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MAXMIN $C(\alpha)$ TESTS AGAINST TWO-SIDED ALTERNATIVES[1]

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Let $\{X_n\}$ be a sequence of i.i.d. random variables, each with probability density function $p(x \mid \theta, \xi)$ subject to certain regularity conditions. Here, θ is an s-dimensional vector of nuisance parameters, and $\xi \in (-r, r)$ is the parameter under test. The first N members of the sequence $\{X_n\}$ are to be used for testing the hypothesis, $H_0: \xi = 0$, against the alternative, $H_1: \xi \neq 0$, while θ remains unspecified. The particular case considered is that in which the left-hand and right-hand derivatives, with respect to ξ, of the logarithm of the density function are unequal. It is shown that the class of $C(\alpha)$ tests based on linear combinations of the left and right derivatives, is an essentially complete class of these tests. The asymptotic power functions of these tests depend upon the coefficients of the linear combination. The maxmin test is deduced and compared with strongly symmetric and weakly symmetric tests. The motivation for the study is the vague notion of "fair" tests which do not arbitrarily favor detection of "positive" or "negative" alternatives.

1. Introduction. Let $\{X_n\}$ be a sequence of i.i.d. random variables each with probability density function $p(x \mid \theta, \xi)$, where $\theta \in \Theta \subset R^s$ is an s-dimensional vector of nuisance parameters, and $\xi \in (-r, r)$ is a test parameter. The problem considered is to test the hypothesis $H_0: \xi = 0$, against the alternative $H_1: \xi \neq 0$ while the nuisance parameters are left unspecified. Specifically a test is sought which has good power against "positive" ($\xi > 0$) as well as "negative" ($\xi < 0$) alternatives. The particular case studied is that in which the density functions $p(x \mid \theta, \xi)$ satisfy the Cramér regularity conditions (see Neyman (1959)) with the exception that

$$\psi_1 = \frac{\partial}{\partial \xi} \log p(x \mid \theta, \xi)\Big|_{\xi=0-} \neq \frac{\partial}{\partial \xi} \log p(x \mid \theta, \xi)\Big|_{\xi=0+} = \psi_2 .$$

Only the class of $C(\alpha)$ tests will be considered. Let $\mathscr{X}$ be the sample space of the random variables $\{X_n\}$. A $C(\alpha)$ test is specified by three things: a test function $f: \mathscr{X} \times \theta \to R$, a sequence of estimators, $\{\hat{\theta}_n\}$, and a rejection region, $S_\alpha \subset R$. For conditions on the test function f and $\{\hat{\theta}_n\}$ see Neyman (1959). The rejection region has the property that the indicator function I_{S_α} is continuous a.e., and if Z is an $N(0, 1)$ random variable, then $P[Z \in S_\alpha] = \alpha$. The $C(\alpha)$ test based on the first N elements of $\{X_n\}$ is given by the rule:

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Reject H_0 when

$$Z_N(\hat{\theta}_N) = \frac{1}{N^{\frac{1}{2}}} \sum f(X_i, \hat{\theta}_N) \in S_\alpha .$$

Such tests are asymptotically of size α, and have an asymptotic power function

(1) $\beta(\tau) = \lim_{n\to\infty} P[Z_n(\hat{\theta}_n) \in S_\alpha \mid \theta, \xi_n] = P[Z(c\tau) \in S_\alpha]$ when $n^{\frac{1}{2}}\xi_n \to \tau$.

Here $Z(c\tau)$ is a random variable with $N(c\tau, 1)$ distribution, and

$$\begin{aligned} c &= Ef\psi_1 \quad \text{if } \tau < 0 \\ &= Ef\psi_2 \quad \text{if } \tau > 0 . \end{aligned}$$

Here and further below all expectations are taken assuming $\xi = 0$.

Under the condition that $\psi_1 \neq \psi_2$, there is no uniformly asymptotically most powerful $C(\alpha)$ test of $H_0: \xi = 0$ against $H_1: \xi \neq 0$. In general, tests which have good power against "positive" alternatives have poor power against "negative" alternatives and vice-versa. In the present problem, detection of "positive" and "negative" alternatives is considered equally important. In particular a "fair" test, i.e. one that does not arbitrarily favor detection of "positive" or "negative" alternatives, is desired. In the search for "fair" tests, Neyman (1969) introduced the concepts of "strongly" and "weakly" symmetric optimal $C(\alpha)$ tests. These tests are characterized by the requirements:

I. *Strong symmetry*:

$$\beta(\tau) = \beta(-\tau) \qquad \forall\tau , \quad \forall\theta .$$

II. *Weak symmetry*:

$$\frac{\partial}{\partial\tau}\{\beta(\tau) + \beta(-\tau)\}\Big|_{\tau=0} \quad \text{is a maximum.}$$

In the present paper a third concept of "fair" test is proposed, the maxmin $C(\alpha)$ test. The defining property of this test is that:

III. *Maxmin $C(\alpha)$ test*: maximizes $\min\{\beta(\tau), \beta(-\tau)\}$.

The maxmin test is, by definition, uniformly asymptotically more powerful than the optimal strongly symmetric test. There exist families of distributions in which for any positive number, M, the relative increase in power is greater than M for some set of values of the nuisance parameters. A practical example of such a family will be given in Section 7.

Prior to deducing the maxmin $C(\alpha)$ test and comparing its asymptotic power with the asymptotic power of the optimal strongly symmetric and weakly symmetric tests, an essentially complete class of $C(\alpha)$ tests will be determined; this is done in Section 2. In Section 3, two natural classes of $C(\alpha)$ tests $\mathscr{C}_1$ and $\mathscr{C}_2$ are described, and a function $H: R^+ \to [0, 1]$, which is used to compare the asymptotic power of tests in $\mathscr{C}_1$ with the asymptotic power of tests in $\mathscr{C}_2$, is defined. In Section 4, the maxmin $C(\alpha)$ test in $\mathscr{C}_1$ and the maxmin $C(\alpha)$ test in $\mathscr{C}_2$ are deduced. The test functions deduced in Section 4 do not satisfy the

Cramér regularity conditions, under which Neyman (1959) proved three theorems necessary for the construction of the class of $C(\alpha)$ tests. In Section 5, it is noted that the three theorems are true under weaker regularity conditions. These weaker conditions are satisfied by the functions deduced in Section 4. The asymptotic power functions of the three types of "fair" tests are compared in Section 6. In order not to interrupt the continuity of the paper the longer proofs are placed in Section 8.

