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# ASYMPTOTIC RESULTS FOR INFERENCE PROCEDURES BASED ON THE $r$ SMALLEST OBSERVATIONS[1]

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We consider procedures for statistical inference based on the smallest $r$ observations from a random sample. This method of sampling is of importance in life testing. Under weak regularity conditions which include the existence of a q.m. derivative for the square root of the ratio of densities, we obtain an approximation to the likelihood and establish the asymptotic normality of the approximation. This enables us to reach several important conclusions concerning the asymptotic properties of point estimators and of tests of hypotheses which follow directly from recent developments in large sample theory. We also give a result for expected values which has importance in the theory of rank tests for censored data.

**1. Introduction and summary.** Suppose that $n$ items are placed on life test and that the test is censored at the time the $r$th failure occurs. Below, we consider asymptotic properties of statistical procedures based on the first $r$ order statistics $Y_1, \cdots, Y_r$ from a random sample of size $n$ from a cdf $F_\theta$ where $\theta \in R^k$. The law $P_{n,\theta}$ of the $r$ order statistics is related to a pdf

$$(1.1) \qquad \frac{n!}{(n-r)!} f_\theta(y_1) \cdots f_\theta(y_r)[1 - F_\theta(y_r)]^{n-r} ,$$

where $f_\theta$ is the population pdf. All of our results follow directly from an expansion of the likelihood ratio

$$(1.2) \qquad \Lambda_{r,n}[\theta_n, \theta_0] = \sum_{j=1}^r \log \frac{f_{\theta_n}(Y_j)}{f_{\theta_0}(Y_j)} + (n-r) \log \frac{1 - F_{\theta_n}(Y_r)}{1 - F_{\theta_0}(Y_r)} ,$$

for sequences $\{\theta_n\}$ with $\theta_n = \theta_0 + h_n n^{-\frac{1}{2}}$, $h_n \to h$. In particular, Theorem 3.1 establishes that, as $n \to \infty$, $r/n \to p$

$$(1.3) \qquad \Lambda_{r,n}[\theta_n, \theta_0] - h'\Delta_n(\theta_0) \to_{P_{n,\theta_0}} -\tfrac{1}{2}h'\Gamma_p(\theta_0)h ,$$

where $\Gamma_p(\theta_0)$ is the Fisher information for the censored case (see 3.16) and

$$(1.4) \qquad \Delta_n(\theta_0) = \sum_{j=1}^r \frac{2}{n^{\frac{1}{2}}} \dot\varphi(Y_j) - (n-r) \frac{\int_{-\infty}^{Y_r} 2\dot\varphi f_{\theta_0}}{1 - F_{\theta_0}(Y_r)} ,$$

with $\dot\varphi$ defined in assumptions (A). Corollary 3.2 shows that $\mathscr{L}[\Delta_n \mid P_{n,\theta_0}]$ converges to a normal distribution with zero mean vector and covariance $\Gamma_p$.

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Several important conclusions can be drawn from the convergence (1.3) and the asymptotic normality of $\Delta_n(\theta_0)$. These in turn employ assumptions (A) and (B) below together with the assumption that $\Gamma_p(\theta_0)$ is positive definite.

The first set of conclusions, dealing with asymptotic properties of point estimators, follows immediately from the representation theorem of Hájek (1970) (see also Roussas and Soms (1971)). Hájek's proof requires only (1.3) and the asymptotic normality of $\Delta_n(\theta_0)$. The main conclusions for estimators are given by (E1), (E2) and (E3).

*Asymptotic efficiency of point estimators.* These results apply to any sequence of estimators $T_n = T_n(Y_1, \cdots, Y_r)$ such that $\mathscr{L}[n^{\frac{1}{2}}(T_n - \theta_0) - h \mid P_{n,\theta_n}] \to L(v)$, for all $h \in R^k$, at the continuity points of $L(v)$. $L(v)$ need not be a normal distribution.

(E1) $L(v)$ has the representation $L(v) = \int \Phi_{\Gamma_p}(v - u)\, dG(u)$ where $G(u)$ is a distribution in $R^k$ and $\Phi_{\Gamma_p}$ is the normal cdf with zero mean vector and covariance $\Gamma_p^{-1}(\theta_0)$.

(E2) $$\limsup P_{n,\theta_0}[n^{\frac{1}{2}}(T_n - \theta_0) \in C] \leqq \int_C d\Phi_{\Gamma_p}$$

for all convex symmetric sets $C$ in $R^k$.

(E3) $$\liminf E[n^{\frac{1}{2}}h'(T_n - \theta_0)]^2 \geqq h'\Gamma_p^{-1}(\theta_0)h\,, \qquad \text{all } h \in R^k\,,$$

so that the limit covariance $D$, if it exists, satisfies $D - \Gamma_p^{-1}$ nonnegative definite.

The conclusions regarding asymptotically optimal tests of hypotheses follow from the development in Johnson and Roussas (1969), (1970), (1971), since the relevant proofs there do not use the Markovian character of the observations. Because this is not in as convenient a form for our purposes as Hájek (1970), we review the main steps in the development before presenting the main conclusions on testing hypothesis.

It is first shown in Section 4 of Johnson and Roussas (1970) that $\rho(\Delta_n) \to \rho(\Delta)$, in law, where $\Delta$ is distributed as $N(0, \Gamma_p)$, under $P_{\theta_0}$, and $\rho(\cdot)$ is the usual Euclidean norm. A truncated version $\Delta_n^*$ of $\Delta_n$ is then constructed, using $\rho(\Delta_n)$, in such a way that $P_{\theta_0}[\Delta_n^* \neq \Delta_n] \to 0$ and the moment generating function of $\rho(\Delta_n^*)$ converges to that of $\rho(\Delta)$.

Based on the sequence $\Delta_n^*$, an exponential family

$$dR_{n,h} = e^{-B_n(h)} e^{h\Delta_n^*}\, dP_{n,\theta_0}$$

is constructed which approximates $P_{n,\theta_n'}$. Theorem 5.1 shows that $\sup\{\|R_{n,h} - P_{n,\theta_n}\|,\ h \varepsilon \text{ bounded set}\} \to 0$ where $\|\cdot\|$ is the total variation. This exponential family approximation is also crucial to Hájek's proof of the representation theorem for point estimators.

