrix of the ARMA $(0, q)$ process specified by (2) with $\beta_q \neq 0$. Then

$$\begin{aligned} \sigma_{\mathrm{MA}}(s, t) &= \sigma_{\mathrm{MA}}(s - t) \\ &= v \sum_{j=0}^{q-|s-t|} \beta_j \beta_{j+|s-t|} , \qquad & |s-t| = 0, 1, \cdots, q , \\ &= 0 , & |s-t| = q+1, q+2, \cdots . \end{aligned} \tag{9}$$

Moreover, let $\boldsymbol{\Sigma}_{\mathrm{AR}} = (\sigma_{\mathrm{AR}}(s, t))$ denote the $T \times T$ covariance matrix of the ARMA $(q, 0)$ process given by (2) with $\beta_0, \cdots, \beta_q$ in place of $\alpha_0, \cdots, \alpha_q$. Then if $T - 2q > 0$ the elements of $\boldsymbol{\Sigma}_{\mathrm{AR}}^{-1} = (\sigma^{\mathrm{AR}}(s, t))$ are (see Siddiqui (1958))

$$\begin{aligned} \sigma^{\mathrm{AR}}(s, t) &= \sigma^{\mathrm{AR}}(T+1-t, T+1-s) \\ &= \frac{1}{v} \sum_{j=0}^{\min(s,t)-1} \beta_j \beta_{j+|s-t|} , & s, t = 1, \cdots, q , \\ &= \frac{1}{v} \sum_{j=0}^{q-|s-t|} \beta_j \beta_{j+|s-t|}, \ \max(s, t) > q, \ \min(s, t) \leqq T - q , \\ & & |s-t| = 0, 1, \cdots, q , \\ &= 0 , & |s-t| = q+1, q+2, \cdots . \end{aligned} \tag{10}$$

If $\boldsymbol{\Sigma}$ is chosen to be $\boldsymbol{\Sigma}_{\mathrm{MA}}$, then $\boldsymbol{\Gamma}$ is $v^{-2}\boldsymbol{\Sigma}_{\mathrm{AR}}$, by the discussion following (2).

Inspection of (9) and (10) reveals that $v^{-1}\boldsymbol{\Sigma}_{\mathrm{MA}}$ and $v\boldsymbol{\Sigma}_{\mathrm{AR}}^{-1}$ are identical except for the $q \times q$ submatrices in the upper left and lower right corners. Specifically,

$v^{-1}\mathbf{\Sigma}_{\text{MA}} = v\mathbf{\Sigma}_{\text{AR}}^{-1} + \mathbf{E}$, with $\mathbf{E} = (e_{st})$ given by

$$
\begin{aligned}
e_{st} &= e_{T+1-t,T+1-s} \\
&= \textstyle\sum_{j=\min(s,t)}^{q-|s-t|} \beta_j \beta_{j+|s-t|}\,, \qquad & s, t = 1, \cdots, q, \\
&= 0\,, & \max(s, t) > q,\ \min(s, t) \leqq T - q\,.
\end{aligned} \tag{11}
$$

Then $v^{-2}\mathbf{\Sigma}_{\text{MA}}\mathbf{\Sigma}_{\text{AR}} = \mathbf{I} + v^{-1}\mathbf{E}\mathbf{\Sigma}_{\text{AR}} = \mathbf{I} + \mathbf{A}$ and $\mathbf{A} = (a_{st})$ is

$$
\begin{aligned}
a_{st} &= a_{T+1-s,T+1-t} \\
&= v^{-1} \textstyle\sum_{r=1}^{q} \sum_{j=\min(s,r)}^{q-|s-r|} \beta_j \beta_{j+|s-r|}\sigma_{\text{AR}}(r - t)\,, \qquad & s = 1, \cdots, q\,, \\
&= 0\,, & s = q + 1, \cdots, T - q\,.
\end{aligned} \tag{12}
$$

In a similar fashion to the above we can consider $v^2\mathbf{\Sigma}_{\text{AR}}^{-1}\mathbf{\Sigma}_{\text{MA}}^{-1}$. We have $v^2\mathbf{\Sigma}_{\text{AR}}^{-1}\mathbf{\Sigma}_{\text{MA}}^{-1} = \mathbf{I} - v\mathbf{E}\mathbf{\Sigma}_{\text{MA}}^{-1} = \mathbf{I} + \mathbf{B}$ and rows $q + 1, \cdots, T - q$ of $\mathbf{B}$ are $\mathbf{0}$.