2. An essentially complete class of $C(\alpha)$ tests. One of the conditions on the test function $f\colon \mathscr{X} \times \Theta \to R$ used to specify a $C(\alpha)$ test is that f satisfy

$$(2) \qquad E\{f\phi_j \mid H_0\} = 0 \qquad \text{where } \phi_j = \frac{\partial}{\partial\theta_j} \log p(x \mid \theta, \xi)\Big|_{\xi=0} \quad \text{and } j = 1, \cdots, s\,.$$

Let

$$g = \phi_1 - \sum a_j \phi_j\,, \qquad h = \phi_2 - \sum b_j \phi_j$$

where $\{a_j\}$ and $\{b_j\}$ are chosen so that g and h satisfy (2).

THEOREM 1. *The class of $C(\alpha)$ tests based on functions of the form $f = ag + bh$ is essentially complete with respect to asymptotic power, within the class of all $C(\alpha)$ tests.*

This theorem (proved in Section 8) provides a convenient classification of $C(\alpha)$ tests. It is particularly useful to compare the asymptotic power functions of various $C(\alpha)$ tests by considering $Ef\psi_1$ and $Ef\psi_2$ as functions of the coefficients of the linear combination. The optimal strongly symmetric and weakly symmetric tests developed by Neyman (1969) belong to this essentially complete class.

3. The partition of $C(\alpha)$ tests into two natural classes. In order to compare the asymptotic powers of various $C(\alpha)$ tests we shall distinguish two classes of these tests, $\mathscr{C}_1$ and $\mathscr{C}_2$. The most powerful unbiased rejection region is one-sided for $\mathscr{C}_1$ and two-sided for $\mathscr{C}_2$.

The class $\mathscr{C}_1$ consists of all $C(\alpha)$ tests with a test function f which has the following property

$$Ef\psi_1 > 0 \quad \text{for all} \quad \theta \in \Theta \qquad \text{and} \qquad Ef\psi_2 < 0 \quad \text{for all} \quad \theta \in \Theta\,.$$

For this class of tests the most powerful rejection region is $S_\alpha = (-\infty, -\nu_\alpha)$ and the asymptotic power is given by

$$(3) \qquad \beta_f(\tau) = \Phi(-\nu_\alpha - \tau Ef\psi_i)\,, \qquad \begin{aligned} i &= 1 \quad \text{if} \quad \tau < 0\,, \\ &= 2 \quad \text{if} \quad \tau > 0\,, \end{aligned}$$

where Φ is the $N(0, 1)$ distribution function and $\Phi(\nu_\alpha) = 1 - \alpha$.

$\mathscr{C}_1{}^*$ is the class of $C(\alpha)$ tests specified by functions f^* with the property that the $C(\alpha)$ test specified by $-f^*$ belongs to $\mathscr{C}_1$. These tests are obviously equivalent to tests in $\mathscr{C}_1$ and will not be discussed.

The class $\mathscr{C}_2$ consists of all $C(\alpha)$ tests not in $\mathscr{C}_1$ or $\mathscr{C}_1{}^*$. The tests in $\mathscr{C}_2$ are specified by test functions with the property that either sgn $Ef\psi_1 = $ sgn $Ef\psi_2$ for some $\theta \in \Theta$, or sgn $E\{f\psi_i \mid \theta\} \neq$ sgn $E\{f\psi_i \mid \theta'\}$ for some θ, θ'; $i = 1$ or 2.

For this class any unbiased rejection region must be symmetric about 0. The most powerful unbiased rejection region is given by $S_\alpha = (-\infty, -\nu_{\alpha/2}) \cup (\nu_{\alpha/2}, \infty)$. The asymptotic power of a test in this class is given by

$$\beta_f(\tau) = \Phi(-\nu_{\alpha/2} + \tau Ef\psi_i) + \Phi(-\nu_{\alpha/2} - \tau Ef\psi_i)\,, \qquad i = 1 \text{ if } \tau < 0\,, \quad 2 \text{ if } \tau > 0\,. \tag{4}$$

In order to compare the asymptotic power of the tests in $\mathscr{C}_1$ with the asymptotic power of tests in $\mathscr{C}_2$ it is necessary to consider the function $H: R^+ \to [0, 1]$, defined as follows:

$$\Phi(-\nu_\alpha + xH(x)) = \Phi(-\nu_{\alpha/2} + x) + \Phi(-\nu_{\alpha/2} - x)\,.$$

It may be shown (Section 8) that:

LEMMA 1. *The function H has the following properties*:

(i) $\lim_{x\to 0} H(x) = 0$;
(ii) $\lim_{x\to\infty} H(x) = 1$;
(iii) *H is increasing.*

Using the above three properties and equations (3) and (4) it is easy to prove

THEOREM 2. *Let f_1 determine a test in $\mathscr{C}_1$, and f_2 determine a test in $\mathscr{C}_2$, with $c_{ij} = |Ef_i\psi_j|$ for $i, j = 1, 2$. Then there exist numbers $\tau_j^* \in [0, \infty]$ such that*

$$\beta_{f_1}(\tau) \geqq \beta_{f_2}(\tau) \qquad for \quad -\tau_1^* \leqq \tau \leqq \tau_2^*\,,$$
$$\beta_{f_1}(\tau) \leqq \beta_{f_2}(\tau) \qquad for \quad \tau \geqq \tau_2^* \quad or \quad \tau \leqq -\tau_2^*\,.$$

The numbers τ_j^* are given by:

$$\begin{aligned} \tau_j^* &= H^{-1}(c_{1j}/c_{2j})/c_{2j} && \text{if } c_{1j} < c_{2j}\,, \\ &= \infty && \text{if } c_{1j} \geqq c_{2j}\,. \end{aligned} \tag{5}$$

A numerical evaluation of the functions H and H^{-1} (H inverse) may be found in Tables 1 and 2 for $\alpha = .1$, .05, and .01.

No uniformly asymptotically most powerful strongly symmetric $C(\alpha)$ test exists in $\mathscr{C}_1 \cup \mathscr{C}_2$, but there is a UAMP strongly symmetric test in $\mathscr{C}_1$, and a different UAMP strongly symmetric test in $\mathscr{C}_2$. The asymptotic power of the two tests is easily compared through the use of the function H. The optimal weakly symmetric test exists and is in $\mathscr{C}_2$, but it has a one-sided rejection region, and is therefore biased for some values of the nuisance parameters. With this exception we will only consider tests with optimal unbiased rejection regions. The asymptotic power of $C(\alpha)$ tests does not depend upon which sequence of estimators of the nuisance parameters is used, provided, of course, that the sequence of estimators satisfies the conditions in Neyman (1959). For the purpose of this paper then, a $C(\alpha)$ test is completely specified by a test function f.