Next, Theorem 6.1 shows that $E_{\theta_0}[Z_n \mid \Delta_n]$ has the same local power, asymptotically, as any test function $Z_n$. Furthermore, Theorem 6.3 establishes that test functions based on $\Delta_n$ and $\Delta_n^*$, respectively, have the same asymptotic local power functions. Hence, asymptotically, we can restrict ourselves to tests based on $\Delta_n^*$ or, if we prefer, $\Delta_n$.

The method of establishing the optimality of tests based on $\Delta_n(\theta_0)$ is by assuming the contrary holds and then obtaining a contradiction for the test based on $\Delta_n^*$ rather than $\Delta_n$. This conclusion employs the fact that $\Delta_n^*$ is sufficient for the approximating exponential family $R_{n,h}$. The conclusions below for one-sided tests can be obtained from the Neyman–Pearson lemma as in Johnson and Roussas (1969). Alternatively, these could be derived in the same manner as those for two-sided tests using the exponential approximation and the fact that a test based on $\Delta_n^*$ is UMP unbiased. The conclusions for tests concerning multidimensional parameters, in turn, use the admissibility of the class of tests based on $\Delta_n^*$ which reject outside of convex sets. The complete details appear in Johnson and Roussas (1971). All of this development uses only (1.3) and the limit law of $\Delta_n$.

We now state our main conclusions regarding statistical tests of hypotheses based on censored data.

*Asymptotically optimal tests of hypotheses.* The following are established for local alternatives of order $\sqrt{n}$ away from $\theta_0$. To obtain global results, we would need an assumption like (A5) of Johnson and Roussas (1969).

(T1) Let $\theta \subset R$ and let $\eta_\alpha$ denote the upper $\alpha$th point of a standard normal. Then, the test $\phi_n$ which rejects $H_0$: $\theta = \theta_0$ for $\Delta_n(\theta_0) > \eta_\alpha \Gamma_p^{\frac{1}{2}}$ is asymptotically most powerful for local alternatives in that for any other sequence of tests $\{\lambda_n\}$, with $E_{\theta_0} \lambda_n \to \alpha$,

$$\limsup \left[\sup_{0 \leqq h < c} (E_{\theta_n} \lambda_n - E_{\theta_n} \phi_n)\right] = 0 .$$

(T2) The test $\phi_{1n}$ which rejects for $|\Delta_n(\theta_0)| > \eta_{\alpha/2} \Gamma_p^{\frac{1}{2}}$ is asymptotically most powerful unbiased. For any other sequence $\{\lambda_n\}$ of tests which is asymptotically of level $\alpha$ and $\liminf \{\inf E_\theta \lambda_n\} \geqq \alpha$, $\limsup \{\sup [E_\theta \lambda_n - E_\theta \phi_{1n}]\} \leqq 0$ where inf and sup are over bounded sets of $h$ and $\theta = \theta_0 + hn^{-\frac{1}{2}}$.

(T3) For testing $H_0$: $\theta = \theta_0$ vs. $H_1$: $\theta \neq \theta_0$ when $\theta \subset R^k$, the test which rejects for $\Delta_n' \Gamma_p^{-1} \Delta_n$ large has asymptotically best average power over certain ellipsoids and is asymptotically most stringent (see Johnson and Roussas (1971) for the relevant definitions).

The regularity conditions imposed here are much weaker than those imposed in previous papers on life testing. For instance, they include the normal, log normal, Weibull, exponential and gamma. In the latter cases, the location parameter fixes the support and this must be known, otherwise $n$ is not necessarily the correct normalization. See David (1970), Chapter 6, for a survey of more applications. Chernoff, Gastwirth and Johns (1967) established lower bounds for the variance of point estimators of location and scale parameters. They require that the partials of $\log f$ exist everywhere and a condition on $f_{\theta_0}''$. This is slightly stronger than our assumptions even for this special case. The results here extend their optimal estimator to a wider class than those which are asymptotically normal.

Incidentally, it can be readily observed from the discussion in Section 3 that the limiting covariance $\Gamma_p(\theta_0)$, given by (3.16), is also the covariance for fixed time censoring.

Section 4 contains a lemma which shows the equality of the expected value of the last term in $\Delta_n$ and the expected value of the scores evaluated at the unobserved order statistics. This result is also of importance in the derivation of locally most powerful rank tests for a general parameter. It is also shown that $E[\Delta_n(\theta_0)]$ is the zero vector for each $n$. We conclude with an application to the double exponential with $p = \frac{1}{2}$.

**2. Assumptions and preliminary results.** We first make some smoothness assumptions regarding the law of the univariate distributions. These are similar to those employed in Johnson and Roussas (1970) except that they are further specialized to Lebesgue measure. Although most of the results hold without this specialization, there would be difficulty in defining the censoring scheme without it.

Let $\Theta$ be an open subset of $R^k$, $(\mathscr{X}, \mathscr{A})$ a measurable space and, for each $\theta \in \Theta$, $Q_\theta$ a probability on $(\mathscr{X}, \mathscr{A})$ such that $X_1, X_2, \cdots, X_n, \cdots$ are independent and identically distributed with $X_n$ taking values in the Borel real line $(R, \mathscr{B})$. Set $\mathscr{A}_n$ equal to the $\sigma$-field induced by $(X_1, \cdots, X_n)$ and let $Q_{n,\theta}$ denote the restriction of $Q_\theta$ to $\mathscr{A}_n$. We can, if we wish, transfer to the coordinate space.

*Assumptions* (A).

(A1) The law of $X_1$ has pdf $f_\theta(x)$ with respect to Lebesgue measure and the set where it is positive does not depend on $\theta$.

(A2) Set $\varphi(\theta, \theta^*) = [f_{\theta^*}(x)/f_\theta(x)]^{\frac{1}{2}}$. Then

(i) For each $\theta \in \Theta$, $\varphi(\theta, \theta^*)$ is differentiable in q.m. at $(\theta, \theta)$ when $P_{1,\theta}$ is employed. Denote this derivative by $\dot\varphi(\theta)$.

(ii) $\dot\varphi(\theta)$ is $X_1^{-1}(\mathscr{B}) \times \mathscr{C}$ measurable where $\mathscr{C}$ is the class of Borel subsets of $\Theta$.

