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PROOF OF CONJECTURES ABOUT THE EXPECTED VALUES OF THE ELEMENTARY SYMMETRIC FUNCTIONS OF A NONCENTRAL WISHART MATRIX

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This paper provides proof of conjectures about the expected values of the jth elementary symmetric function of the roots (i.e. $E(\text{tr}_j A)$) of a noncentral Wishart matrix. Utilizing this result we have obtained computationally convenient formula for the expected value of the jth elementary symmetric function of the roots of central positive definite quadratic form in matrix argument.

1. Introduction. Let $X: p \times n$ be a matrix whose column vectors are independently and identically distributed in multivariate normal distributions having $E(X) = M$ and variance covariance matrix Σ. The matrix $XX' = A$ is distributed as a noncentral Wishart distribution $W_p(n, \Sigma, \Omega; A)$ with n degrees of freedom and noncentrality parameter $\Omega = \Sigma^{-1}MM'$.

In the next section we provide the mathematical proof of conjectures made be de Waal [3] about (i) $E(\text{tr}_p A)$ and (ii) $E(\text{tr}_j A)$, $j = 1, 2, \cdots, p-1$. In the last section, we have extended the above results to the case of central positive definite quadratic form in matrix argument.

2. Central and noncentral Wishart case.

LEMMA 1. *For a central Wishart matrix* $W_p(n, \Sigma, 0; S)$

$$E(\text{tr}_p S) = E(|S|) = n^{(p)}|\Sigma| , \tag{2.1 (a)}$$

$$E(\text{tr}_j S) = n^{(j)} \text{tr}_j \Sigma , \qquad j = 1, 2, \cdots, p-1 , \tag{2.2 (b)}$$

where $n^{(p)} = n(n-1) \cdots (n-p+1)$.

PROOF. (a) This is a well-known result and proof is given in many standard text books.

(b) By definition, $\text{tr}_j S$ is the sum of all principal minors of order j of matrix S. Thus,

$$\begin{aligned} E(\text{tr}_j S) &= E(\text{sum of all principal minors of order } j \text{ of matrix } S) \\ &= n^{(j)}(\text{sum of all principal minors of order } j \text{ of matrix } \Sigma) \\ &= n^{(j)} \text{tr}_j \Sigma . \end{aligned}$$

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This establishes Lemma 1. We may note here that de Waal [3] has established the lemma by providing zonal polynomial argument.

LEMMA 2. *For a noncentral Wishart matrix* $W_p(n, \Sigma, \Omega; A)$ *with* $\Omega = \Sigma^{-1}W$, $W = MM'$,

$$\text{(2.3)} \qquad \text{(a)} \quad E(\operatorname{tr}_p A) = E(|A|) = n^{(j)}|\Sigma| + \textstyle\sum_{i=1}^{p} (n-i)^{(p-i)}|\Sigma| \operatorname{tr}_i \Omega$$

$$\text{(2.4)} \qquad \text{(b)} \quad E(\operatorname{tr}_j A) = n^{(j)} \operatorname{tr}_j \Sigma + \textstyle\sum_{k=1}^{j} (n-k)^{(j-k)} \sum_{i(j)} |\Sigma(i(j))| \operatorname{tr}_k \{[\Sigma(i(j))]^{-1} W(i(j))\}, \qquad j = 1, 2, \cdots, p-1,$$

where $\Sigma(i(j))$ *and* $W(i(j))$ *are submatrices obtained by considering* $i_1, i_2, \cdots, i_j$ *rows and* $i_1, i_2, \cdots, i_j$ *columns of matrices* Σ *and* W *respectively and* $\sum_{i(j)} = \sum_{i_1=1; i_1>i_2>\cdots>i_j}^{p} \sum_{i_j=1}^{p}$.