4. Maxmin $C(\alpha)$ tests. It is not possible to find a $C(\alpha)$ test which is maxmin for all τ and all θ. There is, however, a test in $\mathscr{C}_1$ which is maxmin for all θ and all τ among all tests in $\mathscr{C}_1$, and a different test in $\mathscr{C}_2$ which is maxmin among all tests in $\mathscr{C}_2$.

For tests in $\mathscr{C}_1$ the asymptotic power is given by equation (3). Consequently,

$$\min\{\beta(\tau), \beta(-\tau)\} = \Phi(-\nu_\alpha + |\tau| \min\{|Ef\psi_1|, |Ef\psi_2|\})$$

and it is enough to find the function $f = ag + bh$ which maximizes

$$\min\{Ef\psi_1, -Ef\psi_2\}. \tag{6}$$

Maximizing (6) with respect to all coefficients a, b which satisfy $E(ag + bh)^2 = 1$ results in

THEOREM 3. *The linear combination $f^* = a^*g + b^*h$, in class $\mathscr{C}_1$, which maximizes* $\min\{\beta(\tau), \beta(-\tau)\}$ *is given by*

$$\begin{aligned} a^* &= \max\{0, \min(1/\sigma_1, a_2)\} && \textit{if} \quad \sigma_{12} \leqq 0, \\ &= a_1 && \textit{if} \quad \sigma_{12} \geqq 0, \\ b^* &= -a^*\sigma_1\rho/\sigma_2 - [1 - a^{*2}(1-\rho^2)\sigma_1^2]^{\frac{1}{2}}/\sigma_2. \end{aligned} \tag{7}$$

Here and elsewhere σ_1^2, σ_2^2, σ_{12} and ρ are respectively the variance of g, the variance of h and their covariance and correlation taken under H_0.

The constants a_1 and a_2 are given by:

$$\begin{aligned} a_1 &= (\sigma_2 + |\rho|\sigma_1)/[\sigma_1^2(1-\rho^2)(\sigma_1^2 + 2|\sigma_{12}| + \sigma_2^2)]^{\frac{1}{2}} \\ a_2 &= (\sigma_2 - |\rho|\sigma_1)/[\sigma_1^2(1-\rho^2)(\sigma_1^2 - 2|\sigma_{12}| + \sigma_2^2)]^{\frac{1}{2}}. \end{aligned}$$

(The proof is in Section 8.)

A glance at equation (4) shows that the test in $\mathscr{C}_2$ which maximizes $\min\{\beta(\tau), \beta(-\tau)\}$ is that which maximizes $\min\{|Ef\psi_1|, |Ef\psi_2|\}$. This maximum may again be found by finding the proper linear combination $ag + bh$.

THEOREM 4. *The linear combination $f_m = a_m g + b_m h$, in $\mathscr{C}_2$ which maximizes* $\min\{\beta(\tau), (-\tau)\}$ *is given by*:

$$\begin{aligned} a_m &= \max\{0, \min(1/\sigma_1, a_2)\} \\ b_m &= -a_m\rho\sigma_1/\sigma_2 + \operatorname{sgn}(\rho)[1 - a_m^2(1-\rho^2)\sigma_1^2]^{\frac{1}{2}}. \end{aligned}$$

(This theorem is proved in essentially the same way as Theorem 3.)

Notice that if $\sigma_{12} < 0$ then $a^*g + b^*h = a_m g + b_m h$. If $\sigma_{12} > 0$ it is easy to show that $|Ef_m\psi_i| > |Ef^*\psi_i|$ for $i = 1, 2$. Using Theorem 2, we see that if $\sigma_{12} \leqq 0$, the maxmin test in $\mathscr{C}_1$ is asymptotically more powerful than the maxmin test in $\mathscr{C}_2$ for all values of τ. If $\sigma_{12} > 0$, the maxmin test in $\mathscr{C}_1$ is asymptotically more powerful than the maxmin test in $\mathscr{C}_2$ for $\tau \in (-\tau_1^*, \tau_2^*)$ and less powerful for τ outside this interval, where τ_1^* and τ_2^* are defined as in Theorem 2. For "most" families of density functions $\sigma_{12} \leqq 0$ for some values of the nuisance parameters, and $\sigma_{12} > 0$ for other values of the nuisance parameters, thus for "most" families of density functions neither maxmin test dominates the other.

5. Generalization of the Cramér conditions. In Section 4, the linear combination $a_m g + b_m h$, which maximizes $\min\{|Ef_m\psi_1|, |Ef_m\psi_2|\}$ was constructed. The coefficients a_m and b_m are functions of σ_1^2, σ_2^2, σ_{12}, and ρ, which are in turn functions of the nuisance parameters θ. The coefficients a_m and b_m considered as functions of σ_1^2, σ_2^2, σ_{12} do not have derivatives everywhere, and therefore are not Cramér functions, that is they do not satisfy the Cramér regularity conditions (Neyman (1959)). The question at hand is: "Is it possible to construct a $C(\alpha)$ test specified by the function f_m, even though it is not a Cramér function?" The purpose of this section is to demonstrate that the normed sum

$$Z_N = \frac{1}{N^{\frac{1}{2}}} \sum f_m(X_i, \hat{\theta}_N)$$

has property (1).