(iii) For every $\theta \in \Theta$, $4E_\theta[\dot\varphi(\theta)\dot\varphi'(\theta)]$ is positive definite.

Although $\dot\varphi = \dot\varphi(x, \theta)$, we will write $\dot\varphi(\theta)$ when we want to emphasize the parameter and $\dot\varphi(X_1)$ when we want to consider it as a function of a random variable. We allow a similar abuse of notation with $\varphi = \varphi(X, \theta, \theta_n)$ and sometimes write $\varphi(X_1)$.

Under assumptions (A), we have the following result when the $X_n$ are governed by $Q_{n,\theta_0}$ and the alternatives are of the form $\theta_n = \theta_0 + h_n n^{-\frac{1}{2}}$ with $h_n \to h$.

$$\max_{i\leqq n} |\varphi(X_i) - 1| \to_{Q_{n,\theta_0}} 0\,, \tag{2.1}$$

$$n^{\frac{1}{2}}(\varphi(X_1) - 1) \to_{\text{q.m.}} h'\dot\varphi(\theta_0)\,, \tag{2.2}$$

$$n^{\frac{1}{2}}(\varphi^2(X_1) - 1) \to 2h'\dot\varphi(\theta_0) \quad \text{in 1st mean}\,, \tag{2.3}$$

$$E_{\theta_0}\dot\varphi(X_1) = 0 \quad (k \times 1 \text{ column vector})\,. \tag{2.4}$$

These are (3.1.3), (3.1.2), Lemma 3.1.3 and Lemma 3.1.4 (i) in Roussas (1965).

The existence of a q.m. derivative for $\varphi$ implies the existence of a pointwise partial for the cdf $F_\theta(x)$. For notational convenience, we sometimes write $f$ for $f_{\theta_0}$.

LEMMA 2.1. *Under assumptions* (A), *uniformly on bounded sets of* $h \in R^k$ *and* $z$,

$$n^{\frac{1}{2}}\left[F_{\theta_n}(z) - F_{\theta_0}(z) - \frac{h'}{n^{\frac{1}{2}}}\dot{F}(z)\right] \to 0 , \tag{2.5}$$

*where*

$$h'\dot{F}(z) = \textstyle\int_{-\infty}^{z} 2h'\dot{\varphi} f . \tag{2.6}$$

PROOF. We write $F_{\theta_n}(z) - F_{\theta_0}(z) = \int_{-\infty}^{z} (\varphi^2 - 1)f$, so that the difference is

$$n^{\frac{1}{2}} \int_{-\infty}^{z}\left[(\varphi^2 - 1) - \frac{2h'}{n^{\frac{1}{2}}}\dot{\varphi}\right]f$$

which, by (2.2), goes to zero uniformly in bounded sets of $h$.

We now describe the sampling scheme. Only the first $r$ order statistics $(Y_1, \cdots, Y_r)$ are observed where $r$ is selected so that

$$\left|\frac{r}{n} - p\right| < \frac{1}{n} \qquad \text{with} \quad 0 < p < 1 . \tag{2.7}$$

We further assume that $f_{\theta_0}$ is positive in a neighborhood of the $p$th percentile $\xi_p$ so that it is unique.

The next result is a specialization of Bahadur (1966) to the uniform order statistics $F(Y_i)$, $i = 1, \cdots, n$. Let

$$Z_n(t) = \#X_1, \cdots, X_n \leqq \xi_t , \qquad 0 < t < 1 . \tag{2.8}$$

LEMMA 2.2. *With* $r$ *given by* (2.7) *and* $Z_n \equiv Z_n(p)$ *by* (2.8),

$$n^{\frac{1}{2}}\left[\frac{Z_n}{n} - p + F_{\theta_0}(Y_r) - p\right] \to_{Q_{n,\theta_0}} 0 .$$

Below, we investigate the behavior of certain functions over the random set

$$A_{r,n} = \{(Y_r, \xi_p] \text{ if } Y_r < \xi_p \text{ and } (\xi_p, Y_r] \text{ if } Y_r \geqq \xi_p\} . \tag{2.9}$$

We also have the following property for the $Z_n(t)$.

LEMMA 2.3. *Let* $Z_n(p)$ *be defined by* (2.8), *then*

$$\sup_{\xi_s, \xi_t \in A_{r,n}} n^{\frac{1}{2}}\left|\frac{Z_n(s)}{n} - \frac{Z_n(t)}{n} - [F_{\theta_0}(\xi_s) - F_{\theta_0}(\xi_t)]\right| \to_{Q_{n,\theta_0}} 0 .$$

PROOF. Since $Z_n(s) = \#F_{\theta_0}(X_i) \leqq p$ and $F_{\theta_0}(X_i)$ is uniform, it is sufficient to show convergence in probability for uniform variables. It is well known that $V_n(t) = n^{\frac{1}{2}}(Z_n(t)/n - t)$ converges weakly to a Brownian bridge. Therefore, by a characterization of tightness in $C$ (see Theorem 8.2, Billingsley (1968)) the modulus of continuity of a continuous piecewise linear approximation to $\tilde{V}_n(t)$ is small with high probability and the same holds true for $V_n(t)$. In particular, given any $\varepsilon$, $\eta$ there exists a $\delta$, $0 < \delta < 1$, such that

$$Q_{n,\theta}[\sup_{t \leqq s \leqq t+\delta} |V_n(t) - V_n(s)| > \varepsilon] \leqq \eta \qquad \text{all sufficiently large } n .$$

For a direct verification see (13.16) of Billingsley (1968). Since $\{(s, t): \xi_s, \xi_t \in A_{r,n}\} \subset \{(s, t): |s - t| \leqq \delta\}$ for sufficiently large $n$ with probability one, the result follows.

In order to establish our main results, we have to know that $\varphi$ and $\dot{\varphi}$ act smoothly at $\xi_p$. This does not seem to follow from the quadratic mean calculus and we must make additional assumptions. We will first state these in the form needed later and then prove a lemma which leads to sufficient conditions which are easy to verify.

The assumptions may be expressed in terms of the indicator function $I_{A_{r,n}}$.

