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SUFFICIENT STATISTICS AND EXPONENTIAL FAMILIES

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Using a locally Lipschitz function T of $n > 1$ variables one can reduce data consisting of a sample of size n to one real number. If we are given a family of probability measures on the real line which are equivalent to Lebesgue measure then T yields a sufficient data reduction only if the given family is exponential. This result is compared with the results of Brown (1964) and Denny (1970).

0. Introduction and summary. It is well known (see e.g. Koopman [9], Barankin and Maitra [1], Brown [3], [4], and Denny [7]) that, under suitable regularity conditions, a family of probability measures admits a real-valued sufficient statistic for some sample size greater than one if and only if the family is one-dimensional exponential. It is the purpose of this paper to give regularity conditions for the sufficient statistic under which the "only if" part of the above theorem is true for any family of probability measures on the real line, if each probability measure is equivalent to the Lebesgue measure (in the sense that it has the same null sets). This was done before by Brown [3], [4] and Denny [7], but their regularity conditions fail to be handy or natural. Our proofs are strongly influenced by the techniques developed in Brown [3], [4] and Denny [6], [7].

1. The main theorem. Let $\mathscr{B}$ be the Borel field of $\mathbb{R}$ and $\lambda \mid \mathscr{B}$ the Lebesgue measure on $\mathscr{B}$. For $n \in \mathbb{N}$, $n \geqq 2$, let $\| \;\; \|$ be the norm of $\mathbb{R}^n$ and let $\mathscr{B}^n$ be the Borel field of $\mathbb{R}^n$. For a family $\mathfrak{p} \mid \mathscr{B}$ of probability measures let $\mathfrak{p}^n \mid \mathscr{B}^n$ be the set of all independent products of identical components in $\mathfrak{p} \mid \mathscr{B}$. A map $T: \mathbb{R}^n \to \mathbb{R}$ is called locally Lipschitz if for every $z \in \mathbb{R}^n$ there exists $M \in \mathbb{R}$ and open $U \subset \mathbb{R}^n$ containing z such that for all $z_1, z_2 \in U$, $|T(z_1) - T(z_2)| \leqq M\|z_1 - z_2\|$.

The word "sufficient" is used in the sense of Lehmann ([10], pages 47–48).

1.1 THEOREM. *If $\mathfrak{p} \mid \mathscr{B}$ is a family of probability measures each of which is equivalent to $\lambda \mid \mathscr{B}$, and if for some $n \geqq 2$ there exists a map $T: \mathbb{R}^n \to \mathbb{R}$ which is locally Lipschitz and sufficient for $\mathfrak{p}^n \mid \mathscr{B}^n$, then $\mathfrak{p} \mid \mathscr{B}$ is a one-dimensional exponential family.*

This theorem improves Corollary 3.12 in [8] for the particular case $k = 1$ where the sufficient statistic is assumed to have continuous partial derivatives, since the existence of continuous partial derivatives for T implies that T is locally Lipschitz. We remark that our assumption "T is locally Lipschitz" cannot be

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easily compared with Denny's assumption "T does not isolate sets with a common density point;" if, however, there exist m and $M > 0$ such that for all $x \in \mathbb{R}$ and $a < b$, $m(b-a) \leqq |T(b, x) - T(a, x)| \leqq M(b-a)$, then by our Lemma 3.1 and Lemma 4.2 of Brown [4] T does not isolate sets with a common density point. We give an example where our Theorem 1.1 applies but Denny's Theorem 4.1 of [7] does not:

1.2 EXAMPLE. Let $\mathfrak{p} | \mathscr{B}$ be given by the densities

$$h(x, \vartheta) := \vartheta 1_{(-\infty,0)}(x) + (2-\vartheta) 1_{(0,\infty)}(x), \qquad x \in \mathbb{R},\ \vartheta \in [0, 2],$$

with respect to the unit normal distribution $N(0, 1) | \mathscr{B}$. Then $\mathfrak{p}^2 | \mathscr{B}^2$ admits a real-valued sufficient statistic T which is locally Lipschitz (as it has continuous partial derivatives):

$$\begin{aligned} T(x, y) &= \exp\left[-\frac{1}{x} - \frac{1}{y}\right] && \text{if } x > 0 \text{ and } y > 0, \\ &= -\exp\left[+\frac{1}{x} + \frac{1}{y}\right] && \text{if } x < 0 \text{ and } y < 0, \\ &= 0 && \text{elsewhere.} \end{aligned}$$

Here Denny's Theorem 4.1 in [7] does not apply, as the densities $h(\cdot, \vartheta)$ are not λ-continuous for $\vartheta \neq 1$.

2. Proof of the theorem. W.l.g. we may assume that $n = 2$. For $T: \mathbb{R}^2 \to \mathbb{R}$ let $\mathscr{L}(T)$ be the set of all $\mathscr{B}$-measurable functions $\varphi: \mathbb{R} \to \mathbb{R}$ for which there exist a λ^2-null set $N \in \mathscr{B}^2$ and a $\mathscr{B} \cap T(\mathbb{R}^2)$-measurable function $\psi: T(\mathbb{R}^2) \to \mathbb{R}$ such that for $(x, y) \in N^c$ $\sharp \varphi(x) + \varphi(y) = \psi(T(x, y))$. Assume that T is sufficient for $\mathfrak{p}^2 | \mathscr{B}^2$. Let $P_0 | \mathscr{B} \in \mathfrak{p} | \mathscr{B}$ and let $h(\cdot, P)$ be a positive finite density of $P | \mathscr{B}$ with respect to $\lambda | \mathscr{B}$. Then we obtain from the factorization theorem that

$$\log h(\cdot, P) - \log h(\cdot, P_0) \in \mathscr{L}(T) \qquad \text{for all} \quad P | \mathscr{B} \in \mathfrak{p} | \mathscr{B}.$$

Hence, if $L(T)$ is the set of all λ-equivalence classes of functions in $\mathscr{L}(T)$ and if $\dim L(T) \leqq 2$, then $\mathfrak{p} | \mathscr{B}$ is a one-dimensional exponential family. To prove the theorem it therefore suffices to show that T locally Lipschitz implies $\dim L(T) \leqq 2$.