Let $(\mathscr{X}, \mathscr{A}, \mu)$ be a measure space with μ a σ-finite measure. Let $\Theta \times \Xi$ be a parameter space, with $\Theta \subset R^s$ and $\Xi = (-r, r)$. Let $\mathscr{P} = \{P_{\theta,\xi} : \theta \in \Theta, \xi \in \Xi\}$ be a family of probability measures on $(\mathscr{X}, \mathscr{A})$ with the property that no two measures are disjoint, and let $p(\cdot \mid \theta, \xi) = dP_{\theta,\xi}/d\mu$. Then, a function $f: \mathscr{X} \times \Theta \to R$ is a *generalized Cramér function* with respect to the family $\mathscr{P}$ if

(i) For every $\theta \in \Theta$, there exists a neighborhood, V, of θ such that for every $\theta^* \in V$, the function $f_{\theta,\theta^*}: \mathscr{X} \times [0, 1] \to R$ is a Cramér function with respect to the measure space $(\mathscr{X}, \mathscr{A}, \mu)$, the parameter space $[0, 1] \times \Xi$ and the family of probability measures

$$\mathscr{P}_{\theta,\theta^*} = \{P_{t,\xi} : t \in [0, 1], \xi \in \Xi\} \qquad \text{where}$$

$$dP_{t,\xi}/d\mu = p(\cdot \mid \theta + t(\theta^* - \theta), \xi) \qquad \text{all } t \in [0, 1]$$

$$f_{\theta,\theta^*}(x, t) = f[x, \theta + t(\theta^* - \theta)] .$$

(ii) There exists a function $H_\theta : \mathscr{X} \to R$, $H_\theta \geqq 0$, such that

$$\int H_\theta(x) p(x \mid \theta, \xi)\, dx < \infty \qquad \text{for all } \xi \in \Xi ,$$

and

$$\left| \frac{\partial^2}{\partial t^2} f_{\theta,\theta^*}(x, t) \right| \leqq H_\theta(x) \qquad \text{for all } x \in \mathscr{X}, \quad \text{for all } t \in [0, 1], \quad \text{for all } \theta^* \in V .$$

Condition (i) means that the function f satisfies the Cramér regularity conditions when restricted to the straight line joining θ and θ^*. Condition (ii) means that the bounding function H_θ does not depend upon θ^*. Condition (i) insures that it is possible to make a Taylor expansion of $Z_N(\hat{\theta}_N)$ around θ, and that the second term of the Taylor expansion will converge to zero in P_{θ,ξ_N} probability as $N \to \infty$, whenever $\{\xi_N\}$ is a sequence such that $N^{\frac{1}{2}}\xi_N \to \tau < \infty$.

It is easy to see that the function f_m satisfies the generalized Cramér conditions. The coefficient a_m has partial derivatives with respect to σ_1^2, σ_2^2, σ_{12} except at points where $a_2 = 1/\sigma_1$ and $a_2 = 0$. The condition that the family of measures

$\mathscr{P}$ be regular enough so that ψ_1, ψ_2, ϕ_j, for $j = 1, \cdots, s$, be Cramér functions with respect to the family $\mathscr{P}$, implies that σ_1, σ_2, σ_{12} are differentiable functions of θ.

Examination of the proofs of Theorems 1, 2, 3, of Neyman (1959) shows that sufficient conditions for

$$Z_N = \frac{1}{N^{\frac{1}{2}}} \sum f(X_i, \hat{\theta}_N)$$

to satisfy (1) are that f be a generalized Cramér function that satisfies condition (2) and that $\{\hat{\theta}_N\}$ be a sequence of locally root-N-consistent estimators (Neyman (1959)).

6. Comparison of maxmin tests with strongly symmetric and weakly symmetric tests. The asymptotic power, $\beta(\tau)$, of a $C(\alpha)$ test in class $\mathscr{C}_1$ is an increasing function of $Ef\psi_1$ if $\tau < 0$ and of $-Ef\psi_2$ if $\tau > 0$. The power, $\beta(\tau)$, of a test in $\mathscr{C}_2$ is an increasing function of $|Ef\psi_1|$ if $\tau < 0$ and of $|Ef\psi_2|$ if $\tau > 0$.

The tests in $\mathscr{C}_1$ may be compared to those in $\mathscr{C}_2$ by using the function H which is defined in Section 3. In either case the interest centers on the comparisons of $Ef\psi_1$ and $Ef\psi_2$ and only these comparisons will be considered in this section.

Neyman (1969) deduced the optimal strongly symmetric test in $\mathscr{C}_1$ and the optimal strongly symmetric test in $\mathscr{C}_2{}^*$, here $\mathscr{C}_2{}^*$ is the class of all $C(\alpha)$ tests with test functions f such that $\operatorname{sgn} Ef\psi_1 = \operatorname{sgn} Ef\psi_2$ for all $\theta \in \Theta$.

Let f_w, f_1, f_2, f_m denote respectively the test functions of the optimal weakly symmetric test, the optimal strongly symmetric test in $\mathscr{C}_1$, the optimal strongly symmetric test in $\mathscr{C}_2{}^*$, and the maxmin test in $\mathscr{C}_2$. Then straightforward calculations give the following comparisons.

I. $$|Ef_j\psi_i| \leqq |Ef_m\psi_i| \qquad \text{for } i = 1, 2 \text{ and } j = 1, 2 .$$

II. $$\min \{|Ef_m\psi_1|, |Ef_m\psi_2|\} \geqq \min \{|Ef_w\psi_1|, |Ef_w\psi_2|\} .$$

III. If $\sigma_{12} < 0$, or if $\sigma_{12} \geqq 0$ and $\max \{\sigma_1{}^2, \sigma_2{}^2\} > \sigma_{12} + [\sigma_1{}^2\sigma_2{}^2 - \sigma_{12}^2]^{\frac{1}{2}}$, then

$$\max \{|Ef_m\psi_1|, |Ef_m\psi_2|\} < \max \{|Ef_w\psi_1|, |Ef_w\psi_2|\} .$$

IV. If $\sigma_{12} \geqq 0$ and $\max \{\sigma_1{}^2, \sigma_2{}^2\} < \sigma_{12} + [\sigma_1{}^2\sigma_2{}^2 - \sigma_{12}^2]^{\frac{1}{2}}$ then

$$\max \{|Ef_m\psi_1|, |Ef_m\psi_2|\} \geqq \max \{|Ef_w\psi_1|, |Ef_w\psi_2|\} .$$

These comparisons are easily derived by writing $Ef\psi_i$ as functions of $\sigma_1{}^2$, $\sigma_2{}^2$, σ_{12} for the functions f in question.

7. An illustrative example. The following model was developed by Dorothy Marschak, in order to deal with experiments on the phenomenon of "memory boost," which were done by D. Krech and E. Bennett, at the University of California, Berkeley, in 1968.