That $\mathbf{\Sigma}_{\text{MA}}^{-1}$ is approximated by $v^{-2}\mathbf{\Sigma}_{\text{AR}}$ was noted by Anderson (1971 b, Section 3).

We study d and r when $\mathbf{\Sigma}$ is $\mathbf{\Sigma}_{\text{MA}}$ or $\mathbf{\Sigma}_{\text{AR}}$. Then in (7) $\nu_1, \cdots, \nu_T$ are the roots of $v^{-2}\mathbf{\Sigma}_{\text{MA}}\mathbf{\Sigma}_{\text{AR}} - \mathbf{I} = \mathbf{A}$. Since rows $q + 1, \cdots, T - q$ of $\mathbf{A}$ are $\mathbf{0}$, $2q$ of the roots are zero. Designate the remaining roots by $\nu_1, \cdots, \nu_{2q}$. It was noted below (8) that the roots $\nu_1, \cdots, \nu_{2q}$ are > -1. When $\mathbf{\Sigma}$ is $\mathbf{\Sigma}_{\text{MA}}$ or $\mathbf{\Sigma}_{\text{AR}}$ a more precise result is easily derived.

LEMMA. *If $\beta_q \neq 0$ the nonzero roots of $v^{-2}\mathbf{\Sigma}_{\text{MA}}\mathbf{\Sigma}_{\text{AR}} - \mathbf{I} = \mathbf{A}$ are positive.*

PROOF. When $\{x_t\}$ is the ARMA $(q, 0)$ process

$$
\begin{aligned}
d &= v^{-2}\mathbf{x}'\mathbf{\Sigma}_{\text{MA}}\mathbf{x} - \mathbf{x}'\mathbf{\Sigma}_{\text{AR}}^{-1}\mathbf{x} \\
&= v^{-1}\mathbf{x}'\mathbf{E}\mathbf{x} \\
&= v^{-1} \textstyle\sum_{s,t=1}^{q} e_{st}(x_s x_t + x_{T+1-s}x_{T+1-t}) \\
&= v^{-1} \textstyle\sum_{s,t=1}^{q} \sum_{j=\min(s,t)}^{q-|s-t|} \beta_j \beta_{j+|s-t|}(x_s x_t + x_{T+1-s}x_{T+1-t})\,,
\end{aligned} \tag{13}
$$

where the last two lines follow from (11). By algebraic manipulation the last line of (13) can be rewritten to give

$$d = v^{-1} \textstyle\sum_{s=0}^{q-1} \{(\sum_{t=1}^{q-s} \beta_{s+t}x_t)^2 + (\sum_{t=1}^{q-s} \beta_{s+t}x_{T+1-t})^2\}\,.$$

If $\beta_q \neq 0$, this expression is positive unless $x_t = 0$, $t = 1, \cdots, q, T - q + 1, \cdots, T$.

The lemma implies that the $2q$ nonzero roots of $v^2\mathbf{\Sigma}_{\text{AR}}^{-1}\mathbf{\Sigma}_{\text{MA}}^{-1} - \mathbf{I} = \mathbf{B}$ are negative.

THEOREM 1. *Let $\{x_t\}$ be a Gaussian* ARMA $(0, q)$ *process with $\beta_q \neq 0$ and $T \times T$ covariance matrix $\mathbf{\Sigma}_{\text{MA}}$. Let $\mathbf{\Sigma}_{\text{AR}}$ denote the $T \times T$ covariance matrix of the* ARMA $(q, 0)$ *process with the same coefficients and $z_1, \cdots, z_T$ be independent and $N(0, 1)$. Then $d = v^{-2}\mathbf{x}'\mathbf{\Sigma}_{\text{AR}}\mathbf{x} - \mathbf{x}'\mathbf{\Sigma}_{\text{MA}}^{-1}\mathbf{x}$ has the distribution of $\sum_{t=1}^{2q} \nu_t z_t^2$, where $\nu_1, \cdots, \nu_{2q}$, the nonzero roots of $v^{-2}\mathbf{\Sigma}_{\text{MA}}\mathbf{\Sigma}_{\text{AR}} - \mathbf{I}$, are positive. The distribution of $r = v^{-2}\mathbf{x}'\mathbf{\Sigma}_{\text{AR}}\mathbf{x}/\mathbf{x}'\mathbf{\Sigma}_{\text{MA}}^{-1}\mathbf{x}$ is that of $1 + d/\sum_{t=1}^{T} z_t^2$. The same results hold if $\{x_t\}$ is a Gaussian* ARMA $(q, 0)$ *process and $d = v^{-2}\mathbf{x}'\mathbf{\Sigma}_{\text{MA}}\mathbf{x} - \mathbf{x}'\mathbf{\Sigma}_{\text{AR}}^{-1}\mathbf{x}$, $r = v^{-2}\mathbf{x}'\mathbf{\Sigma}_{\text{MA}}\mathbf{x}/\mathbf{x}'\mathbf{\Sigma}_{\text{AR}}^{-1}\mathbf{x}$.*