2.1 DEFINITION. $T: \mathbb{R}^2 \to \mathbb{R}$ fulfills Lusin's condition iff for all $B \in \mathscr{B}$ with $\lambda(B) = 0$ and all $x \in \mathbb{R}$ $\sharp\{T(x, y), T(y, x) : y \in B\}$ is a λ-null set.

2.2 DEFINITION. A function $h: \mathbb{R} \to \mathbb{R}$ is

(a) λ-constant (λ-continuous) on $D \subset \mathbb{R}$ iff there exists a function $f: D \to \mathbb{R}$ which is constant (continuous) on D such that $\{x \in D : f(x) \neq h(x)\}$ is a λ-null set.

(b) locally λ-constant (locally λ-continuous) on $D \subset \mathbb{R}$ iff for all $x \in D$ there exists open $U \ni x$ such that h is λ-constant (λ-continuous) on U.

(c) locally λ-constant λ-a.e. (locally λ-continuous λ-a.e.) iff h is locally λ-constant (locally λ-continuous) on a set D with $\lambda(D^c) = 0$.

2.3 PROPOSITION. *If continuous* $T: \mathbb{R}^2 \to \mathbb{R}$ *fulfills Lusin's condition and if any* $\varphi \in \mathscr{L}(T)$ *is locally λ-continuous* λ-a.e. *then* $\dim L(T) \leqq 2$.

PROOF OF PROPOSITION 2.3.

2.3.1. If every $\varphi \in \mathscr{L}(T)$ is λ-constant on $\mathbb{R}$ then $\dim L(T) = 1$.

2.3.2. Assume now that there exists $\varphi_0 \in \mathscr{L}(T)$ which is not λ-constant on $\mathbb{R}$.

2.3.2a. Assume, furthermore, that every $\varphi \in \mathscr{L}(T)$ which is locally λ-constant λ-a.e. is λ-constant on $\mathbb{R}$. Let $\varphi_1 \in \mathscr{L}(T)$. Then there exists $x_0 \in \mathbb{R}$ and an open interval $U \ni x_0$ such that φ_0 and φ_1 are λ-continuous on U and φ_0 is not λ-constant on U. Let φ_0', φ_1' be the continuous versions of φ_0, φ_1 on U. Then, by Lemma 4 of Denny ([6], page 1231) we have for $(x, y) \in U^2$, $\varphi_j(x) + \varphi_j(y) = \psi_j(T(x, y))$ with continuous $\psi_j: T(U^2) \to \mathbb{R}$ for $j = 0, 1$. According to Barndorff-Nielsen and Pedersen's result (i) ([2], page 198) there exist $a, b \in \mathbb{R}$ such that $\varphi_1' = a\varphi_0' + b$ on U or, equivalently, $\varphi_1 - a\varphi_0$ is λ-constant on U. As φ_0 is not λ-constant on U we have $\lambda\{x \in \mathbb{R}: \#\{T(x, u): u \in U\} = 1\} = 0$. Hence Lemma 3.2 together with assumption 2.3.2a implies $\varphi_1 = a\varphi_0 + b$ λ-a.e.

2.3.2b. Assume now that there exists $\varphi_0 \in \mathscr{L}(T)$ which is locally λ-constant λ-a.e. but not λ-constant on $\mathbb{R}$. In this case we construct open sets $H_1, H_2 \subset \mathbb{R}$ such that $\lambda(H_1 \cup H_2)^c = 0$ and $L(T)$ is spanned by the λ-equivalence classes of 1_{H_1} and 1_{H_2}:

(A) Let M be the complement of a λ-null set such that for $x \in M$ there exists $c(x) \in \mathbb{R}$ and an open set $U \ni x$ such that $\lambda\{y \in U: \varphi_0(y) \neq c(x)\} = 0$. As $\lambda(U) > 0$ for all open nonvoid U, the number $c(x)$ is uniquely determined for $x \in M$. Let $V(x)$ be the largest open interval I for which $x \in I$ and $\lambda\{y \in I: \varphi_0(y) \neq c(x)\} = 0$, $x \in M$. Then the union of all $V(x)$ with $x \in M$, say K, is open, and $\lambda(K^c) = 0$.

