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ON ADMISSIBILITY AND UNIFORM ADMISSIBILITY IN FINITE POPULATION SAMPLING

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Let p_0 be a given sampling design, and p be any sampling design with support contained in that of p_0. It is shown that if an estimator, e, of some finite population quantity is admissible with respect to p_0 it is also admissible with respect to p. Similarly if the pair (e, p_0) is uniformly admissible in the sense of Joshi (1966), then so is the pair (e, p).

1. Introduction. Following standard practice, we identify the elements of the finite population with the integers $1, 2, \cdots, N$, and denote the power set of $U = \{1, 2, \cdots, N\}$ by S. A sampling design is a discrete probability measure on S. Let x_i denote the value of some characteristic for the ith element and suppose that $\mathbf{x} = (x_1, \cdots, x_N) \in X$. We are interested in finding an estimator $e(s, \mathbf{x})$ (i.e. a real-valued function on $S \times X$ that depends on $\mathbf{x}$ only through those coordinates x_i with $i \in S$) for some real-valued quantity $\theta(\mathbf{x})$, given a real-valued loss function $L[e, \theta]$.

The definitions below follow Joshi (1966), Godambe (1969), and Ericson (1970).

DEFINITION 1.1. For a given sampling design p, an estimator e^* is said to *dominate* the estimator e if, for all $x \in X$,

$$\sum_s p(s)L[e^*(s, \mathbf{x}), \theta(\mathbf{x})] \leqq \sum_s p(s)L[e(s, \mathbf{x}), \theta(\mathbf{x})] \tag{1.1}$$

with strict inequality for at least one $\mathbf{x}$ in X. An estimator is said to be *p-admissible* if no other estimator dominates it.

DEFINITION 1.2. A pair (e^*, p^*) consisting of an estimator e^* and a sampling design p^* is said to *dominate* (e, p) *uniformly* if, for all $x \in X$,

$$\sum_s p^*(s)L[e^*(s, \mathbf{x}), \theta(\mathbf{x})] \leqq \sum_s p(s)L[e(s, x), \theta(\mathbf{x})] \tag{1.2}$$

with strict inequality for at least one $\mathbf{x}$ in X. A pair (e, p) is said to be *uniformly admissible* with respect to a class, C, of designs if $p \in C$ and no other pair (e^*, p^*) with $p^* \in C$ dominates (e, p). The definitions can be extended in the obvious way to include randomized estimators. The two classes of designs usually considered are $C_n = \{p \mid \sum_s p(s)n(s) = n\}$, where $n(s)$ is the cardinality of s, and $D_n = \{p \mid p(s) = 0 \text{ if } n(s) \neq n\}$.

In this paper the properties of p-admissibility and uniform admissibility are shown to be essentially independent of the design p. More precisely, if an

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estimator is p_0-admissible for some design p_0 then it is p-admissible for *any* design p that is absolutely continuous with respect to $p_0(p \ll p_0)$. Similarly, if the pair (e, p_0) is uniformly admissible for C_n(or D_n), then (e, p) is uniformly admissible for *any* p in $C_n(D_n)$ with $p \ll p_0$. Since $e(s, x)$ need not even be defined if $p_0(s) = 0$, it is obviously not possible to say anything in general about designs that give positive probability to samples with $p_0(s) = 0$.

2. Results. The proofs of both results follow by embedding the design p in p_0. Let $S_0 = \{s \in S \mid p_0(s) > 0\}$ be the support of p_0 and let $w = \min_{s \in S_0} p_0(s)/p(s)$. If $p \ll p_0$ then $p(s) = 0$ if $s \notin S_0$ and it follows that $0 < w \leqq 1$.