We consider the question of whether d has a proper limiting distribution as $T \to \infty$. A sufficient condition is that the roots $\nu_1, \cdots, \nu_{2q}$ have finite limiting values, not all 0, as $T \to \infty$.

Assume the roots of

$$\sum_{j=0}^{q} \beta_j z^{q-j} = 0 \tag{14}$$

are all less than 1 in absolute value. The roots $\nu_1, \cdots, \nu_{2q}$ depend only on the elements of $\mathbf{A}$ in the $q \times q$ submatrices in the upper left and upper right corners ($a_{st} = a_{T+1-s,T+1-t}$). By (12) the $q \times q$ submatrices in the upper right and lower left corners of $\mathbf{A}$ involve $\sigma_{\mathrm{AR}}(r)$, $|r| \geqq T - q + 1$. Hence the elements in these submatrices tend to zero as $T \to \infty$. Thus the characteristic equation

$$|v^{-2}\mathbf{\Sigma}_{\mathrm{MA}}\mathbf{\Sigma}_{\mathrm{AR}} - \mathbf{I} - \nu\mathbf{I}| = |\mathbf{A} - \nu\mathbf{I}| = 0$$

is for large T approximately $(-\nu)^{T-2q}|\mathbf{A}_{11} - \nu\mathbf{I}|^2 = 0$, where $\mathbf{A}_{11}$ denotes the $q \times q$ submatrix in the upper left corner of $\mathbf{A} = v^{-1}\mathbf{E}\mathbf{\Sigma}_{\mathrm{AR}}$. From the proof of the lemma preceding Theorem 1 we see that the $q \times q$ submatrix in the upper left corner of E is positive definite if $\beta_q \neq 0$. Moreover, the $q \times q$ submatrix in the upper left corner of $\mathbf{\Sigma}_{\mathrm{AR}}$ is a covariance matrix and is positive definite when no root of (14) is on the circle $|z| = 1$. Therefore $\mathbf{A}_{11}$ is nonsingular and thus has no roots equal to 0. That is, as $T \to \infty$ the roots $\nu_1, \cdots, \nu_{2q}$ approach finite positive paired values. ($\mathbf{A}_{11}$ does not depend on T.) Also note T^{-1} times the denominator of $r - 1$ tends to 1 in probability as $T \to \infty$.

THEOREM 2. *Let the conditions of Theorem* 1 *hold and assume the roots of* (14) *are less than* 1 *in absolute value. Then as $T \to \infty$ the nonzero roots $\nu_1, \cdots, \nu_{2q}$ of $v^{-2}\mathbf{\Sigma}_{\mathrm{MA}}\mathbf{\Sigma}_{\mathrm{AR}} - \mathbf{I}$ have positive paired limiting values $\mu_1, \cdots, \mu_q$ and d and $T(r-1)$ have the limiting distribution $\sum_{t=1}^{q} \mu_t w_t$, where $w_1, \cdots, w_q$ are independent χ_2^2 random variables.*

We analyze d and r when $q = 1$. The issue of interest is the accuracy of the approximation of $\mathbf{\Sigma}_{\mathrm{MA}}^{-1}$ by $v^{-2}\mathbf{\Sigma}_{\mathrm{AR}}$ when $\{x_t\}$ is the Gaussian moving average process with $x_t = \varepsilon_t + \beta_1\varepsilon_{t-1}$, a special case of (2). We assume $|\beta_1| < 1$. References to published explicit expressions for $\mathbf{\Sigma}_{\mathrm{MA}}^{-1}$ are given in Section 1.