(B) Let $x_1, x_2 \in M$, $U_i = V(x_i)$, $i = 1, 2$, $U = U_1 \times U_2$, and ∂U be the boundary of U. Then T is constant on ∂U. To prove (B) it suffices to show that for $x_0 \in \partial U_1$, T is constant on $\{x_0\} \times U_2$, and for $x_0 \in \partial U_2$, T is constant on $U_1 \times \{x_0\}$. We prove the assertion with $x_0 \in \partial U_1$. The case $x_0 \in \partial U_2$ runs similarly. Let $u \in U_2$ be such that (x_0, u) is no local extremum of T on $\{x_0\} \times U_2$. Let V' be an open interval containing x_0, and $V'' \subset U_2$ an open interval containing u. Then for every $m \in \mathbb{N}$ there exists $y_m \in M$ with $|y_m - x_0| < 1/m$ and $c(y_m) \neq c(x_1)$. Let $V'(y_m) = V' \cap V(y_m)$. If for all sufficiently large $m \in \mathbb{N}$ $\#\{T(x, y): x \in V'(y_m), y \in V''\} = 1$, then $\#\{T(x_0, y): y \in V''\} = 1$, which contradicts the choice of u. Hence there exists $m_0 \in \mathbb{N}$ with $\#\{T(x, y): x \in V'(y_{m_0}), y \in V''\} \neq 1$. If $\{T(x, y): x \in V'(y_{m_0}), y \in V''\} \subset (T(U))^0$, we have $(T(V'(y_{m_0}) \times V'') \cap T(U))^0 \neq \emptyset$. As T preserves ample sets (see our Lemma 3.4), this implies the existence of $z_1 \in V(y_{m_0})$, $z_2 \in V''$, $z_3 \in U_1$, $z_4 \in U_2$ with $\varphi_0(z_1) = c(y_{m_0})$, $\varphi_0(z_2) = \varphi_0(z_4) = c(x_2)$, $\varphi_0(z_3) = c(x_1)$

and $\varphi_0(z_1) + \varphi_0(z_2) = \varphi_0(z_3) + \varphi_0(z_4)$, which contradicts $c(y_{m_0}) \neq c(x_1)$. Hence $T(V' \times V'') \cap (T(U))^c \supset T(V'(y_{m_0}) \times V'') \cap T(U)^c \neq \emptyset$. As V', V'' can be chosen arbitrarily small, and T is continuous, we have $T(x_0, u) \in \partial T(U)$. As $\{T(x_0, u): u \in U_2\} \subset \{T(x_0, u): u \in U_2, (x_0, u)$ local extremum of T on $\{x_0\} \times U_2\} \cup \partial T(U)$ is an interval which is countable, it contains exactly one point.

(C) Now we show that the single point $T(\partial U)$ does not depend on the special $x_1, x_2 \in M$ chosen. To this aim it suffices to prove that for $x \in M$ and $x_0 \in \partial V(x)$, T is constant on $\{x_0\} \times \mathbb{R} \cup \mathbb{R} \times \{x_0\}$. Now $T(\{x_0\} \times \mathbb{R}) = \bigcup_{x \in M} T(\{x_0\} \times V(x)) \cup T(\{x_0\} \times M^c)$. $\lambda(M^c) = 0$, and Lusin's condition imply that $T(\{x_0\} \times M^c)$ is contained in a λ-null set. As $\bigcup_{x \in M} T(\{x_0\} \times V(x))$ is at most countable by (B), and the fact that there is an at most countable set $M_0 \subset M$ with $\bigcup_{x \in M_0} V(x) = \bigcup_{x \in M} V(x)$, this implies that $T(\{x_0\} \times \mathbb{R})$ is a single point. The proof for $\mathbb{R} \times \{x_0\}$ is completely analogous.

(D) The set $D = \bigcup_{x \in M} \{c(x)\}$ has exactly two elements. As $\lambda(M^c) = 0$ and φ_0 is not λ-constant on $\mathbb{R}$, we have $\sharp D \geqq 2$. Assume now that there are $x_1, x_2, x_3 \in M$ with $c(x_1) < c(x_2) < c(x_3)$. Let $U_i = V(x_i)$, $c_i = c(x_i)$ for $i = 1, 2, 3$, and $Z_v = U_{v_1} \times U_{v_2}$ for $v = (v_1, v_2) \in Y$, $Y = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$. Let N_0 be a λ^2-null set such that for all $v \in Y$ and $(x, y) \in N_0^c \cap Z_v$ $\varphi_0(x) + \varphi_0(y) = \psi_0(T(x, y)) = c_{v_1} + c_{v_2}$ for some $\mathscr{B} \cap T(\mathbb{R}^2)$-measurable $\psi_0: T(\mathbb{R}^2) \to \mathbb{R}$. Then

(D1) $v, v' \in Y$, $v \neq v'$ implies $T(Z_v \cap N_0^c) \cap T(Z_{v'} \cap N_0^c) = \emptyset$. As T preserves ample sets we obtain from (D1): For $v, v' \in Y$, $v \neq v'$ $(T(Z_v) \cap T(Z_{v'}))^0 = \emptyset$. As $T(\partial Z_v)$ is a single point, say t_0, not depending on $v \in Y$, there are at most two v's such that $T(Z_v)$ is a nondegenerate interval, one at the left and one at the right side of t_0. This implies that there are at least two v's in Y such that $T(Z_v)$ is a degenerate interval. But this contradicts (D1).

(E) From the above we obtain that there exist $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, and open sets $H_1, H_2 \subset \mathbb{R}$ such that $H_1 \cup H_2 = K$ and $\lambda\{x \in H_i: \varphi_0(x) \neq c_i\} = 0$ for $i = 1, 2$. To prove the proposition it has to be shown that any $\varphi \in \mathscr{L}(T)$ is λ-constant on H_i for $i = 1, 2$. To this aim we choose $V_i \subset H_i$ such that $V_i = V(x_i)$ for some $x_i \in M$, $i = 1, 2$. Let $\{t_0\} = T(\partial(V_i \times V_j))$, $i, j = 1, 2$. From $2c_1 < c_1 + c_2 < 2c_2$ we know that $T(V_1^2)$, $T(V_1 \times V_2)$ and $T(V_2^2)$ are three intervals such that any two of them have an empty common interior. And at most one of them is degenerate. Hence exactly one of them is degenerate. We discuss the two cases:

(E1) $T(V_1 \times V_2) = \{t_0\}$

(E2) $T(V_1^2) = \{t_0\}$ or $T(V_2^2) = \{t_0\}$.