For the experiment in question the response, X, of the experimental animal is classified into three types, $X = 1$ (low), $X = 2$ (medium), and $X = 3$ (high). It is assumed that a certain proportion θ_1 of untreated animals would score low, that a proportion θ_2 would score medium, and a proportion $\theta_3 = 1 - \theta_1 - \theta_2$, would score high. The test parameter ξ represents the effect of treatment. If the treatment is beneficial ($\xi > 0$), then a proportion ξ of low scoring animals would score medium, and the same proportion of medium-scoring animals would score high. If the effect of the treatment is detrimental ($\xi < 0$), then the model is symmetric; a proportion $-\xi$ of high-scoring animals would score medium, and the same proportion of medium-scoring animals would score low.

A Bernoulli random variable T ($P[T = 1] = .5$) is associated with each experimental animal. If $T = 1$, the animal is assigned to the treatment group. If $T = 0$, the animal is assigned to the control group. The response X of each experimental animal is a trinomial random variable, with parameters which depend upon the nuisance parameters θ, the test parameter ξ, and the randomization variable T. The model has the following mathematical representation:

$$p_{X,T}(x, t) = \pi^t(1-\pi)^{1-t} \prod_{i=1}^{3} [\theta_i(\xi t)]^{I_i(x)} \qquad \text{where } I_i(x) = 1 \text{ if } x = i\,, \quad = 0 \text{ if } x \neq i\,,$$

$$\begin{array}{ll} \xi < 0 & \xi > 0 \\ \theta_1(\xi t) = \theta_1 - \xi t \theta_2 & \theta_1(\xi t) = \theta_1[1 - \xi t] \\ \theta_2(\xi t) = \theta_2 + \xi t[\theta_2 - \theta_3] & \theta_2(\xi t) = \theta_2 + \xi t[\theta_1 - \theta_2] \\ \theta_3(\xi t) = \theta_3[1 + \xi t] & \theta_3(\xi t) = \theta_3 + \xi t \theta_2\,. \end{array}$$

For this model $\psi_1 \neq \psi_2$ and

$$g = (t-\pi)(-I_1(x)\theta_2/\theta_1 + I_2(x)[\theta_2 - \theta_3]/\theta_2 + I_3(x))\,,$$
$$h = (t-\pi)(-I_1(x) + I_2(x)[\theta_1 - \theta_2]/\theta_2 + I_3(x)\theta_2/\theta_3)\,,$$
$$\sigma_1^2 = \theta_2^2/\theta_1 + (\theta_2 - \theta_3)^2/\theta_2 + \theta_3$$
$$\sigma_2^2 = \theta_1 + (\theta_1 - \theta_2)^2/\theta_2 + \theta_2^2/\theta_3$$
$$\sigma_{12} = 2\theta_2 + (\theta_1 - \theta_2)(\theta_2 - \theta_3)/\theta_2\,.$$

The maxmin $C(\alpha)$ test in $\mathscr{C}_2$ is specified by the function $f_m = a_m g + b_m h$, where

$$\begin{aligned} f_m &= g/[\theta_2^2/\theta_1 - (\theta_2 - \theta_3)^2/\theta_2 + \theta_3]^{\frac{1}{2}} && \text{if } \sigma_1^2 < |\sigma_{12}|\,, \\ &= f_s && \text{if } \min\{\sigma_1^2, \sigma_2^2\} > |\sigma_{12}|\,, \\ &= h/[\theta_1 + (\theta_1 - \theta_2)^2/\theta_2 + \theta_2^2/\theta_3]^{\frac{1}{2}} && \text{if } \varphi_2^2 < |\sigma_{12}|\,, \end{aligned}$$

where

$$f_s = \frac{(\sigma_2^2 - |\sigma_{12}|)g + \operatorname{sgn}(\sigma_{12})(\sigma_1^2 - |\sigma_{12}|)}{[(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)(\sigma_1^2 - 2|\sigma_{12}| + \sigma_2^2)]^{\frac{1}{2}}}\,.$$

The strongly symmetric $C(\alpha)$ test in class $\mathscr{C}_2$ is specified by the function f_s, which is defined above. Whenever $f_m = f_s$, the asymptotic power of the strongly symmetric test and the asymptotic power of the maxmin test are the same. Whenever $f_s \neq f_m$, the asymptotic power of the maxmin test is greater than the asymptotic power of the strongly symmetric test. Figure 1 shows the regions of the parameter space in which $f_m \neq f_s$. The greatest differences in asymptotic power between the maxmin test and the strongly symmetric test are in the corner of the parameter space where θ_1 is close to 1, and θ_2 and θ_3 are close to zero, and symmetrically in the corner of the parameter space where θ_1 and θ_2 are close to zero and θ_3 is close to 1. For any positive number, M, there exists an $\varepsilon > 0$, such that $(|Ef_m\psi_1| - |Ef_s\psi_1|)/|Ef_s\psi_1| > M$ and $(|Ef_m\psi_2| - |Ef_s\psi_2|)/|Ef_s\psi_2| > M$ for $\theta_1 = 1 - 2\varepsilon$, $\theta_2 = \varepsilon$, $\theta_3 = \varepsilon$.

The highest values of $|Ef_m\psi_i|$ are in the region of the parameter space where θ_1 is close to zero, θ_2 is close to 1, and θ_3 is close to zero. In some experimental situations, the experimenter may be willing to approximate the unknown mechanism of the treatment effect by this model. In such a case the range of possible scores will be somewhat arbitrarily partitioned into high, medium and low categories. Of course there is only one, if any, such partition which will exactly fit the data. This "correct" partition of the range is in general unknown, and a partition which would result in small $\theta_1(0)$ and $\theta_3(0)$ may be chosen. The evaluation of the effectiveness of incorrect or poorly fitting models is beyond the scope of this paper.

8. Proofs of Theorem 1, Lemma 1, and Theorem 3.

THEOREM 1. *The class of $C(\alpha)$ tests based on functions of the form $f = ag + bh$ is essentially complete with respect to asymptotic power, within the class of all $C(\alpha)$ tests.*

PROOF. Let f^* be any normed Cramér function satisfying (2), and let $c_i = Ef^*\psi_i$ for $i = 1, 2$. It is sufficient to find linear combinations $f_i = a_i g + b_i h$ with the following properties:

$$Ef_i\psi_1 = c_1, \quad \text{for} \quad i = 1, 2. \qquad Ef_1\psi_2 > c_2 \qquad \text{and} \qquad Ef_2\psi_2 < c_2. \tag{8}$$

Without loss of generality assume that $c_1 > 0$. There are two cases to consider.