If $q = 1$ the characteristic equation $|\mathbf{A} - \nu\mathbf{I}| = 0$ has nonzero roots $a_{11} \pm a_{1T}$. They are both positive and designated by $0 < \nu_2 < \nu_1$. By (12)

$$\nu_1, \nu_2 = \frac{\beta_1^2}{1 - \beta_1^2}(1 \pm |\beta_1|^{T-1}) . \tag{15}$$

The exact distribution of $s = r - 1$ has been studied by von Neumann (1941). If $T = 2n$, the density of s is a polynomial of degree at most $n - 2$ in $0 < s < \nu_2$. The derivative of order $n - 1$ of the density in the interval $\nu_2 < s < \nu_1$ is (T even)

$$\frac{(1 - \beta_1^2)(-1)^{n-1}(n-1)!}{\pi s^{\frac{1}{2}T-1}[\beta_1^{2T+2} - \{(1 - \beta_1^2)s - \beta_1^2\}^2]^{\frac{1}{2}}} .$$

If T is odd the distribution can be derived from the distribution for $T - 1$. See Corollary 6.7.4 of Anderson (1971a), e.g.

The exact distribution of d is that of

$$\frac{\beta_1^2}{1-\beta_1^2}[z_1^2 + z_2^2 + |\beta_1|^{T-1}(z_1^2 - z_2^2)] .$$

When T is large and $|\beta_1|$ is not too close to 1, ν_1 and ν_2 are both approximately $\beta_1^2/(1-\beta_1^2)$. However, if T is large and $\beta_1 = 1 - c/T$, $c > 0$, then

$$\nu_1, \nu_2 \approx \frac{T(T-c)}{c(2T-c)}\left(1 - \frac{c}{T} \pm e^{-c}\right).$$

Therefore ν_1, ν_2, and $\nu_1 - \nu_2$ can assume rather large values.

Approximate distributions for d and s are obtained from Theorem 2, which states that both d and Ts are approximately $\beta_1^2/(1-\beta_1^2)$ times χ_2^2 if T is large.

For general q the roots ν_t, $t = 1, \cdots, 2q$, can assume large positive values if any of the roots of (14), all assumed to be inside the unit circle $|z| = 1$, lie close to the circle.

If $q = 2$ the equation $|\mathbf{A} - \nu\mathbf{I}| = 0$ is $(-\nu)^{T-4}$ times a fourth-degree polynomial in ν with coefficients that are functions of $a_{st}(s = 1, 2,\ t = 1, 2, T-1, T)$. We note the limiting values μ_1, μ_2 of the nonzero roots, specified by Theorem 2. They are the roots of

$$\begin{vmatrix} a_{11} - \nu & a_{12} \\ a_{21} & a_{22} - \nu \end{vmatrix} = 0 .$$

Therefore

$$\begin{aligned} \mu_1, \mu_2 &= \tfrac{1}{2}(a_{11} + a_{22}) \pm \tfrac{1}{2}\{(a_{11} - a_{22})^2 + 4a_{21}a_{12}\}^{\frac{1}{2}} \\ &= -\tfrac{1}{2}v^{-2}\{\sigma_{\mathrm{MA}}(1)\sigma_{\mathrm{AR}}(1) + 2\sigma_{\mathrm{MA}}(2)\sigma_{\mathrm{AR}}(2)\} \\ &\quad \pm \tfrac{1}{2}v^{-2}[\sigma_{\mathrm{MA}}^2(1)\sigma_{\mathrm{AR}}^2(1) + 4\sigma_{\mathrm{MA}}(2)\sigma_{\mathrm{AR}}(1)\{\sigma_{\mathrm{MA}}(1)\sigma_{\mathrm{AR}}(2) + \sigma_{\mathrm{MA}}(2)\sigma_{\mathrm{AR}}(3)\}]^{\frac{1}{2}} . \end{aligned}$$

Detailed computation gives

$$\mu_1, \mu_2 = \frac{\beta_1^2(1-\beta_2) + 2\beta_2^2(1+\beta_2) \pm \beta_1\{\beta_1^2(1-\beta_2)^2 + 4\beta_2^2\}^{\frac{1}{2}}}{2(1-\beta_2)\{(1+\beta_2)^2 - \beta_1^2\}} . \tag{16}$$

The region in the β_1, β_2 plane where the roots z_1, z_2 are less than 1 in absolute value is the interior of the triangle formed by the lines $\beta_2 = 1$, $\beta_2 = \beta_1 - 1$, and $\beta_2 = -\beta_1 - 1$. It is easy to verify that μ_1, μ_2 defined by (16) are positive inside this triangle, except along the line $\beta_2 = 0$ (see the lemma preceding Theorem 1).

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