We remark that we can interchange the indices 1 and 2 without any trouble if we simultaneously replace φ_0 by $-\varphi_0$ (to preserve the inequality $c_1 < c_2$).

(E1) In this case we know that $T(V_i^2)$ is nondegenerate for $i = 1, 2$. If $x, y \in M$, $V(x) \subset H_1$, $V(y) \subset H_2$, then $T(V(x) \times V(y)) = \{t_0\}$, because otherwise $T(V(x) \times V(y))$ has a nonempty common interior with $T(V_1^2)$ or $T(V_2^2)$ which is impossible. Hence from $H_1 \times H_2 = \bigcup \{V(x) \times V(y): c(x) = c_1$ and $c(y) = c_2, x, y \in M\}$ we get $T(H_1 \times H_2) = \{t_0\}$. Then there exists $c \in \mathbb{R}$ such that $\varphi(x) + \varphi(y) = c$ for λ-a.a. $(x, y) \in H_1 \times H_2$ which implies the assertion.

(E2) W.l.g. we may assume that $T(V_1^2) = \{t_0\}$, that $T(V_1 \times V_2)$ and $T(V_2^2)$ are nondegenerate, and $\sup T(V_1 \times V_2) = \inf T(V_2^2) = t_0$. As in (E1) we get $T(H_1^2) = \{t_0\}$. Hence φ is λ-constant on H_1. Let $c \in \mathbb{R}$ be such that $\lambda\{x \in H_1 : \varphi(x) \neq c\} = 0$. Let N be a λ^2-null set such that for suitable ψ_1, $\varphi(x) + \varphi(y) = \psi_1(T(x, y))$ for all $(x, y) \in N^c$. Let $N_x = \{y \in \mathbb{R} : (y, x) \in N\}$ and $N_1 = \{x \in \mathbb{R} : \lambda(N_x) > 0\}$. We have $\lambda(N_1) = 0$.

(E2a) If $\lambda\{y \in H_2 : T(\{y\} \times H_1) \neq \{t_0\}\} = 0$ fix $x_1 \in H_1 \cap N_1^c$. Then $\lambda\{y \in H_2 : \varphi(x_1) + \varphi(y) \neq \psi_1(t_0)\} \leqq \lambda\{y \in H_2 : \varphi(x_1) + \varphi(y) \neq \psi_1(T(x_1, y))\} + \lambda\{y \in H_2 : T(x_1, y) \neq t_0\} = 0$, which proves that φ is λ-constant on H_2.

(E2b) Let now $\lambda\{y \in H_2 : T(\{y\} \times H_1) \neq \{t_0\}\} > 0$. If $\lambda\{y \in H_2 : T(\{y\} \times H_1) = \{t_0\}\} > 0$ there exists $x \in H_1$ with $\varphi_0(x) = c_1$, $y \in H_2$ with $\varphi_0(y) = c_2$, and $\varphi_0(x) + \varphi_0(y) = \psi_1(t_0)$. However, $T(V_1^2) = \{t_0\}$ implies $\psi_1(t_0) = 2c_1$ which is contradictory. Hence we must have $\lambda\{y \in H_2 : T(\{y\} \times H_1) = \{t_0\}\} = 0$. There exists $x_2 \in H_2 \cap N_1^c$ such that $T(\{x_2\} \times H_1) \neq \{t_0\}$. Let $y_2 \in H_2 \cap N_1^c$ with $T(\{y_2\} \times H_1) \neq \{t_0\}$, and $z \in \{x_2, y_2\}$. Then $T(\{z\} \times H_1)$ contains an interval of the form (t, t_0) with $t < t_0$: As $H_1 = \bigcup \{V(x) : x \in H_1\}$ there exists $z' \in H_1$ such that $T(\{z\} \times V(z')) \neq \{t_0\}$. Together with $T(\{z\} \times \partial V(z')) \subset T(\partial(V(z) \times V(z'))) = \{t_0\}$ this implies that $T(\{z\} \times V(z'))$ is a nondegenerate interval having t_0 as an accumulation point. It remains to be shown that $T(\{z\} \times V(z')) \cap (-\infty, t_0) \neq \emptyset$. It suffices to show that $T(\{z\} \times V(z')) \cap (t_0, \infty) \equiv S = \emptyset$. If $S \neq \emptyset$, connectedness of $T(V(z) \times V(z'))$ and assumption (E2) imply $(T(V(z) \times V(z')) \cap T(V_2^2))^0 \neq \emptyset$. As T preserves ample sets this implies $c(z) + c(z') = 2c_2$. But $c(z) + c(z') = c_1 + c_2$, a contradiction. As $z \in \{x_2, y_2\}$ was arbitrary we have $(T(\{x_2\} \times H_1) \cap T(\{y_2\} \times H_1))^0 \neq \emptyset$. Let $W = \{x \in H_1 : (x_2, x) \in N \vee (y_2, x) \in N \vee \varphi(x) \neq c\}$. Then from $\{x_2, y_2\} \subset N_1^c$ we get $\lambda(W) = 0$. By Lusin's condition $T(\{x_2\} \times W)$ and $T(\{y_2\} \times W)$ are both contained in a λ-null set. As $T(\{x_2\} \times H_1) \cap T(\{y_2\} \times H_1)$ has positive λ-measure we obtain $T(\{x_2\} \times (H_1 \cap W^c)) \cap T(\{y_2\} \times (H_1 \cap W^c)) \neq \emptyset$. Hence there are $z_1, z_2 \in H_1 \cap W^c$ with $T(x_2, z_1) = T(y_2, z_2)$, thus $\varphi(x_2) + c = \varphi(x_2) + \varphi(z_1) = \psi_1(T(x_2, z_1)) = \psi_1(T(y_2, z_2)) = \varphi(y_2) + \varphi(z_2) = \varphi(y_2) + c$. As $y_2 \in N_1^c \cap \{y \in H_2 : T(\{y\} \times H_1) \neq \{t_0\}\}$ was arbitrary we obtain $\lambda\{y \in H_2 : \varphi(y) \neq \varphi(x_2)\} = 0$, and hence φ is λ-constant on H_2.