*Assumptions* (B).

$$\text{(B1)} \qquad \textstyle\sum_1^n (\varphi - 1) I_{A_{r,n}} - n \int_{A_{r,n}} (\varphi - 1) f \rightarrow_{Q_{n,\theta_0}} 0 ,$$

$$\text{(B2)} \qquad \sum_1^n \frac{h'\dot{\varphi}}{n^{\frac{1}{2}}} I_{A_{r,n}} - n^{\frac{1}{2}} \int_{A_{r,n}} (h'\dot{\varphi}) f \rightarrow_{Q_{n,\theta_0}} 0 .$$

In order to see what conditions on the pdf's would imply (B1) and (B2), we prove the following.

LEMMA 2.4. *For all sufficiently large $n$, let $\psi_n$ be the difference of two non-decreasing functions over the interval $\xi_p \pm n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}$ where each is essentially bounded by $M$. Then*

$$\sum_1^n \frac{\psi_n}{n^{\frac{1}{2}}} I_{A_{r,n}} - n^{\frac{1}{2}} \int_{A_{r,n}} \psi_n f \rightarrow_{Q_{n,\theta_0}} 0 .$$

PROOF. Without loss of generality, we assume that $\psi_n$ is non-decreasing on the interval and we define $\psi_n^{-1}$ by $\inf\{x: \psi_n(x) \geqq a\}$. For fixed $n$ and arbitrary $\varepsilon$, consider a partition $\{a_l\}$ of $[-M, M]$ with norm less than $\varepsilon$ and not more than $2M\varepsilon^{-1} + 3$ terms. Set $b_l = \psi_n^{-1}(a_l)$ and

$$\text{(2.10)} \qquad M_l = \operatorname{ess\,sup}_{[b_l, b_{l+1}]} \psi_n(x) , \qquad m_1 = \operatorname{ess\,inf}_{[b_l, b_{l+1}]} \psi_n(x) .$$

Then,

$$\text{(2.11)} \qquad M_l - m_l \leqq \varepsilon , \qquad \text{each } l .$$

Furthermore, if $Z_{nl}^* = \#X_i$ belonging to $[b_l, b_{l+1}]$, then

$$\begin{aligned} \text{(2.12)} \qquad & \sum_1^n \frac{\psi_n}{n^{\frac{1}{2}}} I_{[b_l, b_{l+1})} - n^{\frac{1}{2}} \int_{b_l}^{b_{l+1}} \psi_n f \\ & \leqq M_l Z_{nl}^* n^{-\frac{1}{2}} - m_l n^{\frac{1}{2}}[F_{\theta_0}(b_{l+1}) - F_{\theta_0}(b_l)] \\ & \leqq M_l\{Z_{nl}^* n^{-\frac{1}{2}} - n^{\frac{1}{2}}[F_{\theta_0}(b_{l+1}) - F_{\theta_0}(b_l)]\} \\ & \qquad + \varepsilon n^{\frac{1}{2}}[F_{\theta_0}(b_{l+1}) - F_{\theta_0}(b_l)] , \qquad \text{all } l . \end{aligned}$$

The lower bound has $M_l$ replaced by $m_l$ and $\varepsilon$ by $-\varepsilon$. Define $Z_n(t)$ by (2.8) so that, setting $b_l = \xi_{p+\Delta_l}$,

$$\text{(2.13)} \qquad Z_{nl}^* = Z_n(p + \Delta_{l+1}) - Z_n(p + \Delta_l) .$$

Then, for all $l$ such that $b_l, b_{l+1} \in A_{r,n}$,

$$n^{\frac{1}{2}}\left|\frac{Z_{nl}^*}{n} - [F_{\theta_0}(b_{l+1}) - F_{\theta_0}(b_l)]\right|$$
$$\leqq \sup_{\xi_s, \xi_t \in A_{r,n}} n^{\frac{1}{2}}\left|\frac{Z_n(s)}{n} - \frac{Z_n(t)}{n} - [F_{\theta_0}(\xi_s) - F_{\theta_0}(\xi_t)]\right|,$$

and the r.h.s., which does not depend on the partition within $A_{r,n}$, converges in probability to zero by Lemma 2.3. Now $Y_r$ belongs to the interval $\xi_p \pm n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}$, for all sufficiently large $n$, with probability one. If for each $n$, we include $\xi_p$ and $Y_r$ in the partition, the result follows if we add the inequalities that correspond to the interval between $\xi_p$ and $Y_r$, and employ the asymptotic normality of $n^{\frac{1}{2}}[F_{\theta_0}(Y_r) - p]$ and the fact that $\varepsilon$ is arbitrary.

COROLLARY 2.A. *If, for all sufficiently large $n$,*

(B′1) *$n^{\frac{1}{2}}(\varphi - 1)$ is the difference of two non-decreasing functions on the interval $\xi_p \pm n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}$ and each is essentially bounded by $M$, then* (B1) *is satisfied. The two functions and $M$ may depend on $h$. If there is a version of $\dot{\varphi}$ such that $\dot{\varphi}$ has one-sided limits at $\xi_p$, then* (B2) *is satisfied for each $h \in R^k$.*

PROOF. Inspection of the previous proof shows that (2.11) and (2.12) can be established, under (B′2), with a single interval $(Y_r, \xi_p]$ or $(\xi_p, Y_r]$.

REMARK. For a location or scale parameter, a simple sufficient condition for (B′1) and (B′2) is that $f'_{\theta_0}$ is continuous at $\xi_p$. The existence of a continuous derivative insures one-sided monotoneness, for sufficiently large $n$, and the mean value theorem gives uniform boundedness. This includes most one-parameter applications.

In the remaining sections, we will often employ the joint distribution $P_{n,\theta}$ of the first $r$ order statistics since the probabilities can be computed under $Q_{n,\theta}$ or $P_{n,\theta}$.

**3. Proof of main results.** In this section, we employ the previous results to obtain the expansion of the likelihood and its asymptotic distribution. We first note that, from Lemma 2.1, $|[1 - F_{\theta_n}(Y_r)]/[1 - F_{\theta_0}(Y_r)] - 1| \to 0$ in probability since $Y_r \to \xi_p$ in probability. The expansion

$$\log Z = (Z-1) - \tfrac{1}{2}(Z-1)^2 + c(Z-1)^3, \quad |c| \leqq 3 \text{ for } |Z-1| \leqq \tfrac{1}{2} \tag{3.1}$$

is then applied to each term of $\Lambda_{r,n}$.