PROOF. Part (a): The noncentral moments of the generalized variance $|A|$ have been given by Anderson [1] for the case of Ω of rank $\leqq 2$, and in general by Herz [4] who quoted by Constantine [2] as:

$$\text{(2.5)} \qquad E(\operatorname{tr}_p A) = |2\Sigma|\Gamma_p(\tfrac{1}{2}n + 1)\{\Gamma_p(\tfrac{1}{2}n)\}^{-1} {}_1F_1(-1; \tfrac{1}{2}n; -\tfrac{1}{2}\Omega),$$

where ${}_pF_q(a, b; S) = \sum_{k=0}^{\infty} \sum_{\kappa} ((a)_\kappa/(b)_\kappa)(C_\kappa(S)/k!)$, $\sum_\kappa$ is taken over all the partitions $\kappa = (k_1, k_2, \cdots, k_p)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0$ so that $k_1 + k_2 \cdots + k_p = k$, $(a)_\kappa = \prod_1^p \{a - (i-1)/2\}_{k_i}$, $(x)_k = x(x+1) \cdots (x+k-1)$, $C_\kappa(s)$ is a polynomial of degree k in the latent roots of the matrix S and $\Gamma_p(x) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\{x - (j-1)/2\}$. The difficulty, as quoted by de Waal ([3], page 346), is to simplify expression (2.5). From definitions

$$\text{(2.6)} \qquad |2\Sigma|\Gamma_p(\tfrac{1}{2}n + 1)\{\Gamma_p(\tfrac{1}{2}n)\}^{-1} = n^{(p)}|\Sigma|, \qquad \text{and}$$

$$\text{(2.7)} \qquad {}_1F_1(-1; \tfrac{1}{2}n; -\tfrac{1}{2}\Omega) = \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} \dfrac{(-1)_\kappa}{(\frac{1}{2}n)_\kappa} \dfrac{C_\kappa(-\Omega/2)}{k!}.$$

Now $(-1)_\kappa = 0$ if $k_1 > 1$. Hence $(-1)_\kappa = 0$ if $k > p$. Further, $(-1)_\kappa = \prod_{i=1}^{k} \{-1 - (i-1)/2\}$, if $k_1 = k_2 = \cdots = k_k = 1$ and other $k_j = 0$ for $k \leqq p$. Hence $(-1)_\kappa = (-1)^k (k+1)!\,(2^{-k})$, for $\kappa = \{1^k, 0\}$. Therefore,

$$\text{(2.8)} \qquad {}_1F_1(-1; \tfrac{1}{2}n; -\tfrac{1}{2}\Omega) = \textstyle\sum_{k=0}^{p} (k+1)\{n^{(k)} 2^k\}^{-1} C_{\{1^k, 0\}}(\Omega).$$

For every partition $\kappa = (k_1, k_2, \cdots, k_p)$, $k_1 \geqq \cdots \geqq k_p \geqq 0$, the zonal polynomials $C_\kappa(\Omega)$ can be written as (see [5], equation 18, page 478)

$$\text{(2.9)} \qquad C_\kappa(\Omega) = \{\chi_{[2\kappa]}(1) 2^k k!\} Z_\kappa(\Omega)/(2k).$$

Equation (4.6) is to be evaluated at $\kappa = \{1^k, 0\}$. Using equation 19, page 479 of James [5], we have

$$\text{(2.10)} \qquad \chi_{[2\{1^k, 0\}]}(1) = (2k)!\,\{k!\,(k+1)!\}^{-1},$$

and noting (120) and (121) of James [5] on page 492, we have

$$\text{(2.11)} \qquad Z_{\{1^k, 0\}}(\Omega) = k! \operatorname{tr}_k \Omega.$$

Substituting the values of equations (2.10) and (2.11) in (2.9), we obtain

$$C_{\{1^k,0\}}(\Omega) = 2^k(\operatorname{tr}_k \Omega)(k+1)^{-1}. \tag{2.12}$$

Thus,

$$ {}_1F_1(-1; \tfrac{1}{2}n; -\tfrac{1}{2}\Omega) = \textstyle\sum_{k=0}^{p} \operatorname{tr}_k \Omega / n^{(k)}. \tag{2.13}$$

Combining expressions (2.12) and (2.13) establishes a conjectured result.