2.4 LEMMA. *Assume that $T: \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz, and for some $\varphi \in \mathscr{L}(T)$, $x_0 \in \mathbb{R}$ we have*

$$\text{ess} \liminf \varphi(x_0) < \text{ess} \limsup \varphi(x_0) .$$

Then $\# T(\mathbb{R} \times \{x_0\}) = 1$.

PROOF. (A1) Let I be an open bounded interval containing x_0. Then there exists $M \in \mathbb{R}$ such that for all $z_1, z_2 \in I^2$

$$|T(z_1) - T(z_2)| \leqq M\|z_1 - z_2\| .$$

For $x \in I$ let $v(x) | \mathscr{B}$ be the signed measure defined by

$$v(x)([a, b)) = T(b, x) - T(a, x) \qquad \text{for} \quad a, b \in I, a < b, \quad \text{and} \quad v(x)(B) = 0$$

for $B \in \mathscr{B}$ with $B \subset I^c$. Then we have

2.4.1. $|v(x)(B)| \leqq M\lambda(B \cap I)$ for all $B \in \mathscr{B}$ and $x \in I$;
2.4.2. $x \to v(x)(B)$ is continuous for all $B \in \mathscr{B}$;
2.4.3. $x \to v(x)\{y \in C: T(y, x) \in B\}$ is continuous for all $B \in \mathscr{B}$, $C \in \mathscr{B}$.

PROOF OF 2.4.1. Fix an $x \in I$ and let $\mathscr{F}$ be the set of all $B \in \mathscr{B}$ for which $|v(x)(B)| \leqq M\lambda(B \cap I)$. Then $\mathscr{F} = \mathscr{B}$ as $\mathscr{F}$ contains all intervals $[a, b)$ with $a, b \in \mathbb{R}$, $a < b$, and is a monotone system which is closed under disjoint unions.

PROOF OF 2.4.2. Let $x_n \in I$ for $n = 1, 2, \cdots$ with $x_n \to z \in I$, and $\mathscr{F}$ be the set of all $B \in \mathscr{B}$ for which we have $\lim_{n \in \mathbb{N}} v(x_n)(B) = v(z)(B)$. Then $\mathscr{F}$ contains all intervals $[a, b)$ with $a, b \in \mathbb{R}$ and $a < b$. Furthermore, $\mathscr{F}$ is closed under disjoint unions. Hence for $B \in \mathscr{B}$ and $\varepsilon > 0$ there exists $A \in \mathscr{F}$ with $M\lambda(A \triangle B) < \varepsilon/2$. So we obtain $\limsup_{n \in \mathbb{N}} |v(x_n)(B) - v(z)(B)| \leqq \limsup_{n \in \mathbb{N}} (|v(x_n)(B - A)| + |v(x_n)(A - B)| + |v(x_n)(A) - v(z)(A)| + |v(z)(B - A)| + |v(z)(A - B)|) \leqq \varepsilon$.

PROOF OF 2.4.3. Let $C \in \mathscr{B}$ be fixed, let $x_n \in I$ for $n \in \mathbb{N}$ with $x_n \to z \in I$, and $\mathscr{F}$ be the set of all $B \in \mathscr{B}$ for which we have $\lim_{n \in \mathbb{N}} v(x_n)\{y \in C: T(y, x_n) \in B\} = v(z)\{y \in C: T(y, z) \in B\}$. We first show that $\mathscr{F}$ contains all open $B \in \mathscr{B}$ with $\lambda\{y \in I: T(y, z) \in \partial B\} = 0$, where ∂B is the boundary of B. As the measure $B \to \lambda\{y \in I: T(y, z) \in B\}$ is regular for all $\varepsilon > 0$ there exist compact $B_1 \subset B$, $B_2 \subset (\bar{B})^c$ such that $M\lambda\{y \in I: T(y, z) \in B \cap B_1^c\} < \varepsilon$ and $M\lambda\{y \in I: T(y, z) \in (\bar{B})^c \cap B_2^c\} < \varepsilon$. Then

$$\begin{aligned}
|v(x_n)\{y \in C: T(y, x_n) \in B\} &- v(x_n)\{y \in C: T(y, z) \in B\}| \\
&\leqq |v(x_n)\{y \in C: T(y, x_n) \in B \wedge T(y, z) \notin B\}| \\
&\quad + |v(x_n)\{y \in C: T(y, x_n) \notin B \wedge T(y, z) \in B\}| \\
&\leqq M\lambda\{y \in C: T(y, x_n) \in B \wedge T(y, z) \notin B\} \\
&\quad + M\lambda\{y \in I: T(y, x_n) \notin B \wedge T(y, z) \in B\} \\
&\leqq M\lambda\{y \in I: T(y, x_n) \in \bar{B} \wedge T(y, z) \in B_2\} \\
&\quad + M\lambda\{y \in I: T(y, x_n) \in B^c \wedge T(y, z) \in B_1\} + 2\varepsilon\,.
\end{aligned}$$