LEMMA 3.1. *Under Assumptions* (A),

$$\begin{aligned}\Lambda_{r,n} = &\left\{\textstyle\sum_{j=1}^r 2[\varphi(Y_j) - 1] + (n-r)\left[\frac{F_{\theta_0}(Y_r) - F_{\theta_n}(Y_r)}{1 - F_{\theta_0}(Y_r)}\right]\right\} \\ &- \frac{1}{2}\left\{\textstyle\sum_{j=1}^r 2[\varphi(Y_j) - 1]^2 + (n-r)\left[\frac{F_{\theta_0}(Y_r) - F_{\theta_n}(Y_r)}{1 - F_{\theta_0}(Y_r)}\right]^2\right\} \\ &+ W_n,\end{aligned} \tag{3.2}$$

*where $W_n$ converges in probability to zero.*

PROOF. The expansion follows from (3.1) and the result for $W_n$ from (2.1) and the next two lemmas which show that the terms in the second bracket converge to constants.

LEMMA 3.2. *Under Assumptions* (A),

$$(n-r)\left[\frac{F_{\theta_n}(Y_r) - F_{\theta_0}(Y_r)}{1 - F_{\theta_0}(Y_r)}\right]^2 \rightarrow_{P_{n,\theta_0}} \frac{(h'\dot{F}(\xi_p))^2}{1-p} = \frac{[2\int_{-\infty}^{\xi_p} h'\dot{\varphi}f]^2}{1-p}. \tag{3.3}$$

PROOF. Since $Y_r \to p$ in probability and $F_{\theta_0}$ is continuous, it is sufficient to show that $n^{\frac{1}{2}}[F_{\theta n}(Y_r) - F_{\theta_0}(Y_r)] \to h'\dot{F}(\xi_p)$; but this follows from Lemma 2.1.

LEMMA 3.3. *Under Assumptions* (A),

$$\textstyle\sum_{j=1}^r [\varphi(Y_j) - 1]^2 - n^{-1}\sum_{j=1}^r [h'\dot{\varphi}(Y_j)]^2 \rightarrow_{P_{n,\theta_0}} 0, \tag{3.4}$$

$$\textstyle n^{-1}\sum_{j=1}^r [h'\dot{\varphi}(Y_j)]^2 \rightarrow_{P_{n,\theta_0}} \int_{-\infty}^{\xi_p} [h'\dot{\varphi}]^2 f. \tag{3.5}$$

PROOF. The Markov inequality gives the bound

$$\varepsilon^{-1}\textstyle\sum_{j=1}^r E\left|[\varphi(Y_j) - 1]^2 - \left[\frac{h'}{n^{\frac{1}{2}}}\dot{\varphi}(Y_j)\right]^2\right| \leqq \varepsilon^{-1}nE\left|[\varphi(X_1) - 1]^2 - \left[\frac{h'}{n^{\frac{1}{2}}}\dot{\varphi}(X_1)\right]^2\right|$$

for the probability that the r.h.s. of (3.4) exceeds $\varepsilon > 0$ and this bound converges to zero (see Roussas (1965), equation (3.1.19)).

Next, (3.5) follows from the law of large numbers since

$$\textstyle n^{-1}\sum_{j=1}^r [h'\dot{\varphi}(Y_j)]^2 - n^{-1}\sum_{i=1}^n [h'\dot{\varphi}(X_i)I_{(-\infty,\xi_p]}]^2 \to 0 \quad \text{in probability.}$$

This last difference is dominated by $\sum_1^n n^{-1}h'\dot{\varphi}I_{(\xi_p-\delta,\xi_p+\delta)}$ for all sufficiently large $n$ with $\delta$ arbitrary.

For notational convenience, we set $B_p = (-\infty, \xi_p]$ and introduce two statistics corresponding to the case of censoring at a fixed percentile $\xi_p$.

$$S_p = \textstyle\sum_{i=1}^n \{2(\varphi - 1)I_{B_p} + (1-p)^{-1}[F_{\theta_0}(\xi_p) - F_{\theta n}(\xi_p)]I_{B_p^c}\}, \tag{3.6}$$

$$\dot{S}_p = \textstyle\sum_{i=1}^n \left\{\frac{2h'}{n^{\frac{1}{2}}}\dot{\varphi}I_{B_p} - \frac{h'\dot{F}(\xi_p)}{n^{\frac{1}{2}}(1-p)} I_{B_p^c}\right\}. \tag{3.7}$$

These will later be compared with the statistics for censoring at the $r$th order statistic. Namely,

$$S_r = \textstyle\sum_{j=1}^r 2[\varphi(Y_j) - 1] + (n-r)\left[\frac{F_{\theta_0}(Y_r) - F_{\theta_n}(Y_r)}{1 - F_{\theta_0}(Y_r)}\right], \tag{3.8}$$

$$\dot{S}_r = \textstyle\sum_{j=1}^r \frac{2h'}{n^{\frac{1}{2}}}\dot{\varphi}(Y_j) - \frac{(n-r)}{n^{\frac{1}{2}}}\frac{h'\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)}. \tag{3.9}$$

LEMMA 3.4. *Under Assumptions* (A),

$$E_{\theta_0}[S_p - \dot{S}_p] = -nE[(\varphi(X_1) - 1)^2 I_{B_p}] \to -E(h'\dot{\varphi})^2 I_{B_p},$$
$$\operatorname{Var}[S_p - \dot{S}_p] \to 0.$$