THEOREM 2.1. *If $e(s, \mathbf{x})$ is p_0-admissible and $p \ll p_0$, then $e(s, \mathbf{x})$ is p-admissible.*

PROOF. Suppose not. Then we can find an estimator $e^*(s, \mathbf{x})$ satisfying (1.1) for all $\mathbf{x} \in X$, with strict inequality for at least one $\mathbf{x}$ in X. Let $\Pi(s) = wp(s)/p_0(s)$ for any $s \in S_0$ and consider the (randomized) estimator

$$\begin{aligned} e'(s, x) &= e^*(s, \mathbf{x}) \qquad \text{with probability} \quad \Pi(s)\,. \\ &= e(s, \mathbf{x}) \qquad \text{with probability} \quad 1 - \Pi(s)\,. \end{aligned}$$

Now for any $\mathbf{x} \in X$, the expected loss for this estimator with design p_0 is

$$\begin{aligned} &\textstyle\sum_s p_0(s)\{\Pi(s)L[e^*(s, \mathbf{x}), \theta(\mathbf{x})] + (1 - \Pi(s))L[e(s, \mathbf{x}), \theta(\mathbf{x})]\} \\ &\quad = w \textstyle\sum_s p(s)L[e^*(s, \mathbf{x}), \theta(\mathbf{x})] + \sum_s p_0(s)(1 - \Pi(s))L[e(s, \mathbf{x}), \theta(\mathbf{x})] \\ &\quad \leqq w \textstyle\sum_s p(s)L[e(s, \mathbf{x}), \theta(\mathbf{x})] + \sum_s p_0(s)(1 - \Pi(s))L[e(s, \mathbf{x}), \theta(\mathbf{x})] \qquad \text{(by 1.1)} \\ &\quad = \textstyle\sum_s p_0(s)L[e(s, \mathbf{x}), \theta(\mathbf{x})] \end{aligned}$$

with strict inequality for at least one $\mathbf{x}$ in X. But this is impossible since e is p_0-admissible.

THEOREM 2.2. *If (e, p_0) is uniformly admissible* w.r.t. $C_n(D_n)$ *then (e, p) is uniformly admissible* w.r.t. $C_n(D_n)$ *for any $p \in C_n(D_n)$ with $p \ll p_0$.*

PROOF. Suppose not. Then we can find a pair (e^*, p^*) with $p^* \in C_n(D_n)$ satisfying (1.2) for all $\mathbf{x} \in X$ with strict inequality for at least one $\mathbf{x}$ in X. Let

$$p'(s) = p_0(s) + w(p^*(s) - p(s))\,.$$

Since $p_0(s) - wp(s) \geqq 0$, $p'(s) > 0$ for $s \in S$ and $\sum_s p'(s) = 1$. Thus p' is a sampling design. Moreover $p' \in C_n(D_n)$.
Let $\Pi(s) = wp^*(s)/p'(s)$ and define the randomized estimator

$$\begin{aligned} e'(s, \mathbf{x}) &= e^*(s, \mathbf{x}) \qquad \text{with probability} \quad \Pi(s) \\ &= e(s, \mathbf{x}) \qquad \text{with probability} \quad 1 - \Pi(s)\,. \end{aligned}$$

Then, for any $\mathbf{x} \in X$, the expected loss for the pair (e', p') is

$$\begin{aligned} &\textstyle\sum_s p'(s)\{\Pi(s)L[e^*(s, \mathbf{x}), \theta(\mathbf{x})\} + [1 - \Pi(s)]L[e(s, \mathbf{x}), \theta(\mathbf{x})]\} \\ &\quad = w \textstyle\sum_s p^*(s)L[e^*(s, \mathbf{x}), \theta(\mathbf{x})] + \sum_s [p_0(s) - wp(s)]L[e(s, \mathbf{x}), \theta(\mathbf{x})] \\ &\quad \leqq w \textstyle\sum_s p(s)L[e(s, \mathbf{x}), \theta(\mathbf{x})] + \sum_s [p_0(s) - wp(s)]L[e(s, \mathbf{x}), \theta(\mathbf{x})] \qquad \text{by (1.2)} \\ &\quad = \textstyle\sum_s p_0(s)L[e(s, \mathbf{x}), \theta(\mathbf{x})] \end{aligned}$$

with strict inequality for at least one $\mathbf{x} \in X$. Again this is impossible since (e, p_0) is admissible w.r.t. $C_n(D_n)$.

An immediate corollary of this last theorem is that if (e, p_0) is uniformly admissible and p_0 corresponds to simple random sampling then (e, p) is uniformly admissible for *any* design in D_n.

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