Let $d = \inf\{|x - y| : x \in \bar{B}, y \in B_2 \text{ or } x \in B^c, y \in B_1\}$. Then $d > 0$ and

$$\begin{aligned}
|v(x_n)\{y \in C: T(y, x_n) \in B\} &- v(x_n)\{y \in C: T(y, z) \in B\}| \\
&\leqq 2\varepsilon + 2M\lambda\{y \in I: |T(y, x_n) - T(y, z)| \geqq d\}\,.
\end{aligned}$$

As T is uniformly continuous on I^2 this implies

2.4.4. $\limsup_{n \in \mathbb{N}} |v(x_n)\{y \in C: T(y, x_n) \in B\} - v(x_n)\{y \in C: T(y, z) \in B\}| = 0$.
As for $A \in \mathscr{B}$ we have

$$\begin{aligned}
|v(x_n)\{y \in C: T(y, x_n) \in A\} &- v(z)\{y \in C: T(y, z) \in A\}| \\
&\leqq |v(x_n)\{y \in C: T(y, x_n) \in A\} - v(x_n)\{y \in C: T(y, z) \in A\}| \\
&\quad + |v(x_n)\{y \in C: T(y, z) \in A\} - v(z)\{y \in C: T(y, z) \in A\}|\,.
\end{aligned}$$

2.4.4 together with 2.4.2 implies $B \in \mathscr{F}$.

To prove the assertion $\mathscr{F} = \mathscr{B}$ it suffices to show that $\mathscr{F}$ is a monotone system.

As $\mathbb{R} \in \mathscr{F}$, $\mathscr{F}$ is closed under complements. Hence it suffices to show that $F_k \in \mathscr{F}$ for $k \in \mathbb{N}$, $F_k \downarrow F_0$ implies $F_0 \in \mathscr{F}$. By Lemma 3.1 we have for all $x \in I$, $B \in \mathscr{B}$

$$|v(x)\{y \in \mathbb{R}: T(y, x) \in B\}| \leqq \lambda(B \cap T(I^2)) .$$

Hence

$$\begin{aligned}\limsup_{n \in \mathbb{N}} &|v(x_n)\{y \in C: T(y, x_n) \in F_0\} - v(z)\{y \in C: T(y, z) \in F_0\}| \\ &\leqq \limsup_{n \in \mathbb{N}} |v(x_n)\{y \in C: T(y, x_n) \in F_k\} \\ &\qquad - v(z)\{y \in C: T(y, z) \in F_k\}| + 2\lambda((F_k - F_0) \cap T(I^2))\end{aligned}$$

for all $k \in \mathbb{N}$. Then the assertion follows from $F_k \in \mathscr{F}$ for $k \in \mathbb{N}$ and $\lim_{k \in \mathbb{N}} \lambda((F_k - F_0) \cap T(I^2)) = 0$.

(A2) Let $s_1, s_2 \in \mathbb{R}$ be such that

$$\text{ess lim inf } \varphi(x_0) < s_1 < s_2 < \text{ess lim sup } \varphi(x_0) .$$

Then for all open $U \ni x_0$ we have

2.4.5. $\lambda\{x \in U: \varphi(x) < s_1\} > 0$ and $\lambda\{x \in U: \varphi(x) > s_2\} > 0$.

Let $0 < \varepsilon < s_2 - s_1$ and $\mathbb{R} = X^+ \cup X^-$ a Hahn–Jordan decomposition of $\mathbb{R}$ with respect to $v(x_0) \mid \mathscr{B}$. Choose $k \in \mathbb{Z}$ and let

$$\begin{aligned} C^+ &= X^+ \cap \{x \in \mathbb{R}: k\varepsilon \leqq \varphi(x) < (k+1)\varepsilon\} , \\ C^- &= X^- \cap \{x \in \mathbb{R}: k\varepsilon \leqq \varphi(x) < (k+1)\varepsilon\} . \end{aligned}$$

Then we have

2.4.6. $v(x_0)(C^+) = v(x_0)(C^-) = 0$.

Proof of 2.4.6. Let N_1 be a λ-null set such that for $x \in N_1^c$ $\lambda\{y \in \mathbb{R}: \varphi(x) + \varphi(y) \neq \psi(T(x, y))\} = 0$ for an appropriate measurable ψ, and $B = \psi^{-1}(-\infty, (k+1)\varepsilon + s_1)$. Then $B \in \mathscr{B}$, and for $x \in \mathbb{R}$ with $\varphi(x) < s_1$ we have

$$\begin{aligned} C^\delta &\supseteqq \{y \in C^\delta: \varphi(x) + \varphi(y) \in (-\infty, (k+1)\varepsilon + s_1)\} \\ &\supseteqq \{y \in C^\delta: \varphi(y) \in (-\infty, (k+1)\varepsilon)\} = C^\delta , \qquad \text{for } \delta = +, - . \end{aligned}$$

Similarly, for $x \in \mathbb{R}$ with $\varphi(x) > s_2$ we have

$$\{y \in C^\delta: \varphi(x) + \varphi(y) \in (-\infty, (k+1)\varepsilon + s_1)\} = \emptyset \qquad \text{for } \delta = +, - .$$

By 2.4.5, for all open $U \ni x_0$ there exists $x \in U \cap N_1^c$ such that $\varphi(x) < s_1$, i.e.