PROOF. Set $p_n = F_{\theta_n}(\xi_p)$ and consider the identity

$$\varphi^2 - p_n/p = (\varphi - 1)^2 + 2(\varphi - 1) + 1 - p_n/p .$$

Multiplying both sides by $I_{B_p}$ and taking expected values gives $E[2(\varphi - 1)I_{B_p}] = -E(\varphi - 1)^2 I_{B_p} + p_n - p$ so that $E[S_p - \dot{S}_p] = -nE(\varphi - 1)^2 I_{B_p}$, and this converges to $-E(h'\dot{\varphi})^2 I_{B_p}$ since $n^{\frac{1}{2}}(\varphi - 1) \to h'\dot{\varphi}$ in q.m. by (2.2). Also,

$$\begin{aligned}
&\operatorname{Var}[S_p - \dot{S}_p] \\
&\quad = n \operatorname{Var}\left\{\left[2(\varphi - 1) - \frac{2h'}{n^{\frac{1}{2}}}\dot{\varphi}\right] I_{B_p} + (1 - p)^{-1}\left[p - p_n + \frac{h'}{n^{\frac{1}{2}}}\dot{F}\right] I_{B_p}\right\} \\
&\quad \leqq 16E\{[n^{\frac{1}{2}}(\varphi - 1) - h'\dot{\varphi}]^2 I_{B_p}\} + 4(1 - p)^{-1}\{n^{\frac{1}{2}}(p - p_n) + h'\dot{F}\}^2 ,
\end{aligned}$$

which converges to zero by (2.2) and Lemma 2.1.

We now employ the statistics (3.6), (3.7) and the result for first moments to obtain an approximation to $S_r$ in the expansion of $\Lambda_{r,n}$. Here we require the extra smoothness assumptions on the pdf's at $\xi_p$.

LEMMA 3.5. *Under Assumptions* (A) *and Assumptions* (B) (*or* (B′1) *and* (B′2)),

$$S_r - \dot{S}_r - (S_p - \dot{S}_p) \to_{P_{n,\theta_0}} 0 .$$

PROOF. Let $Z_n \equiv Z_n(p)$ be defined by (2.8). First, we have

$$[(n - Z_n) - (n - r)]\left[F_{\theta_0}(\xi_p) - F_{\theta_n}(\xi_p) + \frac{h'}{n^{\frac{1}{2}}}\dot{F}(\xi_p)\right] \to_{P_{n,\theta_0}} 0 \tag{3.10}$$

by Lemma 2.1 and the asymptotic normality of the binomial variable $Z_n$. Next, we write

$$F_{\theta_0}(z) - F_{\theta_n}(z) + \frac{h'}{n^{\frac{1}{2}}}\dot{F}(z) = -\textstyle\int_{-\infty}^{z}\left[(\varphi^2 - 1) - \frac{2h'}{n^{\frac{1}{2}}}\dot{\varphi}\right] f$$

and employ the asymptotic normality of $F_{\theta_0}(Y_r)$ to give

$$[1 - F_{\theta_0}(Y_r)]^{-1} = (1 - p)^{-1}\{1 + (1 - p)^{-1}[F_{\theta_0}(Y_r) - p] + o_p(n^{-\frac{1}{2}})\} . \tag{3.11}$$

Together, (3.10) and (3.11) give

$$\begin{aligned}
&\frac{(n - r)}{1 - F_{\theta_0}(Y_r)}\left[F_{\theta_0}(Y_r) - F_{\theta_n}(Y_r) + \frac{h'}{n^{\frac{1}{2}}}\dot{F}(Y_r)\right] \\
&\quad - \textstyle\sum_1^n (1 - p)^{-1}\left[p - F_{\theta_n}(\xi_p) + \frac{h'}{n^{\frac{1}{2}}}\dot{F}(\xi_p)\right] I_{B_p^c} \\
&\quad = \frac{(n - r)/n}{1 - p}\left[-n \textstyle\int_{\xi_p}^{Y_r}\left[(\varphi^2 - 1) - \frac{2h'}{n^{\frac{1}{2}}}\dot{\varphi}\right] f\right] + o_p(n^{-\frac{1}{2}}) + o_p(1) \\
&\qquad - \frac{(n - r)/n}{1 - p} \cdot n^{\frac{1}{2}}\left[\frac{F_{\theta_0}(Y_r) - p}{1 - p}\right] \cdot n^{\frac{1}{2}} \textstyle\int_{-\infty}^{Y_r}\left[(\varphi^2 - 1) - \frac{2h'}{n^{\frac{1}{2}}}\dot{\varphi}\right] f \\
&\quad = -n \textstyle\int_{\xi_p}^{Y_r}\left[(\varphi^2 - 1) - \frac{2h'}{n^{\frac{1}{2}}}\dot{\varphi}\right] f + o_p(1) .
\end{aligned} \tag{3.12}$$

The last equality follows since $F_{\theta_0}(Y_r)$ is asymptotically normal and $n^{\frac{1}{2}}(\varphi^2 - 1)$ converges in first mean to $2h'\dot{\varphi}$ by (2.3).

From the definitions (3.6)—(3.9) of the statistics, we now see that it is sufficient to show that

$$(3.13)\qquad \sum_{Y_i \in A_{r,n}} \left[2(\varphi - 1) - \frac{2h'}{n^{\frac{1}{2}}} \dot{\varphi}\right] - n \int_{A_{r,n}} \left[(\varphi^2 - 1) - \frac{2h'}{n^{\frac{1}{2}}} \dot{\varphi}\right] f \to_{P_{n,\theta_0}} 0 ,$$

where $A_{r,n}$ is defined by (2.9). To this end, let $B_\varepsilon$ be an interval about $\xi_p$ such that $\int_{B_\varepsilon} (h'\dot{\varphi})^2 f < \varepsilon$. Then, lim sup $\int_{A_{r,n}} n(\varphi - 1)^2 f < \varepsilon$ by the first mean convergence of $n(\varphi - 1)^2$. Since $(\varphi - 1)^2 = (\varphi^2 - 1) - 2(\varphi - 1)$, we see that it remains to show that

$$(3.14)\qquad \begin{aligned} &\sum_{Y_i \in A_{r,n}} 2(\varphi - 1) - 2n \int_{A_{r,n}} (\varphi - 1) \to_{P_{n,\theta_0}} 0 , \\ &\sum_{Y_i \in A_{r,n}} \frac{2h'}{n^{\frac{1}{2}}} \dot{\varphi} - 2n^{\frac{1}{2}} \int_{A_{r,n}} h'\dot{\varphi} f \to_{P_{n,\theta_0}} 0 . \end{aligned}$$

This follows directly from Corollary 2.4 under the assumptions (B′1) and (B′2) on $\varphi - 1$ and $\dot{\varphi}$.