Case 1. $\rho^2 \neq 1$. Let

$$\begin{aligned}
a_1 &= [c_1(1-\rho^2)^{\frac{1}{2}} - \rho(\sigma_1^2 - c_1^2)^{\frac{1}{2}}]/[(1-\rho^2)^{\frac{1}{2}}\sigma_1^2]\,,\\
b_1 &= (\sigma_1^2 - c_1^2)^{\frac{1}{2}}/(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)^{\frac{1}{2}}\,,\\
a_2 &= [c_1(1-\rho^2)^{\frac{1}{2}} + \rho(\sigma_1^2 - c_1^2)^{\frac{1}{2}}]/[(1-\rho^2)^{\frac{1}{2}}\sigma_1^2]\,,\\
b_2 &= -(\sigma_1^2 - c_1^2)^{\frac{1}{2}}/(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)^{\frac{1}{2}}\,,\\
r_i &= E[f_i f^* \mid H_0] \qquad \text{for} \quad i = 1, 2\,,
\end{aligned}$$

where

$$\sigma_1^2 = Eg^2\,, \qquad \sigma_2^2 = Eh^2\,, \qquad \sigma_{12} = Egh\,, \qquad \rho = \sigma_{12}/\sigma_1\sigma_2\,,$$

all expectations being taken under H_0.

Note that $Ef\phi_j = 0$, for $j = 1, \cdots, s$, implies that $Ef\psi_1 = Efg$, and $Ef\psi_2 = Efh$. Simple algebra shows that $Ef_i g = c_1$ for $i = 1, 2$, and

$$\begin{aligned}
c_2 &= Ef^*h = Ef^*[a_i g + b_i h] - a_i Ef^*g + (1 - b_i)Ef^*h\,,\\
c_2 &= r_i - a_i c_1 + (1 - b_i)c_2\,,\\
c_2 &= (r_i - a_i c_1)/b_i\,.
\end{aligned} \tag{9}$$

When $i = 1$, $b_i > 0$ and the right-hand side of (9) is maximized when $r_i = 1$, therefore, $c_2 \leqq Ef_1 h$.

When $i = 2$, $b_i < 0$ and the right-hand side of (9) is minimized when $r_i = 1$, therefore, $c_2 \geqq Ef_2 h$.

The idea of the proof for Case 2, $\rho^2 = 1$, is the same as for Case 1, but the details are slightly different.

PROOF OF LEMMA 1. Define a function $k: R^+ \to R^+$ by

$$\Phi(-\nu_\alpha + k(x)) = \Phi(-\nu_{\alpha/2} + x) + \Phi(-\nu_{\alpha/2} - x) . \tag{10}$$

Then $H(x) = k(x)/x$ for all $x > 0$. $\Phi(-\nu_\alpha) = 2\Phi(-\nu_{\alpha/2})$ implies that $k(0) = 0$. By L'Hôpital's rule, $\lim_{x\to 0} H(x) = \lim_{x\to 0} k(x)/x = \lim_{x\to 0} k'(x)/1$. Differentiating both sides of (10) and solving for $k'(x)$ we get

$$k'(x) = \frac{\exp\{-\frac{1}{2}(-\nu_{\alpha/2} + x)^2\} - \exp\{-\frac{1}{2}(-\nu_{\alpha/2} - x)^2\}}{\exp\{-\frac{1}{2}[-\nu_\alpha - k(x)]^2\}} \tag{11}$$

$k'(0) = 0$. This demonstrates (i).

The Neyman–Pearson Lemma implies that $k(x) < x$. Equation (10) implies $k(x) - \nu_\alpha > x - \nu_{\alpha/2}$; this gives inequalities (12) which imply (ii).

$$1 - (\nu_{\alpha/2} - \nu_\alpha)/x < H(x) < 1 . \tag{12}$$

We will show that H is increasing, first for $x \leqq \nu_{\alpha/2}$ and then for $x > \nu_{\alpha/2}$.

Case 1. $x \leqq \nu_{\alpha/2}$. For $x > 0$,

$$H'(x) = \frac{d}{dx}[k(x)/x] = k'(x)/x - k(x)/x^2 > 0 \leftrightarrow xk'(x) - k(x) > 0 . \tag{13}$$

Because $k(0) = 0$, and, from a Maclaurin expansion of (11), $k'(x) > 0$ near the origin, it is sufficient to show that $xk'(x) - k(x)$ is an increasing function of x. Differentiating, we see that it is sufficient to show that $k''(x) > 0$. Differentiating both sides of (11) and solving for $k''(x)$ we have

$$\begin{aligned} k''(x) = {} & k'(x)(\nu_{\alpha/2} - x + k'(x)[k(x) - \nu_\alpha]) \\ & + 2\nu_{\alpha/2} \exp\{-\tfrac{1}{2}(\nu_{\alpha/2} + x)^2 + \tfrac{1}{2}[\nu_\alpha - k(x)]^2\} . \end{aligned} \tag{14}$$

The second term of (14) is obviously greater than zero. To see that the first term is greater than or equal to zero, use (11) to get

$$0 < k'(x) < \exp\{\tfrac{1}{2}[\nu_\alpha - k(x)]^2 - \tfrac{1}{2}(\nu_{\alpha/2} - x)^2\} . \tag{15}$$

If $k(x) - \nu_\alpha \leqq 0$, $k'(x) < 1$, because (10) implies that $k(x) - \nu_\alpha > x - \nu_{\alpha/2} \to [k(x) - \nu_\alpha]^2 < (x - \nu_{\alpha/2})^2$, and the first term of (14) is greater than or equal to zero. If $k(x) - \nu_\alpha > 0$, $k'(x) > 0$ and $x \leqq \nu_{\alpha/2}$ imply that the first term of (14) is positive.