$$\begin{aligned} v(x)\{y \in C^\delta: T(y, x) \in B\} &= v(x)\{y \in C^\delta: \varphi(x) + \varphi(y) \in (-\infty, (k+1)\varepsilon + s_1)\} \\ &= v(x)(C^\delta) , \end{aligned}$$

and for all open $U \ni x_0$ there exists $x \in U \cap N_1^c$ such that $\varphi(x) > s_2$, i.e.

$$\begin{aligned} v(x)\{y \in C^\delta: T(y, x) \in B\} &= v(x)\{y \in C^\delta: \varphi(x) + \varphi(y) \in (-\infty, (k+1)\varepsilon + s_1)\} \\ &= v(x)(\emptyset) = 0 , \qquad \text{for } \delta = +, - . \end{aligned}$$

This together with 2.4.3 implies $0 = \lim_{x\to x_0} v(x)\{y \in C^\delta : T(y, x) \in B\} = \lim_{x\to x_0} v(x)(C^\delta)$ for $\delta = +, -$. Now 2.4.2 implies $v(x_0)(C^\delta) = 0$ for $\delta = +, -$, which was asserted in 2.4.6.

As $k \in \mathbb{Z}$ was arbitrary, we have $v(x_0)(X^+) = v(x_0)(X^-) = 0$, which implies that $T(\cdot, x_0)$ is constant on I. As I was an arbitrary open bounded interval containing x_0, we obtain that $T(\cdot, x_0)$ is constant on $\mathbb{R}$ which proves the assertion.

2.5 Proposition. *If $T: \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz, then every $\varphi \in \mathscr{L}(T)$ is locally λ-continuous* λ-a.e.

Proof. If every $\varphi \in \mathscr{L}(T)$ is λ-constant on $\mathbb{R}$ the assertion is obvious. Assume now that there exists $\varphi_0 \in \mathscr{L}(T)$ which is not λ-constant on $\mathbb{R}$. Then the closed set $K = \{x \in \mathbb{R} : \# T(\mathbb{R} \times \{x\}) = 1\}$ is of Lebesgue-measure zero. If for some $\varphi \in \mathscr{L}(T)$ and $x_0 \in \mathbb{R}$ ess lim inf $\varphi(x_0) <$ ess lim sup $\varphi(x_0)$, then $x_0 \in K$ by Lemma 2.4. By Denny's Lemma 3.8 in [7], page 407 there exists a continuous function $\varphi' : K^c \to \mathbb{R}$ such that $\lambda\{x \in K^c : \varphi(x) \neq \varphi'(x)\} = 0$. As φ is real-valued and $\{x \in K^c : |\varphi'(x)| = \infty\}$ is relatively closed, φ is λ-continuous on $K^c \cap \{x \in K^c : |\varphi'(x)| < \infty\}$. As K and $\{x \in K^c : |\varphi'(x)| = \infty\}$ have Lebesgue-measure zero, this proves the assertion.

3. Lemmas. Here we collect the technical lemmas for the sake of reference. Lemma 3.1 seems to be obvious, and Lemma 3.2 is exactly Lemma 2.4 of [8]. Lemma 3.3 and Lemma 3.4 can also be found in [8].

3.1 Lemma. *If $I = (\alpha, \beta)$ is an open bounded interval and $f: \mathbb{R} \to \mathbb{R}$ is Lipschitz on I* (i.e. *there exists $M \in \mathbb{R}$ such that for all $x, y \in I$ $|f(x) - f(y)| \leqq M|x - y|$*), *and if $v \mid \mathscr{B} \cap I$ is the signed measure with $v([a, b)) = f(b) - f(a)$ for all $a, b \in I$ with $a < b$, then for all $B \in \mathscr{B} \cap f(I)$ $|v\{y \in \mathbb{R} : f(y) \in B\}| \leqq \lambda(B)$.*

Proof. Let f' be a Radon–Nikodym derivative of f with respect to the Lebesgue measure on I. Then $|f'(x)| \leqq M$ for λ-a.a. $x \in I$. For continuous $g: \mathbb{R} \to \mathbb{R}$ let G be an indefinite integral with respect to $\lambda \mid \mathscr{B}$. Then $\int g(f(x))v(dx) = \int g(f(x))f'(x)1_I(x)\,dx = G(f(\beta)) - G(f(\alpha))$, as $G \circ f$ has $g \circ f \cdot f'$ as a Radon–Nikodym derivative with respect to the Lebesgue measure on I. Hence we have for all continuous $g: \mathbb{R} \to \mathbb{R}_+$:

$$|\int g(f(x))v(dx)| \leqq \int g(x)1_{f(I)}(x)\,dx\,.$$

Now we easily obtain the assertion by approximating for given $B \in \mathscr{B} \cap f(I)$ the function 1_B with continuous functions g such that $|\int (1_B(x) - g(x))v(dx)| + |\int (1_B(x) - g(x))\,dx|$ is arbitrarily small.

3.2 Lemma. *Assume that continuous $T: \mathbb{R}^2 \to \mathbb{R}$ fulfills Lusin's condition. If $\varphi \in \mathscr{L}(T)$ is λ-constant on some open interval U and $\lambda\{x \in \mathbb{R} : \# T(\{x\} \times U) = 1\} = 0$, then φ is locally λ-constant* λ-a.e.