Part (b): By definition of trace, it is easy to write

$$\operatorname{tr}_j A = \textstyle\sum_{i(j)} |A(i(j))| \tag{2.14}$$

where $A(i(j))$ is a submatrix obtained by considering $i_1, i_2, \cdots, i_j$ rows and $i_1, i_2, \cdots, i_j$ columns of matrix A. We may further note that the submatrix $A(i(j))$ is distributed as noncentral Wishart $W_j(n, \Sigma(i(j)), [\Sigma(i(j))]^{-1}W(i(j)); A(i(j)))$. Applying the result of Lemma 2 part (a) to the submatrix $A(i(j))$, we get

$$\begin{aligned} E(\operatorname{tr}_j A) &= \textstyle\sum_{i(j)} E\{|A(i(j))|\} \cdot \sum_{i(j)} \{n^{(j)} \\ &\quad + \textstyle\sum_{k=1}^{j} (n-k)^{(j-k)} \operatorname{tr}_k [\Sigma(i(j))]^{-1}W(i(j))\}|\Sigma(i(j))| \end{aligned} \tag{2.15}$$

which establishes part (b).

COROLLARY. *When* $\Sigma = \sigma^2 I_p$

$$E(\operatorname{tr}_j A) = \sigma^{2j} \textstyle\sum_{k=0}^{j} (n-k)^{(j-k)} \binom{p-k}{j-k} \operatorname{tr}_k \Omega, \qquad j = 1, 2, \cdots, p. \tag{2.16}$$

PROOF. Since $\Sigma = \sigma^2 I_p$, we have

$$\textstyle\sum_{i(j)} \operatorname{tr}_k [\Sigma(i(j))]^{-1} W(i(j)) = \binom{p-k}{j-k} \operatorname{tr}_k \Omega,$$

which establishes the corollary.

3. Quadratic forms in matrix argument. Let $Q = XLX'$, where $X: p \times n$ is a matrix whose column vectors are independently and identically distributed in multivariate normal distribution having $E(X) = 0$ and variance covariance matrix Σ, and $L: n \times n$ is a positive definite matrix. Previously Khatri [6] gave $E(|Q|)$ involving hypergeometric series representation in matrix argument. This expression is very difficult to evaluate numerically. Lemma 3 gives a simpler and computationally convenient expression.

LEMMA 3. *For a central positive definite quadratic form*

$$\text{(a)}\quad E(|Q|) = n^{(p)}|\Sigma| q^p \textstyle\sum_{i=0}^{p} (-1)^i p^{(i)} \operatorname{tr}_i T / n^{(i)} i!, \tag{3.1}$$

$$\text{(b)}\quad E(\operatorname{tr}_j Q) = n^{(j)} q^j (\operatorname{tr}_j \Sigma) \textstyle\sum_{i=0}^{j} (-1)^i j^{(i)} \operatorname{tr}_i T / n^{(i)} i!, \tag{3.2}$$

where $q > 0$ and $T = I_n - q^{-1}L$.

PROOF. (a) Equation (46) of [6] can be written as,

$$\begin{aligned} E(|Q|) &= n^{(p)}|\Sigma| q^{p(\frac{1}{2}n+1)} |L|^{-p/2} {}_1F_0^{(n)}(\tfrac{1}{2}n + 1; I_n - qL^{-1}, I_p) \\ &= n^{(p)}|\Sigma| |qL^{-1}|^{\frac{1}{2}p} q^p {}_2F_1(\tfrac{1}{2}n + 1, p; \tfrac{1}{2}n; I_n - qL^{-1}) \\ &= n^{(p)}|\Sigma| q^p {}_2F_1(-1, \tfrac{1}{2}p; \tfrac{1}{2}n; T), \end{aligned} \tag{3.3}$$

since ${}_2F_1(a; b; c; S) = |I - S|^{-b} {}_2F_1(c - a, b; c; -S(I - S)^{-1})$. Equation (3.3) can then be simplified in the manner described in Lemma 2, giving a final result as mentioned in (3.1). We may note that when $L = I$, (3.1) reduces to (2.1).

(b) By applying the argument of Lemma 1, we can easily establish Lemma 3(b). This result generalizes Lemma 1.

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