THEOREM 3.1. *Under the Assumptions* (A) *and* (B) (*or* (B′1) *and* (B′2)), *for each alternative* $h_n \to h$,

$$(3.15)\qquad \Lambda_{r,n} - \frac{h'}{n^{\frac{1}{2}}} \left[\sum_{i=1}^{r} 2\dot{\varphi}(Y_i) - \frac{(n-r)}{1 - F_{\theta_0}(Y_r)} \dot{F}(Y_r)\right] \to -\tfrac{1}{2} h' \Gamma_p(\theta_0) h ,$$

*where*

$$(3.16)\qquad \Gamma_p(\theta_0) = 4 \int_{-\infty}^{\xi_p} \dot{\varphi}\dot{\varphi}' f + \frac{1}{1-p} \dot{F}(\xi_p) \dot{F}(\xi_p)'$$

*is the Fisher information for the censored case.*

PROOF. We write $\sim$ when the difference converges in probability to zero. Thus, from Lemma 3.1 together with Lemmas 3.2 and 3.3, we have

$$\Lambda_{r,n} \sim S_r - \tfrac{1}{2}(1-p)^{-1}[h'\dot{F}(\xi_p)]^2 - \int_{-\infty}^{\xi_p} (h'\dot{\varphi})^2 f$$

and Lemma 3.4 combined with Lemma 3.5 gives

$$S_r \sim \dot{S}_r + S_p - \dot{S}_p \sim \dot{S}_r - \int_{-\infty}^{\xi_p} (h'\dot{\varphi})^2 f .$$

The next result yields the asymptotic normality of the statistic which approximates the likelihood in (3.15). That is, of

$$(3.17)\qquad \Delta_n(\theta_0) = n^{-\frac{1}{2}} \left[\sum_{j=1}^{r} 2\dot{\varphi}(Y_j) - \frac{(n-r)\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)}\right],$$

which is central to the derivation of the main result.

THEOREM 3.2. *Under Assumptions* (A) *and* (B2) *or* (B′2),

$$\dot{S}_p - \dot{S}_r \to_{P_{n,\theta_0}} 0 ,$$

*where the $h$, which enters the definitions* (3.7) *and* (3.9) *of* $\dot{S}_p$ *and* $\dot{S}_r$, *respectively, is arbitrary.*

PROOF. Employing (3.11), we expand the last term of $\dot{S}_r$ as

$$(3.18)\qquad n^{-\frac{1}{2}}(n-r)[1-F_{\theta_0}(Y_r)]^{-1}\int_{-\infty}^{Y_r} 2h'\dot{\varphi}f = n^{\frac{1}{2}}\int_{-\infty}^{Y_r} 2h'\dot{\varphi}f + n^{\frac{1}{2}}\left[\frac{F_{\theta_0}(Y_r)-p}{1-p}\right]\int_{-\infty}^{Y_r} 2h'\dot{\varphi}f + o_p(1)\,.$$

Again, setting $Z_n$ equal to the number of observations $\leqq \xi_p$, we expand the corresponding term of $\dot{S}_p$ as

$$(3.19)\qquad n^{-\frac{1}{2}}(n-Z_n)(1-p)^{-1}\int_{-\infty}^{\xi_p} 2h'\dot{\varphi}f = \frac{(1-Z_n/n)-(1-p)}{1-p}\, n^{\frac{1}{2}}\int_{-\infty}^{\xi_p} 2h'\dot{\varphi} + n^{\frac{1}{2}}\int_{-\infty}^{\xi_p} 2h'\dot{\varphi}f\,.$$

Subtracting (3.18) from (3.19) gives

$$(3.20)\qquad n^{\frac{1}{2}}\int_{Y_r}^{\xi_p} 2h'\dot{\varphi}f - n^{\frac{1}{2}}\left\{\frac{Z_n}{n} - p + [F_{\theta_0}(Y_r)-p]\right\} \times (1-p)^{-1}\int_{-\infty}^{\xi_p} 2h'\dot{\varphi}f + o_p(1)\,,$$

since $F_{\theta_0}(Y_r)$ is asymptotically normal and $Y_r \to \xi_p$ in probability. Furthermore, Lemma 2.2 establishes that the second term in (3.20) is $o_p(1)$.

From the definitions of $\dot{S}_r$ and $\dot{S}_p$ it is clearly sufficient to show that

$$n^{\frac{1}{2}}\sum_{Y_i\in A_{r,n}} \frac{2h'\dot{\varphi}}{n^{\frac{1}{2}}} - n^{\frac{1}{2}}\int_{A_{r,n}} 2h'\dot{\varphi}f \to_{P_{n,\theta_0}} 0\,,$$

where $A_{r,n}$ is given by (2.9). However, this follows directly from Corollary 2.4 under the Assumption (B′2).

Applying the central limit theorem to $\dot{S}_p$, for each $h$, we obtain

COROLLARY 3.3. *Under the assumptions of the theorem,*

$$(3.21)\qquad \mathscr{L}[\Delta_n \mid P_{n,\theta_0}] \to \mathscr{N}[0, \Gamma_p(\theta_0)]$$

*where* $\Gamma_p(\theta_0)$ *is defined by* (3.16), $\Delta_n(\theta_0)$ *by* (3.17) *and* $h'\Delta_n(\theta_0) = \dot{S}_r$.

Summarizing, we have the asymptotic normality of $\Delta_n$ from (3.21), and Theorem 3.1 states that

$$\Lambda_{r,n} + h'\Delta_n \to_{Q_{n,\theta_0}} -\tfrac{1}{2}h'\Gamma_p(\theta_0)h\,, \qquad h \in R^k\,.$$

These two results lead to an exponential approximation of the sequence of alternatives from which asymptotic optimality properties may be obtained. In particular, the results for sequences of point estimators are just the conclusions from the theorem of Hájek (1970) which requires only these two results.