Proof. Let $\psi: T(\mathbb{R}^2) \to \mathbb{R}$ be $\mathscr{B} \cap T(\mathbb{R}^2)$-measurable and N be a λ^2-null set such that for $(x, y) \in N^c$ $\varphi(x) + \varphi(y) = \psi(T(x, y))$, and $c \in \mathbb{R}$ such that $H = \{x \in U : \varphi(x) \neq c\}$ is a λ-null set. If for $x \in \mathbb{R}$ $N_x = \{y \in \mathbb{R} : (x, y) \in N\}$ and

$N^1=\{x \in \mathbb{R} : \lambda(N_x)>0\}$ then $\lambda(N^1) = 0$. Fix $x_1 \in M = (N^1)^c \cap \{x \in \mathbb{R} : \sharp T(\{x\} \times U) \neq 1\}$. Then there exist $a, b, \alpha, \beta, \gamma$ such that $T(\{x_1\} \times U) \supset (\alpha, \beta) \supset [a, b]$ with $a < \gamma < b$, $a = T(x_1, u_1)$, $b = T(x_1, u_2)$, $u_1, u_2 \in U$. Let $V = \{x \in \mathbb{R} : T(x, u_1) \in (\alpha, \gamma)$ and $T(x, u_2) \in (\gamma, \beta)$. Now $x_1 \in V$, and for all $x \in V \cap M$ we have $(T(\{x\} \times U) \cap T(\{x_1\} \times U))^0 \neq \emptyset$. As $F = N_{x_1} \cup N_x \cup H$ is a λ-null set and T fulfills Lusin's condition we obtain that $T(\{x\} \times (U \cap F))$ and $T(\{x_1\} \times (U \cap F))$ are both contained in a λ-null set. As $T(\{x\} \times U) \cap T(\{x_1\} \times U)$ has positive Lebesgue measure we obtain $T(\{x\} \times (U \cap F^c)) \cap T(\{x_1\} \times (U \cap F^c)) \neq \emptyset$. This implies $\varphi(x) = \varphi(x_1)$. As M^c is a λ-null set and $x \in V \cap M$ was arbitrary, φ is locally λ-constant λ-a.e.

3.3 LEMMA. *Assume that $T: \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz. Then T fulfills Lusin's condition.*

PROOF. We have to show that for $N \in \mathscr{B}$ with $\lambda(N) = 0$ and $x \in \mathbb{R}$ $T(\{x\} \times N) \cup T(N \times \{x\})$ is a λ-null set. We shall show $\lambda(T(\{x\} \times N)) = 0$. The assertion $\lambda(T(N \times \{x\})) = 0$ follows from $\lambda(\hat{T}(\{x\} \times N)) = 0$ for the map $\hat{T}(x, y) = T(y, x)$. Let U_i, $i \in \mathbb{N}$, be open such that $\mathbb{R} = \bigcup_{i \in \mathbb{N}} U_i$ and T is Lipschitz on $\{x\} \times U_i$. To prove $\lambda(T(\{x\} \times N)) = 0$ it suffices to show that $\lambda(T(\{x\} \times (U_i \cap N))) = 0$ for all $i \in \mathbb{N}$. Let $i_0 \in \mathbb{N}$ be arbitrary and $M \in \mathbb{R}$ such that for $y_1, y_2 \in U_{i_0}$ $|T(x, y_1) - T(x, y_2)| \leqq M|y_1 - y_2|$. Then for every $\varepsilon > 0$ we can find open disjoint intervals (a_n, b_n), $n \in \mathbb{N}$, such that $N \cap U_{i_0} \subset \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ and $\sum_{n \in \mathbb{N}} b_n - a_n < \varepsilon$. Now $\lambda(T(a_n, b_n)) \leqq M(b_n - a_n)$ implies

$$\lambda^*(T(\{x\} \times (N \cap U_{i_0}))) \leqq \lambda^*(T(\{x\} \times \textstyle\bigcup_{n \in \mathbb{N}} (a_n, b_n))) \leqq \sum_{n \in \mathbb{N}} M(b_n - a_n) < M\varepsilon .$$

(λ^* is the outer Lebesgue-measure on $\mathbb{R}$.) This proves the assertion.

3.4. LEMMA. *Assume that $T: \mathbb{R}^2 \to \mathbb{R}$ fulfills Lusin's condition. Then T preserves ample sets, i.e. for every λ^2-null set $N \in \mathscr{B}^2$ and open $U_1, U_2 \subset \mathbb{R}^2$ $(T(U_1) \cap T(U_2))^0 \neq \emptyset$ implies $T(U_1 \cap N^c) \cap T(U_2 \cap N^c) \neq \emptyset$.*

PROOF. Choose $t_0 \in (T(U_1) \cap T(U_2))^0$ such that t_0 is not the image of a local extremum of T. Let $N_0 \equiv \{x \in \mathbb{R} : \lambda\{y \in \mathbb{R} : (x, y) \in N\} > 0\}$, $z_i \in U_i \cap T^{-1}\{t_0\}$; let, furthermore, $z_i \in V_i \subset U_i$, V_i open rectangles, and $x_i^1, x_i^2, y_i^1, y_i^2 \in \mathbb{R}$ such that $(x_i^1, y_i^1), (x_i^2, y_i^2) \in V_i$ and $x_i^1, y_i^2 \in N_0^c$, fulfilling $T(x_i^1, y_i^1) < t_0 < T(x_i^2, y_i^2)$ for $i = 1, 2$. Then if I_i is the interval having y_i^1, y_i^2 as endpoints and if J_i is the interval having x_i^1, x_i^2 as endpoints we obtain $H_i \equiv \{x_i^1\} \times I_i \cup J_i \times \{y_i^2\} \subset V_i$ for $i = 1, 2$. As T fulfills Lusin's condition we have $\lambda(T(H_i \cap N)) = 0$, $i = 1, 2$. However, $\lambda(T(H_1) \cap T(H_2)) > 0$, and therefore $T(H_1 \cap N^c) \cap T(H_2 \cap N^c) \neq \emptyset$. This implies the assertion.

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