Case 2. $x > \nu_{\alpha/2}$ (proof by contradiction). Assume that H is not increasing. Then, the differentiability of H and (12) imply that there are two points x_1 and x_2 such that $H(x_1) = H(x_2) = y$, for some $0 < y < 1$. Now, (12) implies that if

$x > (\nu_{\alpha/2} - \nu_\alpha)/(1 - y)$ then $H(x) > y$. This means that we can find points z_1 and z_2 with $z_1 < z_2$ and $z_1 \geqq \nu_{\alpha/2}$ such that

$$H(z_1) \geqq y \qquad \text{and} \qquad H(z_2) \leqq y\,, \tag{16}$$

$$H'(z_1) = H'(z_2) = 0\,, \tag{17}$$

$$H''(z_1) \leqq 0 \qquad \text{and} \qquad H''(z_2) \geqq 0\,. \tag{18}$$

Now, $H''(x) = k''(x)/x - 2H'(x)/x$. This implies that $k''(z_1) \leqq 0$ and $k''(z_2) \geqq 0$. Statements (16), (17) and (13) imply

$$k'(z_1) = H(z_1) \geqq y \qquad \text{and} \qquad k'(z_2) = H(z_2) \leqq y\,. \tag{19}$$

Combining this with (14) and the fact that (12) implies $yz_1 > \nu_\alpha$, we get

$$\begin{aligned} k''(z_1) \geqq {} & y(\nu_{\alpha/2} - z_1 + y(yz_1 - \nu_\alpha)) \\ & + 2\nu_{\alpha/2}\exp\{-\tfrac{1}{2}(z_1 + \nu_{\alpha/2})^2 + \tfrac{1}{2}(yz_1 - \nu_\alpha)^2\} \end{aligned} \tag{20}$$

$$\begin{aligned} k''(z_2) \leqq {} & y(\nu_{\alpha/2} - z_2 + y(yz_2 - \nu_\alpha)) \\ & + 2\nu_{\alpha/2}\exp\{-\tfrac{1}{2}(z_2 + \nu_{\alpha/2})^2 + \tfrac{1}{2}(yz_2 - \nu_\alpha)^2\}\,. \end{aligned} \tag{21}$$

Subtracting (20) from (21) and using the fact that for $yz > \nu_\alpha$, $\exp\{-\frac{1}{2}(z + \nu_{\alpha/2})^2 + \frac{1}{2}(yz - \nu_\alpha)^2\}$ is a decreasing function of z, we get $(z_1 - z_2)(1 - y^2) \geqq 0$, but $z_1 < z_2$, contradiction.

THEOREM 3. *The linear combination* $f^* = a^*g + b^*h$ *which maximizes* $\min\{\beta(\tau), \beta(-\tau)\}$ *in* $\mathscr{C}_1$, *is given by* (7).

PROOF. It is sufficient to find a^* and b^* such that

$$\min\{d_1(a^*, b^*), -d_2(a^*, b^*)\} = \max_{a,b}\min\{d_1(a, b), -d_2(a, b)\}$$

where $d_i(a, b) = E(ag + bh)\phi_i$ for $i = 1, 2$. The restriction $E(ag + bh)^2 = 1$ implies that for each value of a, there are only two possible values of b. These values are given by:

$$b_i = -a\rho\sigma_1/\sigma_2 - (-1)^i\rho\sigma_1[1 - a^2(1 - \rho^2)\sigma_1^2]^{\frac{1}{2}} \qquad \text{for } i = 1, 2\,.$$

Now in order that $b \in R$, it is necessary that $a \in (-K, K)$ with $K = [(1 - \rho^2)\sigma_1^2]$. Using the fact that $Eg\phi_1 = \sigma_1$, $Eg\phi_2 = \sigma_{12}$, $Eh\phi_2 = \sigma_2$, $Eh\phi_1 = \sigma_{12}$ we can write:

$$\begin{aligned} d_1(a, b_i) &= a(1 - \rho^2)\sigma_1^2 - (-1)^i[1 - a^2(1 - \rho^2)\sigma_1^2]^{\frac{1}{2}}\rho\sigma_1 \qquad \text{and} \\ d_2(a, b_i) &= (-1)^i[1 - a^2(1 - \rho^2)\sigma_1^2]^{\frac{1}{2}}\sigma_2\,. \end{aligned}$$

Note that $d_1(a, b_1) = -d_1(-a, b_2)$ and $d_2(a, b_1) = -d(-a, b_2)$. Therefore it is sufficient to consider only one pair, say (a, b_2). There are two cases to consider: $\rho \leqq 0$ and $\rho > 0$.

Case 1. $\rho \leqq 0$. The graph described by $d_1(a, b_2)$ for $a \in (-K, K)$ is a half ellipse. The function is increasing in a for $a < 1/\sigma_1$, and decreasing in a for $a > 1/\sigma_1$. The graph described by $-d_2(a, b_2)$ is also a half ellipse. It is increasing for $a < 0$, and decreasing for $a > 0$. There is only one point of intersection,

a_2, such that $d_1(a_2, b_2) = -d_2(a_2, b_2)$. The point a_2 is given by (7) and $d_1(a, b_2) < -d_2(a, b_2)$ for $a < a_2$, and $d_1(a, b_2) > -d_2(a, b_2)$ for $a > a_2$. It is easily verified, using the Cauchy–Schwarz inequality, that $a_2 \in (-K, K)$.

There are three subcases to consider.

(i)	$a_2 \leqq 0$	This implies that $a^* = 0$.
(ii)	$0 < a_2 \leqq 1/\sigma_1$	This implies that $a^* = a_2$.
(iii)	$a_2 > 1/\sigma_1$	This implies that $a^* = 1/\sigma_1$.

The proof will be given for case (i) only, the proofs of the other cases being similar; specifically: $a_2 < 0 \rightarrow d_1(0, b_2) < -d_2(0, b_2)$, and $-d_2(0, b_2) = \max_a \{-d_2(a, b_2)\}$.

Case 2. $\rho > 0$. The function $d_1(a, b_2)$ is increasing for $a > -1/\sigma_1$, and $d_1(a, b_2) > 0$ for $a > \rho/[(1 - \rho^2)\sigma_1^2] > 0$. The function $-d_2(a, b_2)$ is increasing for $a < 0$, and is decreasing for $a > 0$. Also, $-d_2(a, b_2) > 0$. Therefore, the value of a which maximizes $\min \{d_1(a, b_2), -d_2(a, b_2)\}$ must be given by a^* with the property that $d_1(a^*, b_2) = -d_2(a^*, b_2)$. The expression for a^* satisfying this equation is given by a_1 in (7). This is easily shown by setting $d_1(a^*, b_2) = -d_2(a^*, b_2)$ and solving for a^*.