As far as testing problems are concerned, the results follow from the approximation of $P_{n,\theta_n}$ by the exponential family

$$R_{n,h}(A) = \exp[-B_n(h)]\int_A \exp[h'\Delta_n^*]\,dP_{n,\theta_0}$$

based on a truncated version $\Delta_n^*$ of $\Delta_n$, as in Section 5 of Johnson and Roussas (1970), since none of the results from Section 4 onward require anything but Assumptions (A) and the approximation of $\Lambda_{r,n}$ by $\Delta_n$ which converges to a normal distribution (see Section 7 for conclusions). The results on multiparameter testing follow from Johnson and Roussas (1971).

Besides the conclusions stated at the start of the paper, just as in Theorem 6.2 of Johnson and Roussas (1970), we also conclude that the alternative distribution satisfies

$$\mathscr{L}[\Delta_n(\theta_0) \mid P_{n,\theta_n}] \to \mathscr{N}(\Gamma_p h, \Gamma_p) . \tag{3.22}$$

As in Proposition 3.1, the measures $\{P_{n,\theta_0}\}$ and $\{P_{n,\theta_n}\}$ are contiguous. The result (3.22) enables one to calculate asymptotic power.

Finally, we note that, from Theorem 3.2, that $\dot{S}_p$ could be used instead of $\dot{S}_r$ to obtain the exponential approximation to $P_{n,\theta_n}$. The statistic $\dot{S}_p$ is the one that approximates the likelihood, under Assumptions (A), for the sampling scheme which observes only the lifetimes which are not greater that $\xi_p$. The above lemma states that the asymptotically sufficient statistics for each of the two cases are asymptotically equivalent. See Kendall and Stuart (1967), page 523, for a statement that maximum likelihood estimation is the same for the two schemes although they do not state their conditions.

**4. An interpretation of $\dot{S}_r$.** We consider two special cases when $\Theta$ is a subset of the real line. For location families with $f_\theta(x) = f(x - \theta)$, Assumptions (A) are satisfied if $0 < \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x)\, dx < \infty$. In this case, $2\dot{\varphi} = -f'(x - \theta_0)/f(x - \theta_0)$ so that

$$\frac{-h\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)} = \frac{hf_{\theta_0}(Y_r)}{1 - F_{\theta_0}(Y_r)} , \tag{4.1}$$

which is the hazard rate evaluated at the $r$th order statistic. For scale alternatives, $f_\theta(x) = \theta^{-1}f(x/\theta)$, $\theta > 0$, it is sufficient for Assumptions (A) to require that $\int_{-\infty}^{\infty} [1 + x(f'(x)/f(x))]^2 f(x)\, dx < \infty$. Then $2\dot{\varphi} = \theta_0^{-1}[-1 - (x/\theta_0)(f(x/\theta_0)/f(x/\theta_0))]$ and

$$\frac{-h\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)} = \frac{(Y_r/\theta_0)f_{\theta_0}(Y_r)}{1 - F_{\theta_0}(Y_r)} . \tag{4.2}$$

Equations (4.1) and (4.2) show the manner in which the hazard rate carries the information on the unobserved order statistics in the case of scale of location alternatives. We also have the following general result which gives another interpretation.

LEMMA 4.1. *If Assumptions* (A) *hold, then*

$$(n - r)E\left[\frac{-h'\dot{F}(Y_r)}{1 - F_{\theta_0}(Y_r)}\right] = \textstyle\sum_{j=r+1}^{n} E[2h'\dot{\varphi}(Y_j)] , \qquad for \ \ 1 \leqq r < n . \tag{4.3}$$

PROOF. Set $b_{n,r} = (n - r)E[-h'\dot{F}(Y_r)/1 - F_{\theta_0}(Y_r)]$. Now $b_{n,n} = 0$ and a direct evaluation of $b_{n,n-1}$, using integration by parts with $dU = F^{n-2}f$ and $h'\dot{F}(y) \to 0$ as $y \to \infty$ $(-\infty)$ (according to (2.4)) gives $b_{n,n-1} = E[2h'\dot{\varphi}(Y_n)]$. Furthermore, an integration by parts with $dU = -f[1 - F]^{n-r-1}$ establishes that

$$(4.4) \qquad \begin{aligned} b_{n,r} &= c_{n,r}[h'\dot{F}(y)]F^{r-1}(y)[1 - F(y)]^{n-r}|_{-\infty}^{\infty} + b_{n,r-1} - E[2h'\dot{\varphi}(Y_r)] \\ &= b_{n,r-1} - E[2h'\dot{\varphi}(Y_r)] \qquad 1 < r < n\,, \end{aligned}$$

where $c_{n,r}$ is the constant for the pdf of $Y_r$. Thus, a general solution of (4.4) is given by

$$b_{n,r} = \sum_{j=r+1}^{N} E[2h'\dot{\varphi}(Y_j)]\,.$$

The result above shows that the "hazard" rate term has an expected value equal to the sum of the expected values of the unobserved scores, which themselves appear in the uncensored version in the general case. Besides giving some intuitive feeling for the manner in which the unobserved scores enter, this relationship is exactly what is needed to obtain the statistic for the locally most powerful rank test for the two sample case. It extends Johnson and Mehrotra (1972) to a general parameter. Furthermore, we also have the exact moments

$$(4.5) \qquad \begin{aligned} E\dot{S}_p &= 0 \\ E\dot{S}_r &= nE[2h'\dot{\varphi}(x_1)] = 0\,, \end{aligned}$$

which follow from the lemma and (2.4).

We conclude with a particular example, the double exponential $f_\theta(x) = \frac{1}{2}\exp[-|x - \theta|]$.

EXAMPLE. It is well known that the pdf $\frac{1}{2}\exp[-|x - \theta|]$ satisfies Assumptions (A) with $\dot{\varphi} = \frac{1}{2}\operatorname{sgn}(x - \theta_0)$. Since $\theta_0$ is a location parameter, we take $\theta_0 = 0$. Then $\varphi = \exp\{\frac{1}{2}|x| - \frac{1}{2}|x - \theta|\}$. We see that $\varphi$ is monotone in $x$. Consider the scheme for $p = \frac{1}{2}$ and $\theta_n = h_n n^{-\frac{1}{2}}$, $h_n \to h$. Then $n^{\frac{1}{2}}|\varphi - 1| \leqq |h_n|e^{|\theta_n|}$ is uniformly bounded. Thus (B'1) and (B'2) are satisfied. Here the censored tests are the same for all $p \geqq \frac{1}{2}$.

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