9. Concluding remarks. In the search for a "fair" test of the hypothesis H_0: $\xi = 0$, against two-sided alternatives, H_1: $\xi \neq 0$, a maxmin $C(\alpha)$ test has been deduced, and it has been shown that the maxmin test is uniformly asymptotically more powerful than the optimal strongly symmetric tests, and that the maxmin test is never dominated by the weakly symmetric test of Neyman. In order to construct the maxmin test it was necessary to consider an essentially complete class of $C(\alpha)$ tests specified by linear combinations $f = ag + bh$. It was shown that the linear combination which specifies the maxmin $C(\alpha)$ test does not satisfy the Cramér regularity conditions, which were used in the proofs of Theorems 1, 2, and 3 of Neyman (1959). However, the function, f_m, which specifies the maxmin $C(\alpha)$ test does satisfy a generalized Cramér condition, and this slightly weaker condition is sufficient for the proofs of those theorems.

In conclusion, it is interesting to note that the function f_m has the following property:

$$\begin{aligned} f_m &= g/\sigma_1 && \text{if } |\sigma_{12}| > \sigma_1^2, \\ &= h/\sigma_2 && \text{if } |\sigma_{12}| > \sigma_2^2, \\ &= f_s && \text{otherwise.} \end{aligned}$$

The $C(\alpha)$ test specified by $f = g/\sigma_1$ is the uniformly asymptotically most powerful $C(\alpha)$ test against alternatives H_1': $\xi < 0$, the test specified by $f = h/\sigma_2$ is the uniformly asymptotically most powerful $C(\alpha)$ test against alternatives H_1'': $\xi > 0$; the test specified by f_s is the optimal strongly symmetric test.

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TABLE 1
The function H

x	$\alpha = .1$	$\alpha = .05$	$\alpha = .01$
.05	.04825	.05542	.06964
.10	.09605	.11014	.13794
.15	.14294	.16354	.20367
.20	.18855	.21503	.26587
.25	.23253	.26418	.32386
.30	.27463	.31067	.37726
.35	.31466	.35428	.42596
.40	.35249	.39492	.47005
.45	.38807	.43260	.50975
.50	.42139	.46737	.54538
.55	.45250	.49937	.57731
.60	.48146	.52875	.60590
.65	.50837	.55569	.63151
.70	.53335	.58038	.65448
.75	.55650	.60299	.67512
.80	.57796	.62372	.69371
.85	.59785	.64272	.71049
.90	.61629	.66017	.72568
.95	.63339	.67622	.73947
1.00	.64926	.69098	.75203
1.05	.66400	.70460	.76350
1.10	.67771	.71718	.77400
1.15	.69047	.72882	.78365
1.20	.70236	.73961	.79253
1.25	.71347	.74962	.80073
1.30	.72384	.75894	.80832
1.35	.73355	.76763	.81537
1.40	.74265	.77574	.82192
1.45	.75118	.78332	.82804
1.50	.75920	.79043	.83375
1.55	.76675	.79710	.83909
1.60	.77386	.80337	.84411
1.65	.78056	.80927	.84883
1.70	.78690	.81483	.85327
1.75	.79289	.82009	.85745
1.80	.79857	.82506	.86141
1.85	.80395	.82976	.86515
1.90	.80906	.83423	.86870
1.95	.81391	.83846	.87207
2.00	.81853	.84249	.87526

H: $R^+ \to [0, 1]$ is the function defined by

$$\Phi(-\nu_\alpha + xH(x)) = \Phi(-\nu_{\alpha/2} + x) + \Phi(-\nu_{\alpha/2} - x).$$

TABLE 2
The function H^{-1}

x	$\alpha = .1$	$\alpha = .05$	$\alpha = .01$
.025	.0259	.0225	.0179
.050	.0518	.0451	.0358
.075	.0779	.0678	.0539
.100	.1042	.0907	.0720
.125	.1307	.1138	.0904
.150	.1576	.1372	.1090
.175	.1850	.1610	.1279
.200	.2128	.1852	.1471
.225	.2413	.2099	.1668
.250	.2705	.2353	.1869
.275	.3004	.2614	.2076
.300	.3314	.2883	.2290
.325	.3634	.3161	.2510
.350	.3966	.3449	.2739
.375	.4313	.3750	.2978
.400	.4675	.4065	.3228
.425	.5056	.4396	.3490
.450	.5459	.4745	.3766
.475	.5885	.5115	.4060
.500	.6340	.5510	.4372
.525	.6829	.5934	.4707
.550	.7356	.6391	.5069
.575	.7929	.6887	.5462
.600	.8556	.7431	.5892
.625	.9250	.8032	.6368
.650	1.0024	.8703	.6898
.675	1.0898	.9461	.7497
.700	1.1898	1.0326	.8182
.725	1.3058	1.1332	.8977
.750	1.4429	1.2520	.9916
.775	1.6083	1.3953	1.1050
.800	1.8130	1.5728	1.2454
.825	2.0744	1.7994	1.4247
.850	2.4215	2.1003	1.6629
.875	2.9063	2.5207	1.9957
.900	3.6330	3.1511	2.4947
.925	4.8440	4.2015	3.3199
.950	6.3578	6.2899	4.9896
.975	12.7156	12.5798	8.7319

H^{-1} is the inverse of the function H: $R^+ \to [0, 1]$ defined by

$$\Phi(-\nu_\alpha + xH(x)) = \Phi(-\nu_{\alpha/2} + x) + \Phi(-\nu_{\alpha/2} - x).$$

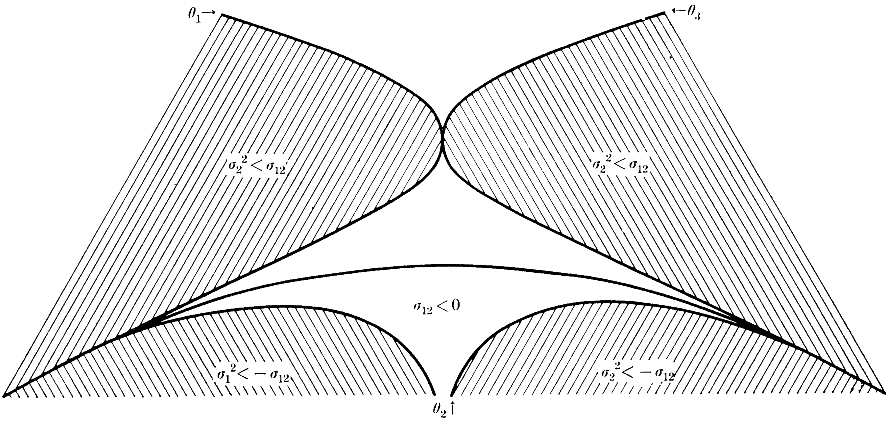

FIG. 1.

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